Improved Inference for the GA0 Distribution

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Abstract

This paper presents adjusted profile likelihoods for α, the roughness parameter of $\mathcal{G}_A^0(\alpha,\gamma,\mathcal{L})$ distribution. This distribution has been widely used in the modelling of data corrupted by speckle noise (SAR images). We focus on point estimation and on signalized likelihood ratio tests. As far as point estimation is concerned, the numerical evidence presented in the paper favors the Cox and Reid's adjustment [1], and in what concerns signalized likelihood ratio tests, the results favor the approximation to Barndorff–Nielsen's adjustment based on the results in [2]. An application to real synthetic aperture radar imagery is presented.

1. Introduction

Imagery obtained with coherent illumination suffers from a noise known as speckle. This is the case of laser, sonar, ultrasound-B and Synthetic Aperture Radar (SAR) images. The noise does not follow the classical Gaussian additive structure, being multiplicative in nature. Classical techniques for image analysis are thus inefficient for extracting information from speckled data.

This phenomenological model states that the observation in every pixel is the outcome of a random variable $Z\colon\Omega\to \mathbb{R}_+$ which is, in turn, the product of two independent random variables: $X\colon\Omega\to\mathbb{R}_+$, the ground truth or backscatter, related to the intrinsic dielectric properties of the target, and $Y\colon\Omega\to\mathbb{R}_+$, the speckle noise, which follows a square root of gamma law. The distribution of the return, $Z=XY$, is completely specified by the distributions of X and Y.

The univariate multiplicative model began as a single distribution, namely the Rayleigh law, was extended by [4] to accomodate the K law and was later further improved by [3] to the G distribution, which generalizes the previous probability distributions.

The GA0 law is an important particular case of the more general G distribution. It can successfully model a wide range of targets through their roughness. If Z is a $\mathcal{G}_A^0(\alpha,\gamma,\mathcal{L})$-distributed random variable, then its probability density function is

$$p(z;\alpha,\gamma,\mathcal{L}) = p(z) = \frac{2\mathcal{L}^{\mathcal{L}}\Gamma(\mathcal{L}-\alpha)z^{2\mathcal{L}-1}}{\gamma^{\alpha}\Gamma(\mathcal{L})\Gamma(-\alpha)(\gamma+\mathcal{L}z^2)^{\mathcal{L}-\alpha}},$$

with $-\alpha,\gamma,z\geq 0$ and $\mathcal{L}\geq 1$. The parameter α is directly related to the roughness of the target. For typical sensors and scenes, if $\alpha\leq -10$ then the area is homogeneous (usually crops or pastures), if $-10<\alpha\leq -5$ then the region is heterogeneous (usually forests or undulated relief), and $-5<\alpha<0$ is associated with extremely heterogeneous targets (usually urban areas). γ is a scale parameter that can be viewed as a nuisance parameter, and $\mathcal{L}$, the number of looks (it will be assumed known in our study), is directly related to the signal-to-noise ratio (the smaller $\mathcal{L}$, the noisier the image).

This paper presents two new results regarding inference under the GA0 model, namely: we obtain analytically improved parameter estimators and develop improved one-sided likelihood ratio inference. Improved parameter estimation is achieved by maximizing an adjusted profiled likelihood function ([1], [2]). We also develop one-sided improved likelihood ratio inference for the GA0 roughness parameter. The chief goal of such inference lies in identifying whether a given scanned region is extremely heterogeneous, heterogeneous or homogeneous. This kind of analysis, that turns data into valuable information for decision making, is one of the ultimate goals of environmental studies.

2. Monte Carlo Results

Images are richly structured data consisting of several underlying classes, that turn into more or less discernible groups of values; these values can be displayed as shades of gray or as colors. A digital image is a function $f: S \to K^p$, where $S \subset \mathbb{Z}^2$ is the (finite) support of the data, $p \in \mathbb{N}$ is the number of bands or dimensionality of the data and $K \subset \mathbb{R}$ is the set of possible values.

Neighborhoods are usually squares of odd side, called 'windows', centered on the pixel being processed. The smallest non-trivial odd window is of size 3×3, but odd windows up to side 11 are frequently used. This defines the sample sizes that we used in the Monte Carlo experiments, namely, 25, 49, 81 and 121 (11×11).

The following values were used for $(-\alpha; \mathcal{L})$: $(1; 1)$, $(1; 3)$, $(5; 3)$, $(5; 8)$, $(8; 3)$, $(8; 8)$, $(10; 3)$, $(10; 8)$, $(15; 3)$, $(15; 8)$. The value of the nuisance parameter was chosen so that the resulting $\mathcal{G}_A^0$-distributed random variable has unit mean.

In what follows we shall present numerical results related to the inference of the roughness parameter α. All Monte Carlo results are based on 10,000 replications. Maximum likelihood estimators obtained from usual likelihood (*MLE*), adjusted profile likelihood based on [1] (*aMLE–1*), and adjusted profile likelihood based on [2] (*aMLE–2*) are considered.

To save space we only consider the situation where $\alpha = -1$, i.e., we simulate observations on the return signal amplitude of an extremely heterogeneous region, e.g., an urban area. For a window of size 7×7 and number of looks ($\mathcal{L}$) equal to one, the least favorable situation, the relative bias of the estimator *aMLE–1* (0.973%) was approximately twenty times smaller than that of *MLE* (20.349%). The mean squared errors of these estimators were 0.097 and 1.421, respectively, that is, the mean squared error of *aMLE–1* was over 14 times smaller than that of *MLE*. The best performing estimator was *aMLE–1*, followed by the estimator *aMLE–2*.

We have also performed one-sided signalized likelihood ratio tests on the roughness parameter using the test statistic: $\mathrm{signal}(\widehat{\alpha} - \alpha)\sqrt{LR}$, where LR denotes the likelihood ratio test statistic based on the usual likelihood (*LRT*) or on the adjusted profile likelihoods, *aLRT–1* ([1]) and *aLRT–2* ([2]), and $\widehat{\alpha}$ denotes the respective maximum likelihood estimate: *MLE*, *aMLE–1* and *aMLE–2*. We performed two tests, namely:

1. homogeneous and heterogeneous regions $\times$ extremely heterogeneous region,
2. homogeneous region $\times$ heterogeneous and extremely heterogeneous regions.

The main goal here is to compare the finite-sample behavior of the different tests. The asymptotic null distribution of all test statistics is standard normal.

To save space we only consider the Test 1 and the null rejection rates of the different signalized likelihood ratio tests at the 10% significance level. The value of α is -5 and we consider the following pair $(n, \mathcal{L}) = (81, 3)$. The tests based on adjusted profile likelihoods displayed the smallest size distortions. The null rejection rates of the tests based on *LRT*, *aLRT–1* and *aLRT–2* were equal to 7.920%, 10.500%, 9.080%, respectively. At the 5% significance level, when the null hypothesis is false, the rejection rates (power) of the test *aLRT–1* were always greater than those of the other tests; the next best performing test was *aLRT–2*.

3. Application to Real Data

Finally, we have analyzed real data from a single look image obtained by the E-SAR airborne sensor over surroundings of München, Germany, originally of 1024×600 pixels with a resolution of the order of one meter. We selected samples from the three typical regions present (crops, forest and urban areas) in the image. The adjusted profile maximum likelihood estimators proved to be more capable of providing useful information about the nature of the ground truth than the usual maximum likelihood estimator. Usual and adjusted profile likelihood ratio test statistics were computed using the same data. Decisions based on the former always suggested that the imaged area was urban, even when that was clearly not so, whereas the adjusted profile likelihood test yielded much more sensible inference.

References

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