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A.9 Three-dimensional Rotations

This section collects some useful facts about *3-D rotations* and their representation. We limit ourselves to a few formulae involved by the discussions in this book, most importantly Chapter 11.

General Properties of Rotation Matrices

A rigid rotation of a 3-D vector $\mathbf{v}$ onto a 3-D vector $\mathbf{v}'$ can be represented by a linear transformation, defined by a 3×3 matrix R:

$$\mathbf{v}' = R\mathbf{v},$$

subject to the constraints

$$RR^\top = R^\top R = I \qquad \det(R) = 1,$$

with I the identity matrix. The first constraint tells you that the *inverse* of R equal its *transpose*. The second that *the transformation preserves the relative orientation of the reference frame*, and therefore its right- or left-handedness.

A matrix R for which $RR^\top = I$ is called *orthogonal*. The orthogonality property can be better appreciated by assuming that $\mathbf{v}$ and $\mathbf{v}'$ are expressed in two orthogonal reference frames, defined by the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ respectively. It can easily be seen that the generic entry r_{ij} of R is *the cosine of the angle formed by the base vector*, $\mathbf{e}_i$, *with the rotated base vector* $\mathbf{e}'_j$:

$$r_{ij} = \mathbf{e}_i^\top \mathbf{e}'_j.$$

Therefore we have that

$$\sum_{j=1}^{3} r_{ij} r_{kj} = \sum_{j=1}^{3} r_{ji} r_{jk} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases} \tag{A.16}$$

that is, *the rows (and columns) of R are mutually orthogonal unit vectors.*

As discussed in section A.6, this property makes the numerically estimation of a rotation matrix a rather tricky task. Even a slight perturbation of the entries of R, due to noise or small errors, destroys the orthogonality property and affects the estimation of the rotation parameters.

Two Useful Parametrizations

A 3×3 orthogonal matrix has nine elements which have to satisfy the six orthogonality constraints (A.16). This reduces the number of *degrees of freedom* of a 3-D rotation to $9 - 6 = 3$ and tells you that describing a rotation matrix through its nine entries is redundant and not really natural; in most cases it is useful and simpler to represent a rotation by means of more *natural* parametrization. In what follows we limit ourselves to the two parametrizations used in the book: *rotations around the coordinate axes* and *axis and angle*. Further parametrizations, well-known in computer vision, include *Euler angles*, *quaternions*, and *pitch,roll,yaw angles* (see References).

Rotations Around the Coordinate Axes. We can express a 3-D rotation as the result of three consecutive rotations around the coordinate axes, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, by angles α, β, and γ respectively. The angles are then the three free parameters of R, and each rotation is expressed as a rotation matrix, R_j, rotating vectors around $\mathbf{e}_j$, that is,

$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_2(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_3(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix R describing the overall rotation is the product of the R_j:

$$R = R_1 R_2 R_3 =$$

$$\begin{bmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma & \cos\alpha\cos\gamma \end{bmatrix}$$

☞ The order of multiplication matters. Different sequences give different results, with the same triplet of angles. Notice that there are six ways to represent a rotation matrix as a product of rotations around three different coordinate axes.

The recovery of α, β, and γ from R is easy (and is left to you as an exercise). However, if the matrix R has been obtained as the output of some numerical computation, you should always make sure that the estimated R is *really* orthogonal; section A.6 gives you the recipe that we employed throughout the book.

Axis and Angle. According to *Euler's theorem*, any 3-D rotation can be described as a rotation by an angle, θ, around an axis identified by a unit vector $\mathbf{n} = [n_1, n_2, n_3]^\top$. The corresponding rotation matrix, R, can then be obtained in terms of θ and the components of $\mathbf{n}$, which gives you a total of four parameters. The redundancy of this parametrization (four parameters for three degrees of freedom) is eliminated by adding the constraint that **n** has unit norm, that is, by dividing each n_i by $\sqrt{n_1^2+n_2^2+n_3^2}$.

The matrix R in terms of θ and **n** is given by

$$R = I\cos\theta + (1-\cos\theta)\begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2^2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3^2 \end{bmatrix} + \sin\theta\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \quad (A.17)$$

Conversely, both θ and **n** can be obtained from the eigenvalues and eigenvectors of R. The three eigenvalues of R are 1, $\cos\theta + i\sin\theta$, and $\cos\theta - i\sin\theta$, where i is the imaginary unit. The unit vector, **n**, is proportional to the eigenvector of R corresponding to the eigenvalue 1; the angle θ can be obtained from either of the two complex eigenvalues. To resolve the ambiguity in the sign of both θ and **n**, you can check the consistency of (A.17).

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