9 Interoperability aspects of fuzzy sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

9.1 Fuzzy sets and its family of α–cuts

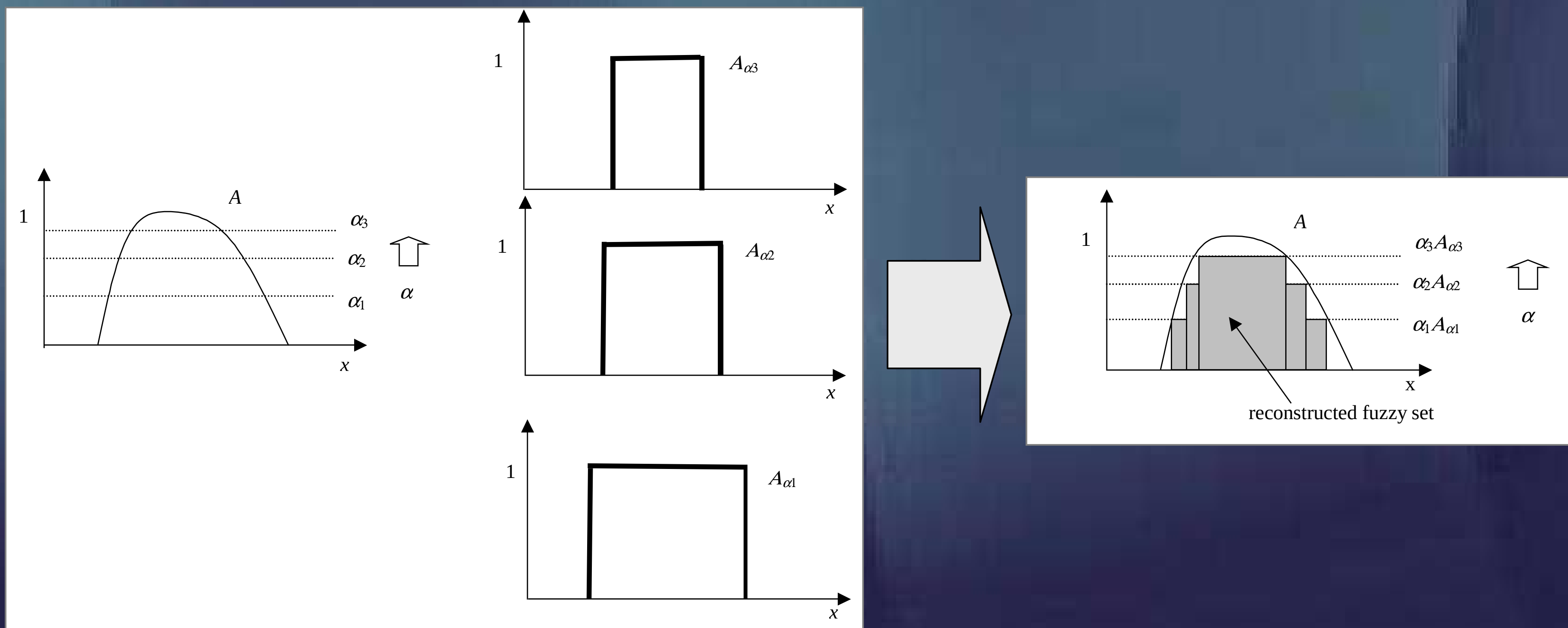
From fuzzy set to a family of sets

- Representation theorem offers an important insight into links between a given fuzzy set and its α-cuts
- Any fuzzy set can be represented as an infinite family of α-cuts

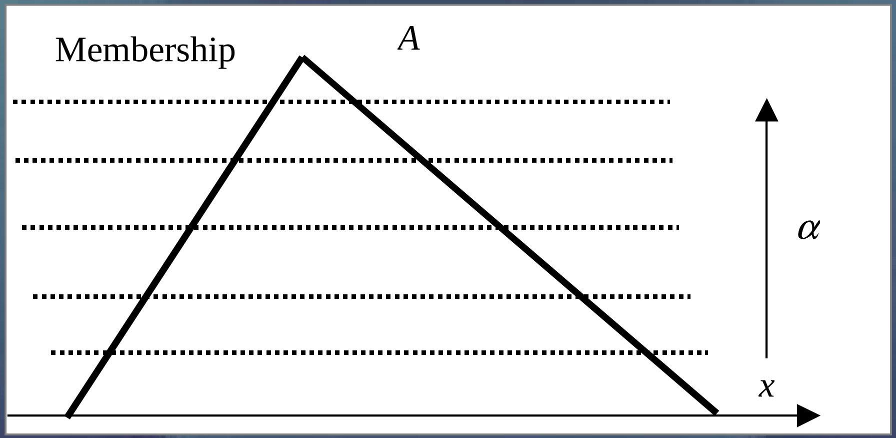
$$A = \bigcup_{\alpha>0} \alpha A_\alpha$$

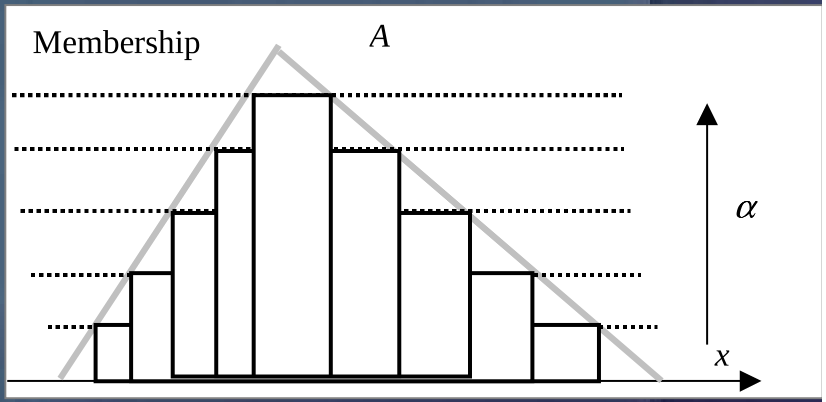
$$A_\alpha = \{x \in \mathbf{X} \mid A(x) \geq \alpha\}$$

Reconstruction



Reconstruction





From fuzzy set to a family of sets: An optimization

- Is there an optimal level a that optimizes a single α-cut of A so that A_a approximates A to the highest extent?

- Performance index

$$Q = \int_{x \notin A_\alpha} A(x)dx + \int_{x \in A_\alpha} (1\text{-}A(x))dx$$

$$\min_\alpha Q = Q(\alpha_{opt})$$

$$\alpha_{opt} = \arg \min_\alpha Q(\alpha)$$

Triangular fuzzy sets optimization

$$A(x) = \max(1-\frac{x}{b},0), \quad x \geq 0$$

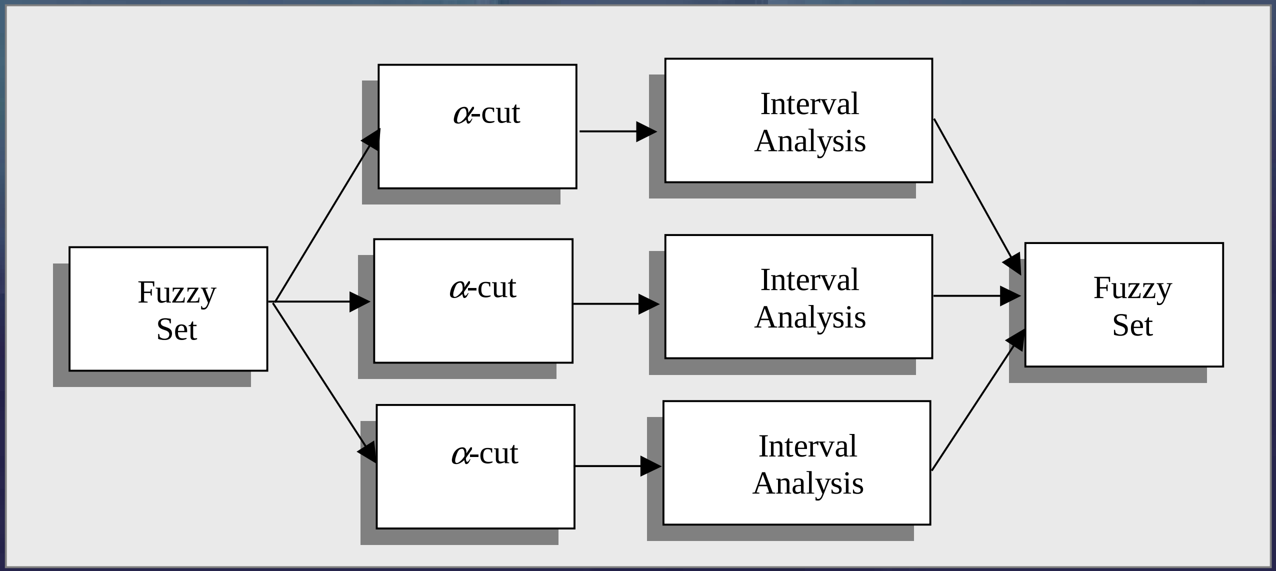
$$Q = \int_{b(1-\alpha)}^{b} \left(1-\frac{x}{b}\right) dx + \int_{0}^{b(1-\alpha)} \left(1-1-\frac{x}{b}\right) dx$$

$$Q = b - b(1-\alpha) + b(1-\alpha)^2 - \frac{b}{2}$$

$$\frac{\partial Q}{\partial \alpha} = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2}$$

Set-based approximation of fuzzy sets

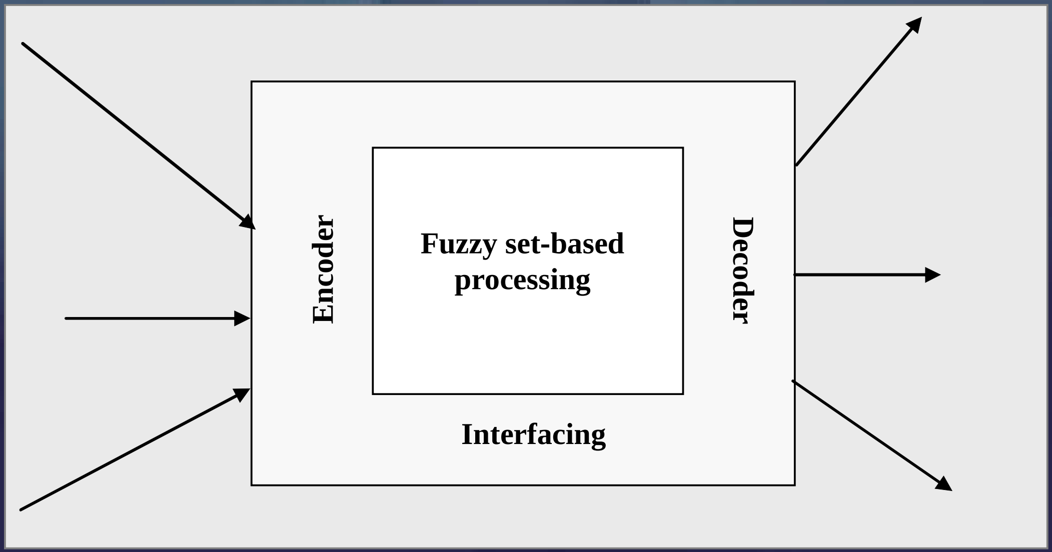
By approximating fuzzy sets by a finite family of sets we can directly exploit well-developed techniques of interval analysis and combine the partial results into a single fuzzy set (result).



9.2 Fuzzy sets and their interfacing with the external world

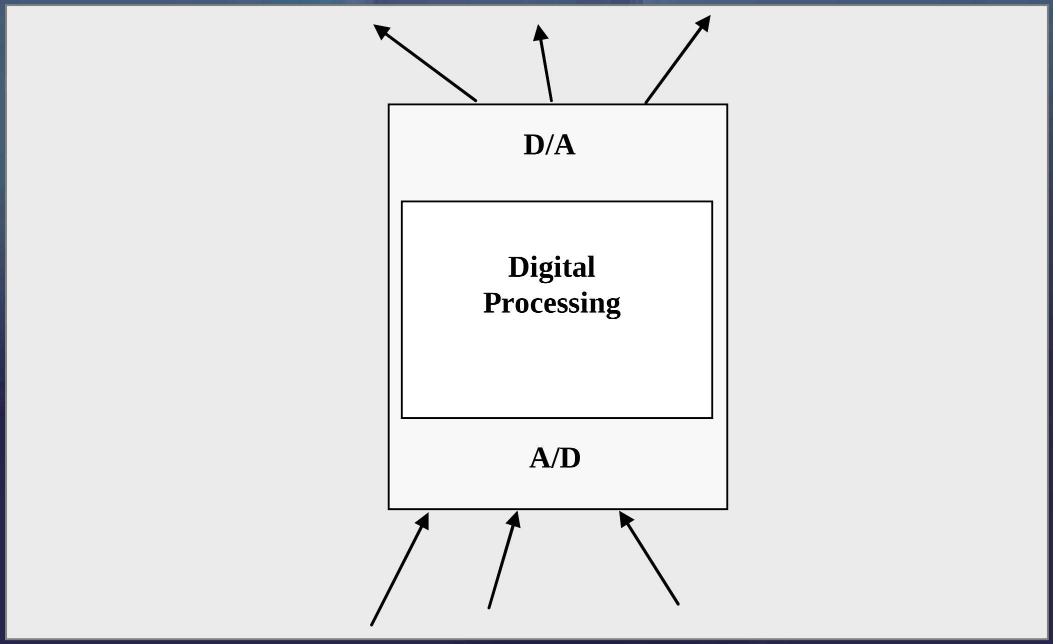
Fuzzy sets and interfaces

- Fuzzy sets do not exist in real-world (sets do not as well)
- To interact with the world one has to construct interfaces (encoders and decoders)



Fuzzy sets and interfaces

- Need for building interfaces exists in case of sets (interval analysis)
- Here we encounter well-known constructs of analog-to-digital (AD) and digital-to-analog (DA) converters.



Fuzzy sets and interfaces

- Two functional modules:
  - Encoders The objective is to translate input data into some internal format acceptable for processing at level of fuzzy sets
  - Decoders The objective is to convert the results of processing of fuzzy sets into some format acceptable by the external world (typically in the form of some numeric quantities)
- For encoding and decoding we engage a collection of fuzzy sets – information granules

Encoding mechanisms

- Given is a collection of fuzzy sets $A_1, A_2, \ldots, A_c$; express some numeric input x in $\mathbf{R}$ in terms of these fuzzy sets

$$x \rightarrow [\, A_1(x) \quad A_2(x) \ldots A_c(x)]$$

- *Nonlinear* mapping from $\mathbf{R}$ to c-dimensional unit hypercube

Decoding mechanisms

- Decoding completed on a basis of a single fuzzy set)
- Decoding realized on a basis of a certain finite family of fuzzy sets and levels of their activation

Decoding process: a single fuzzy set

Single fuzzy set B $\rightarrow$ develop a single numeric representative

$$\hat{x} = \frac{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_p}{p}$$

Mean of maxima

$$\int_{-\infty}^{\hat{x}} B(x)dx = \int_{\hat{x}}^{\infty} B(x)dx$$

Centre of Area

$$\hat{x} = \frac{\int_{\mathbf{X}} B(x)xdx}{\int_{\mathbf{X}} B(x)dx}$$

Centre of gravity

Single fuzzy set decoding: centre of gravity

- Solution to the following optimization problem

$$\min_{\hat{x}} V = \int_{\mathbf{X}} B(x)[x-\hat{x}]^2 dx$$

$$\frac{\partial V}{\partial \hat{x}} = 0 \quad \Rightarrow \quad 2\int_{\mathbf{X}} B(x)[x-\hat{x}]dx = 0$$

Single fuzzy set decoding: augmented strategies

- Augmented centre of gravity

$$\hat{x} = \frac{\int\limits_{x \in \mathbf{X}: B(x) \geq \beta} B(x) x dx}{\int\limits_{x \in \mathbf{X}: B(x) \geq \beta} B(x) dx} \quad \rightarrow \quad \hat{x} = \frac{\int\limits_{x \in \mathbf{X}: B(x) \geq \beta} B^{\gamma}(x) x dx}{\int\limits_{x \in \mathbf{X}: B(x) \geq \beta} B^{\gamma}(x) dx}$$

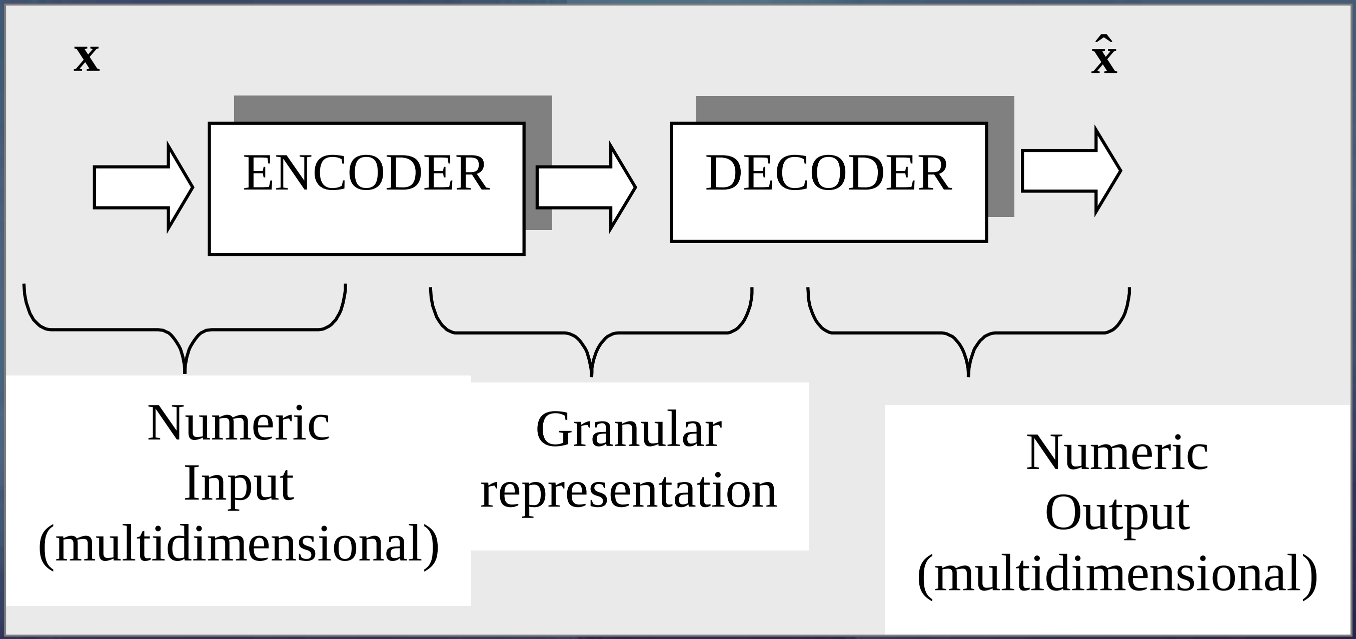
Single fuzzy set decoding: general requirements

- Requirements implied by :
  - monotonicity with respect to changeable membership functions
  - graphically motivated requirements (symmetry, translation, scaling…)
  - use of logic operations and logic modifiers

9.3 Encoding and decoding as an optimization problem of vector quantization

Fuzzy scalar optimization

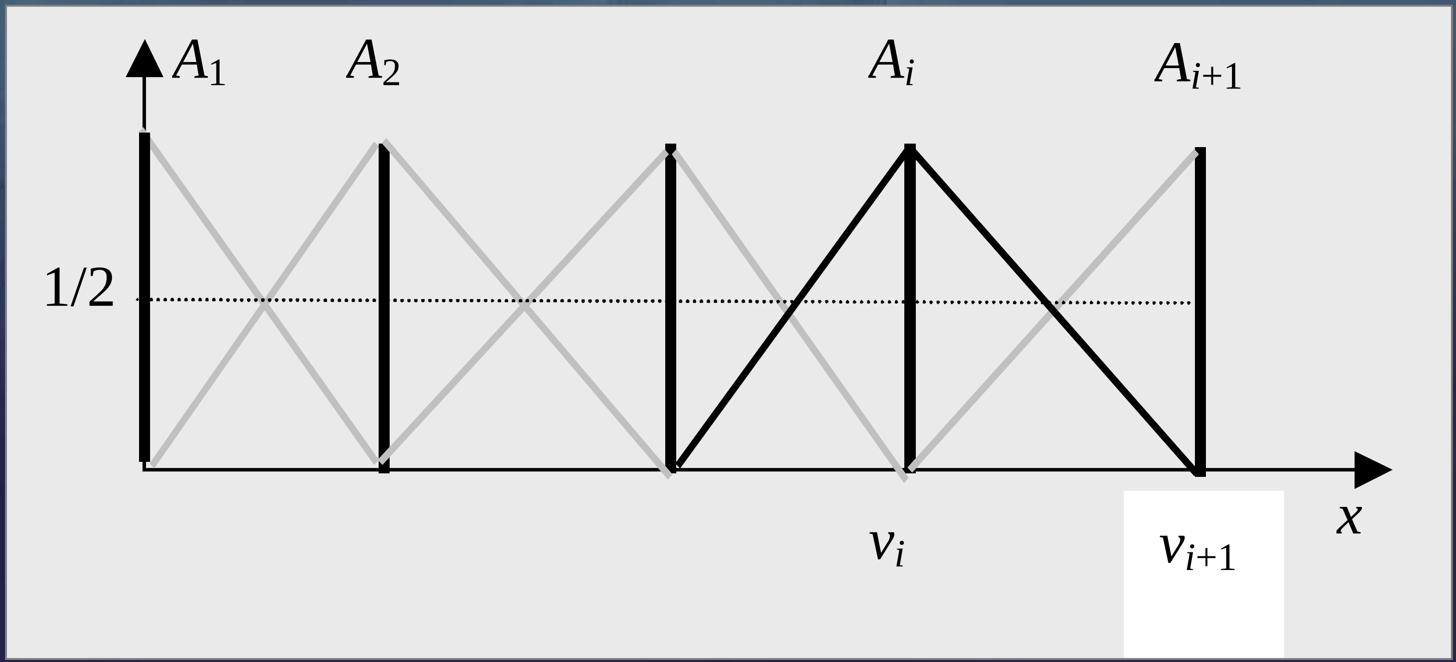
- Decoding: a collection of fuzzy sets



- One-dimensional case
- Multivariable case

Decoding: one-dimensional (scalar) case

Codeboook – a finite family of fuzzy sets $\{A_1, A_2, \ldots, A_c\}$



Proposition

Assume:

a) $\{A_i\}$ $i = 1,\ldots, c$ forms a partition

$$\sum_{i=1}^{c} A_i(x) = 1, \quad \forall x \in \mathbf{X}, \quad \exists i \mid A_i(x) > 0$$

b) $A_i > 0$, $A_{i+1} > 0$ and $A_k = 0 \quad \forall k \neq i, i+1$

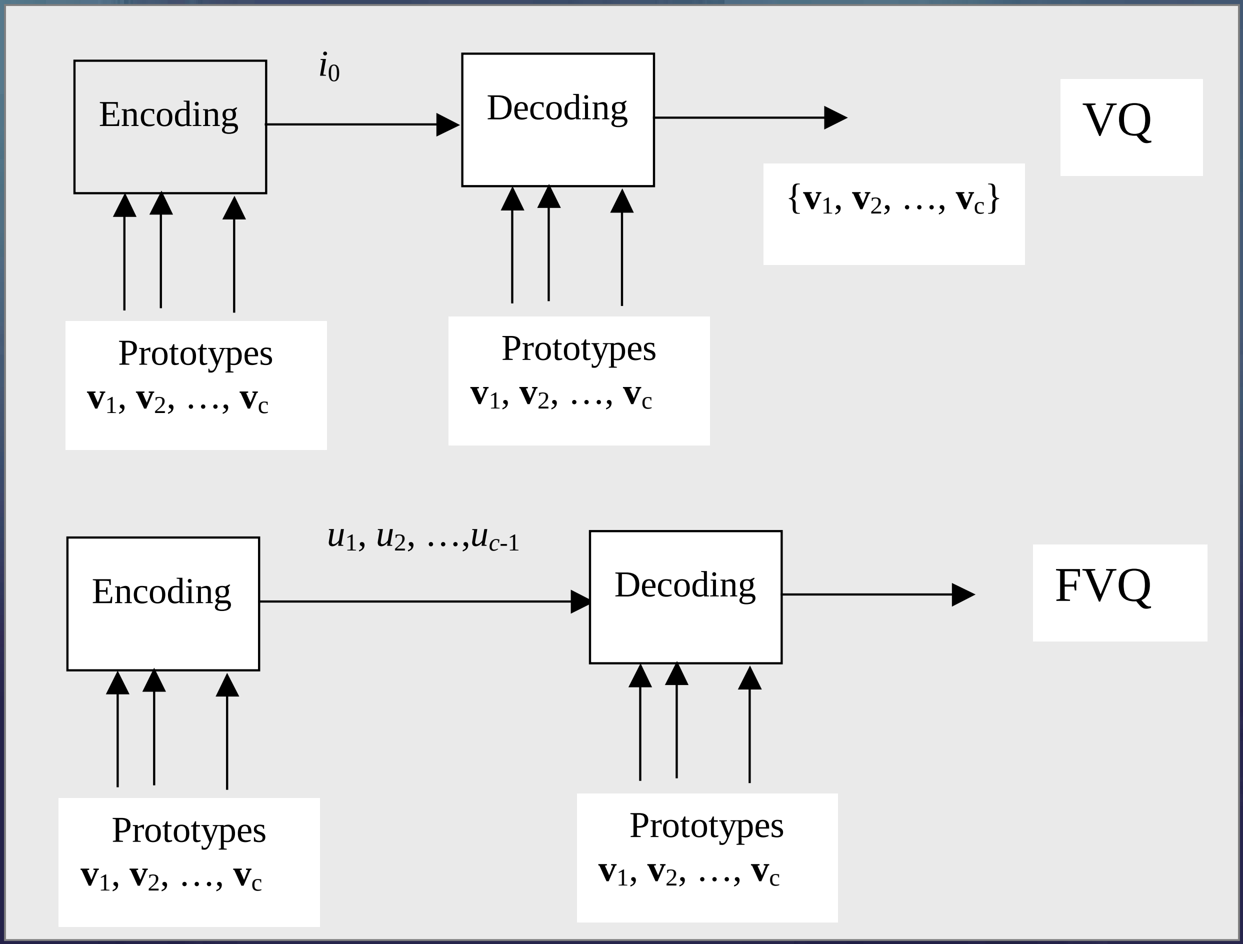
c) decoding is a weighted sum of activation levels and prototypes v_i

$$\hat{x} = \sum_{i=1}^{c} A_i(x) v_i$$

Then

$$A_i(x) = \begin{cases} \dfrac{x - v_{i-1}}{v_i - v_{i-1}} & \text{if} \quad x \in [v_{i-1}, v_i] \\ \dfrac{x - v_{i+1}}{v_i - v_{i+1}} & \text{if} \quad x \in [v_{i-1}, v_{i+1}] \end{cases}$$

Forming mechanisms of fuzzy quantization



use of sets –
Vector Quantization
(VQ)

use of fuzzy sets –
Fuzzy Vector Quantization
(FVQ)

Fuzzy vector quantization

- Codebook formed through fuzzy clustering (FCM) producing a finite collection of prototypes $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c$
- Given any new input $\mathbf{x}$ we realize its encoding and decoding
- Recall
  - **encoding**: representation of x in terms of the prototypes
  - **decoding**: development of external representation of the result of processing realized at the level of information granules

Coding and decoding with fuzzy codebooks

Encoding: optimization problem

$$\sum_{i=1}^{c} u_i^m ||\mathbf{x} - \mathbf{v}_i||^2$$

Minimize w.r.t. u_i subject to

$$u_i(\mathbf{x}) \in [0,1], \quad \sum_{i=1}^{c} u_i(\mathbf{x}) = 1$$

$$u_i(\mathbf{x}) = \frac{1}{\sum \left(\frac{||\mathbf{x} - \mathbf{v}_i||}{||\mathbf{x} - \mathbf{v}_j||} \right)^{\frac{2}{m-1}}}$$

- Decoding: optimization problem

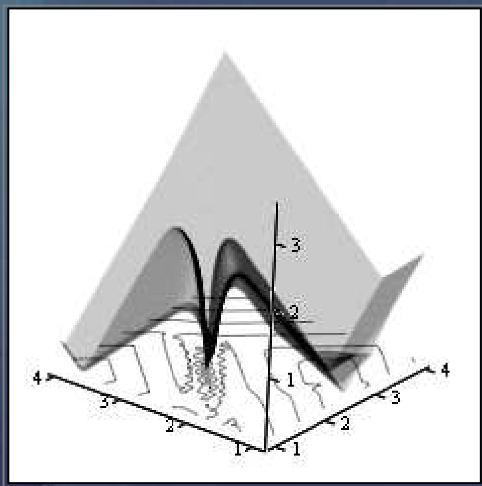
- Reconstruct original mutidimensional input $\mathbf{x}$

$$Q_2(\hat{\mathbf{x}}) = \sum_{i=1}^{c} u_i^m \| \hat{\mathbf{x}} - \mathbf{v}_\mathrm{i} \|^2$$

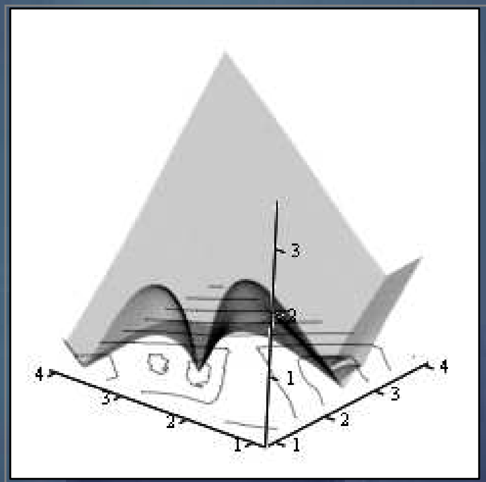
minimize

$$\hat{x} = \frac{\sum_{i=1}^{c} u_i^m \mathbf{v}_i}{\sum_{i=1}^{c} u_i^m}$$

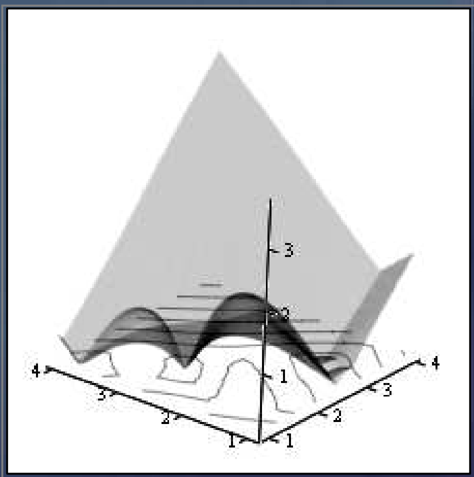
Fuzzy vector quantization: decoding error



$m = 1.2$



$m = 2.0$



$m = 3.5$

9.4 Decoding of a fuzzy set through a family of fuzzy sets

Fuzzy encoding and decoding with possibility and necessity measures

- Consider a family of fuzzy sets $A_1, A_2, \ldots, A_c$
- Input datum X either a fuzzy set or a numeric quantity

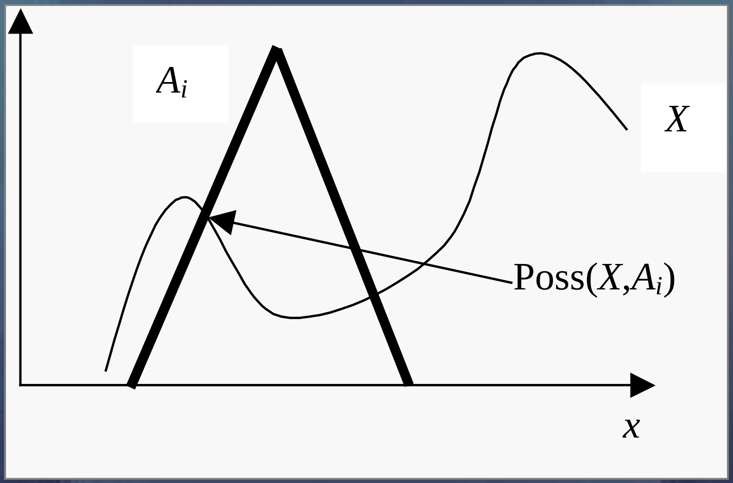
$$\text{Poss}(A_i, X) = \sup_{x \in \mathbf{X}} [X(x)\, t\, A_i(x)]$$

Possibility

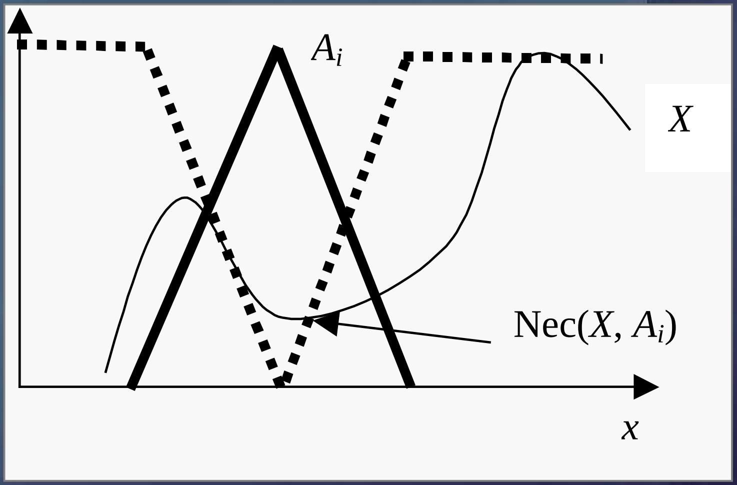
$$\text{Nec}(A_i, X) = \inf_{x \in \mathbf{X}} [X(x)\, s\, (1 - A_i(x))]$$

Necessity

Possibility and necessity



Possibility



Necessity

Possibility and necessity encoding: example

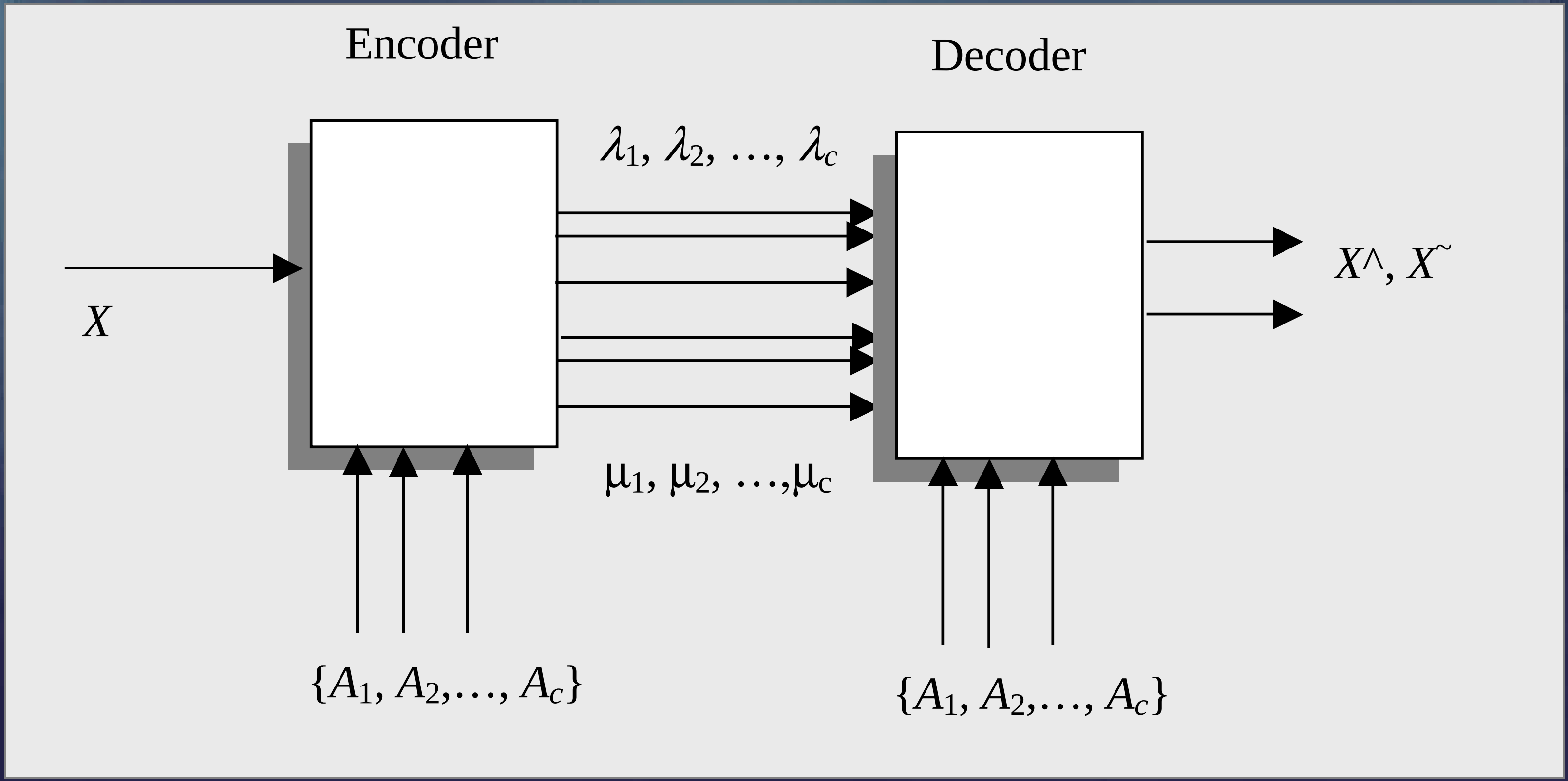
X = [0.0 0.2 0.8 1.0 0.9 0.5 0.1 0.0]

A_i = [0.6 0.5 0.4 0.5 0.6 0.9 1.0 1.0]

Poss (A_i, X) = max (0.0, 0.5, 0.4, 0.5, 0.6, 0.5, 0.1, 0.0) = 0.6

Nec (A_i, X) = min (0.4, 0.5, 0.8, 1.0, 0.9, 0.5, 0.1, 0.0) = 0.0

Encoding and decoding: an overview



Design of the decoder of fuzzy data

- Given the nature of encoding (possibility and necessity measures), the decoding is regarded as a certain "inverse" problem in terms of fuzzy relational equations:
  - Possibility measure: sup-t composition
  - Necessity measure: inf-s composition

Decoding –possibility measure

Possibility measure: sup-t composition

$$\hat{X}(x) = A(x)\varphi\lambda \; = \begin{cases} 1 & \text{if } A(x) \le \lambda \\ \lambda & \text{otherwise} \end{cases}$$

$$\hat{X}(x) = A(x) \to \lambda \; = \sup\{a \in [0,1] \| a\, t\, A(\mathrm{x}) \le \lambda\}$$

$$\hat{X} = \bigcap_{i=1}^{c} \hat{X}_i$$

Decoding –necessity measure

Necessity measure: inf-s composition

$$\tilde{X}(x) = (1 - A\ (x))\varepsilon\,\mu\ \ = \begin{cases} \mu, & \text{if } 1\text{-}\,A\ (x) < \mu \\ 0\,, & \text{otherwise} \end{cases}$$

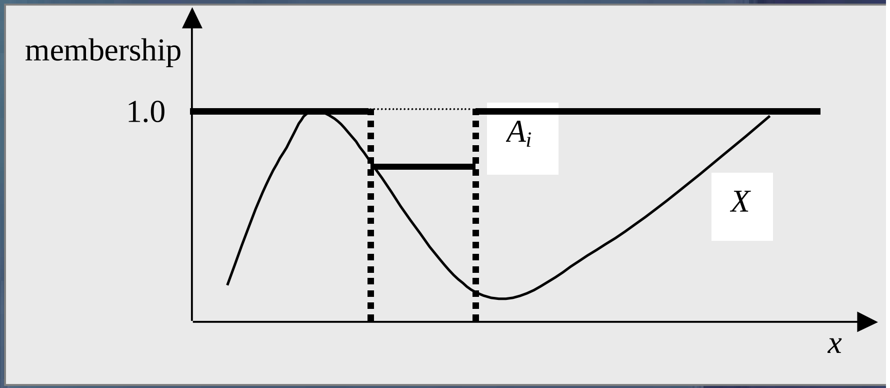
$$\tilde{X}(x) = (1 - A_i(\mathrm{x}))\varepsilon\mu = \inf\{a \in [0,1] \mid \mathrm{as}\,(1 - A(x)) \geq \mu\}$$

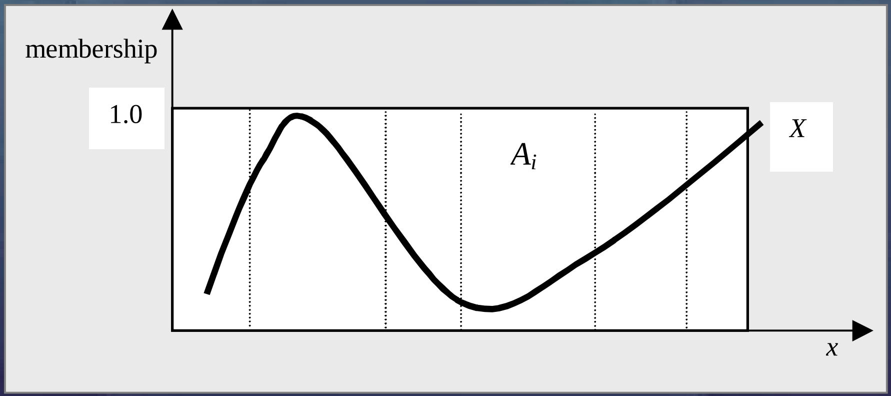
$$\tilde{X} = \bigcup_{i=1}^{c} \tilde{X}_i$$

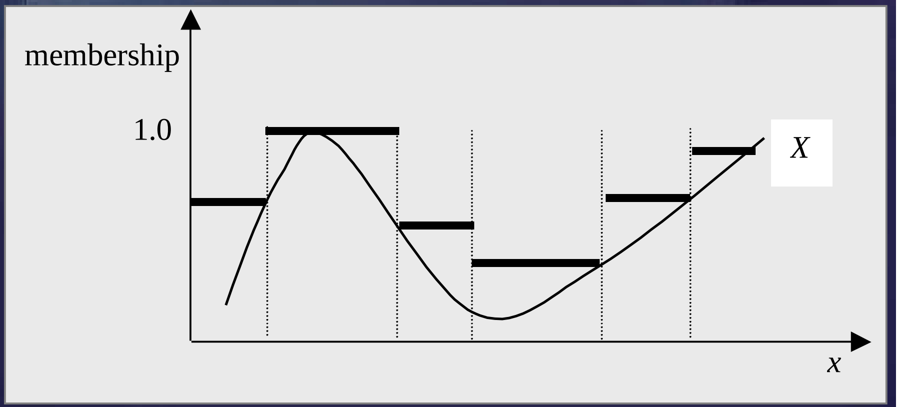
$$\tilde{X} \subseteq X \subseteq \hat{X}$$

Decoding: example

Possibility measure



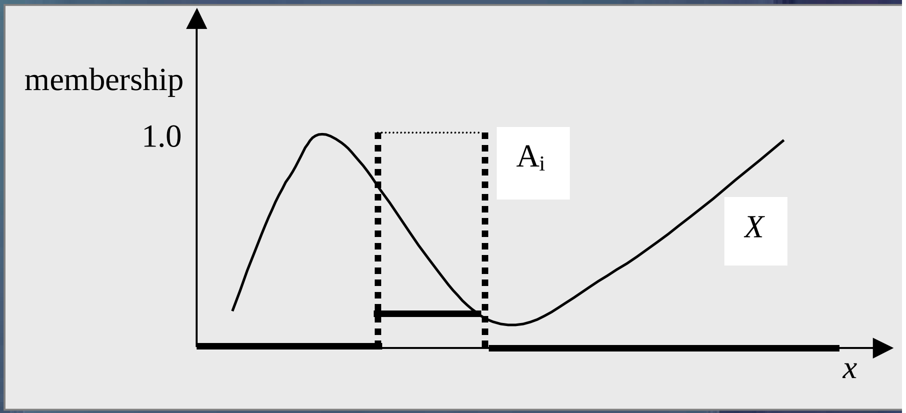


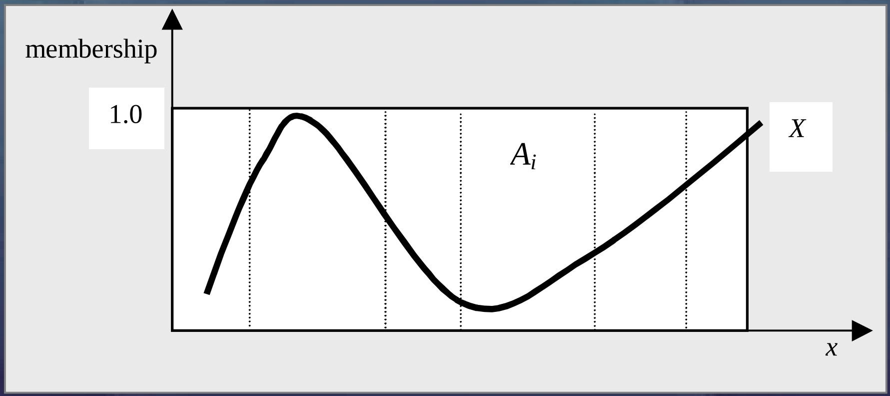


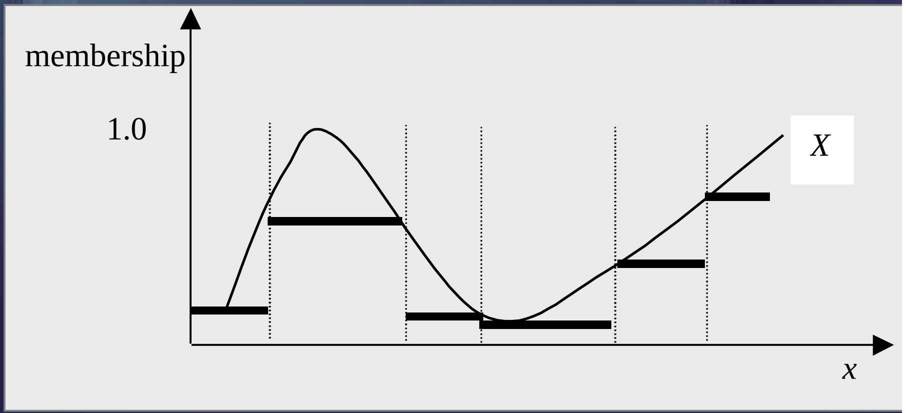
Pedrycz and Gomide, FSE 2007

Decoding: example

Necessity measure

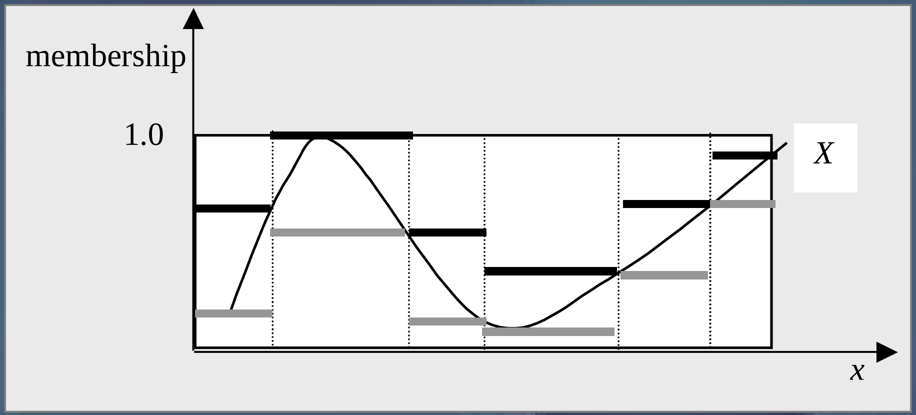


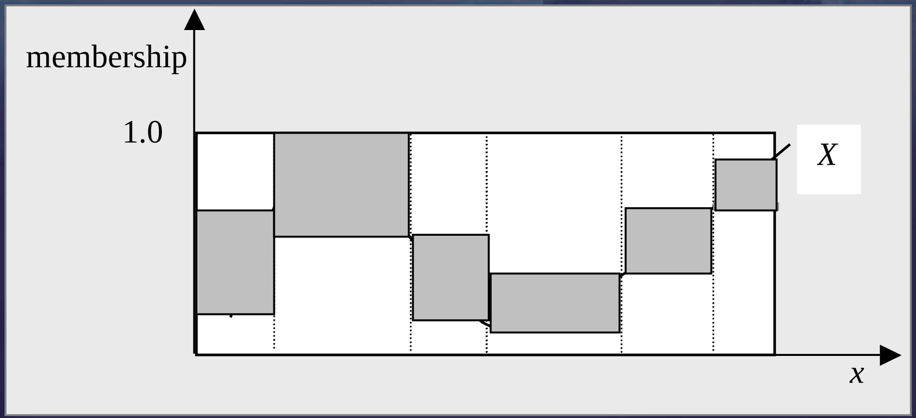




Decoding example

- Bounds of possibility and necessity measure





Taxonomy of data in structure description with shadowed sets

- Core structure
- Shadowed data structure
- Uncertain data structure

- Core data structure

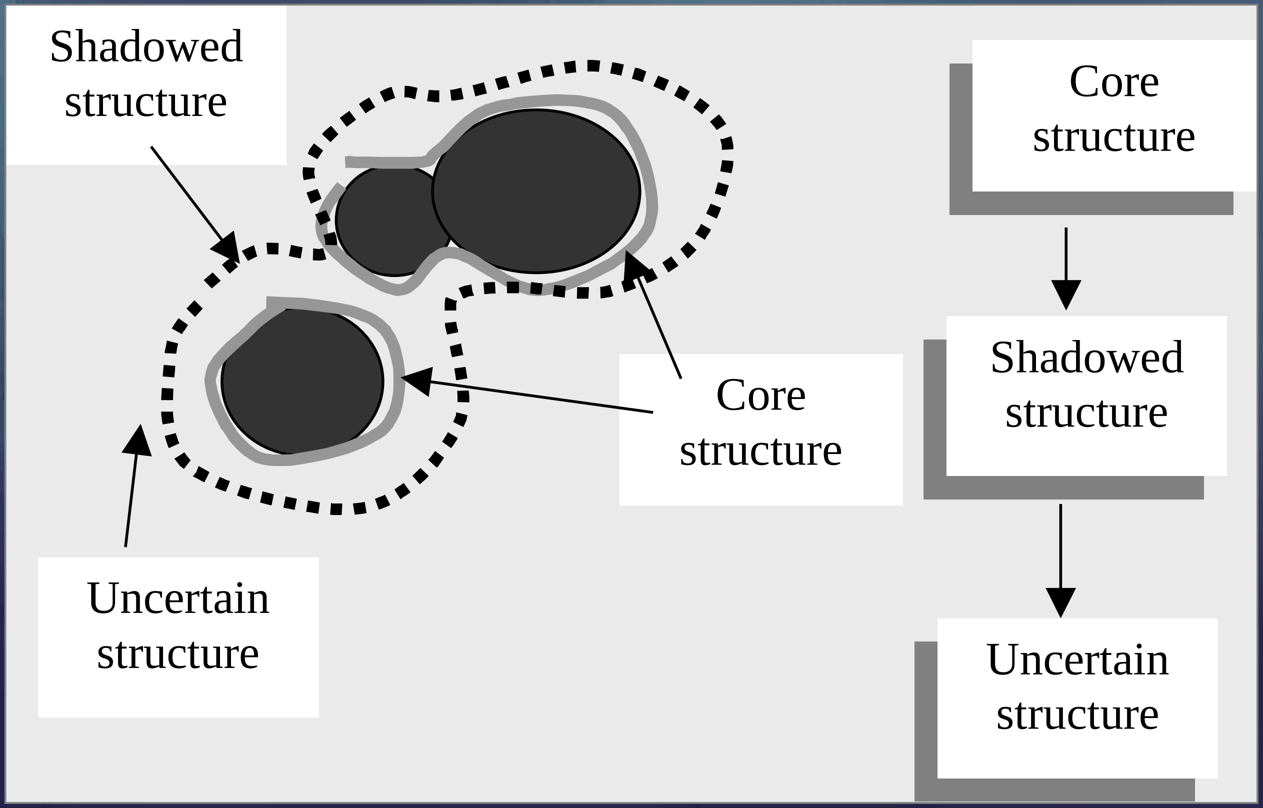
  – patterns that belong to a core of at least one shadowed sets

  – core data structure = $\{ x \mid \exists i \; x \in \mathrm{Core}(A_i) \}$

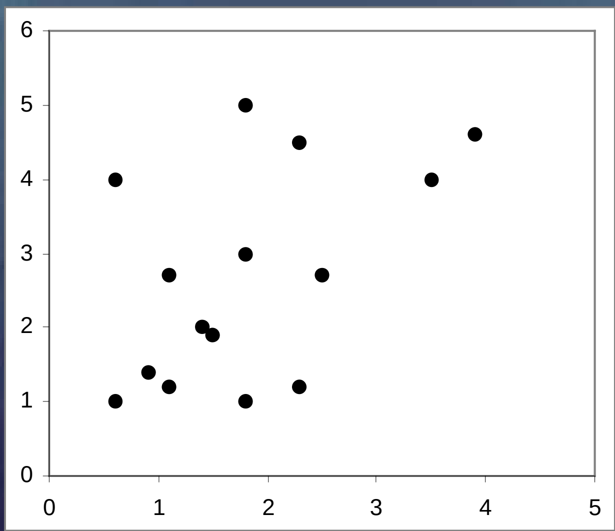
- Shadowed data structure
  - patterns that do not belong to a core of any shadowed set
  - core fall within the shadow of one ofr more shadowed sets
  - shadowed data structure = $\{ x \mid \exists i \; x \in \text{Shadow}(A_i) \text{ and } \forall x \notin \text{Core}(A_i)\}$

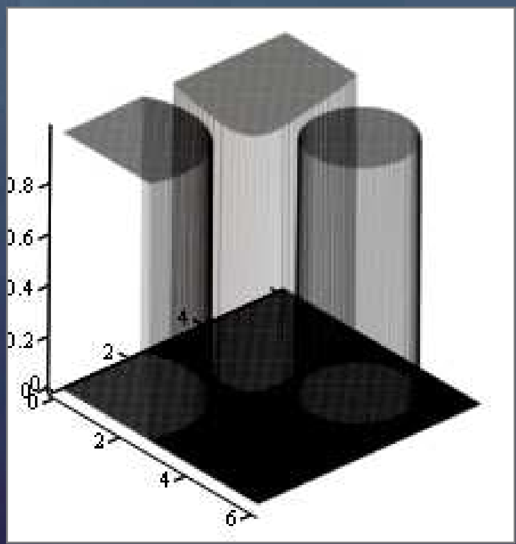
- Uncertain data structure
  - – patterns that left out from all shadows
  - – uncertain data structure = $\{ x \mid \exists i \; x \notin \text{Shadow}(A_i) \text{ and } \forall x \notin \text{Core}(A_i)\}$

Three-valued characterization of data structure with shadowed sets



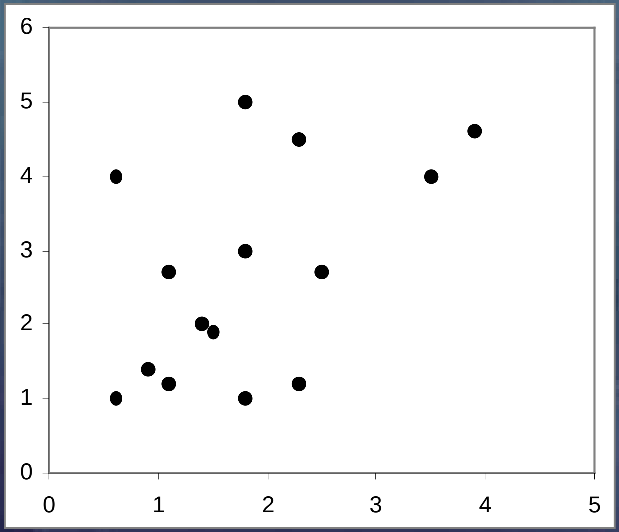
Three-valued characterization of data structure: Example

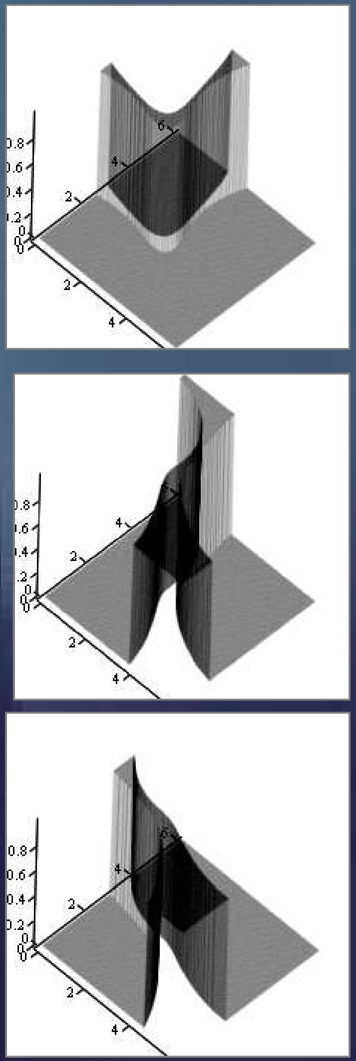




Core

Three-valued characterization of data structure: example





Shadow