8 Generalizations and Extensions of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

8.1 Fuzzy sets of higher order

Fuzzy sets: a retrospective view

- So far we distinguished between
  - *implicit*, and
  - *explicit*

  description of phenomena when dealing with fuzzy sets

- Typically explicit fuzzy sets we discussed so far were defined in some universe of discourse:
  - each elopement of the universe is associated with a membership degree

Fuzzy sets of order 2

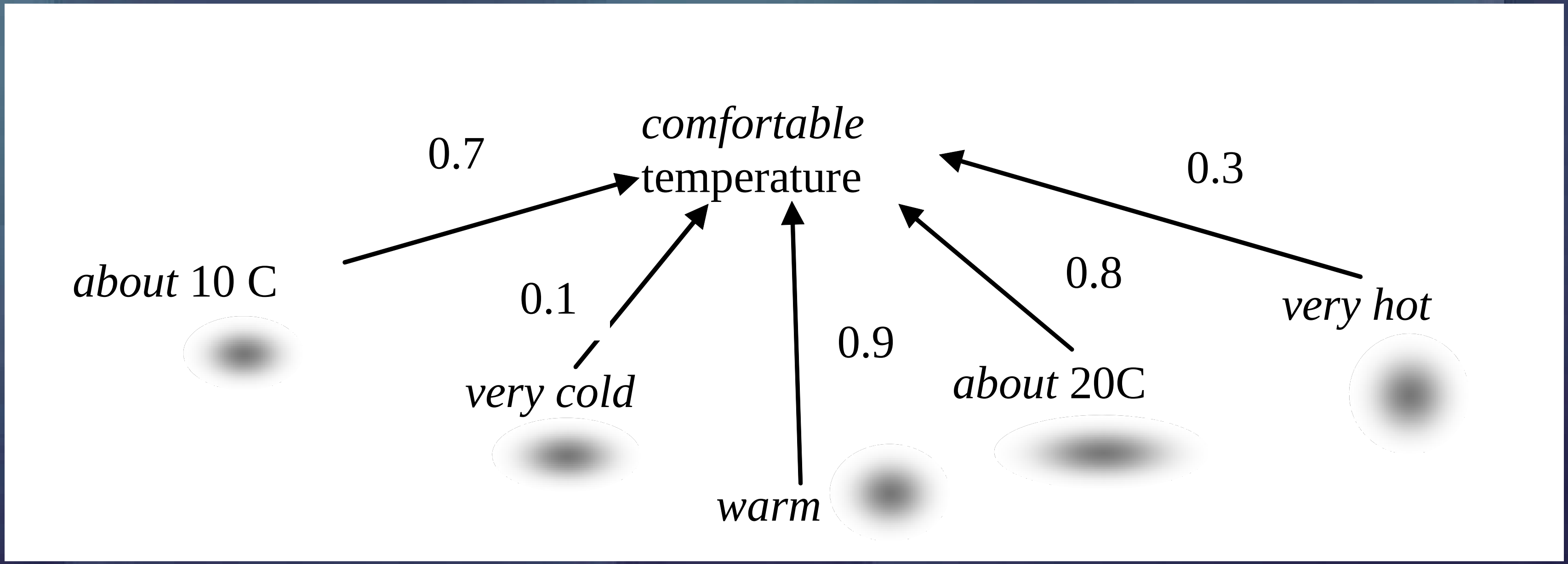
- Defining fuzzy set over a finite family of fuzzy sets
- Example

Describe comfortable temperature given a collection of generic terms (reference fuzzy sets) such as

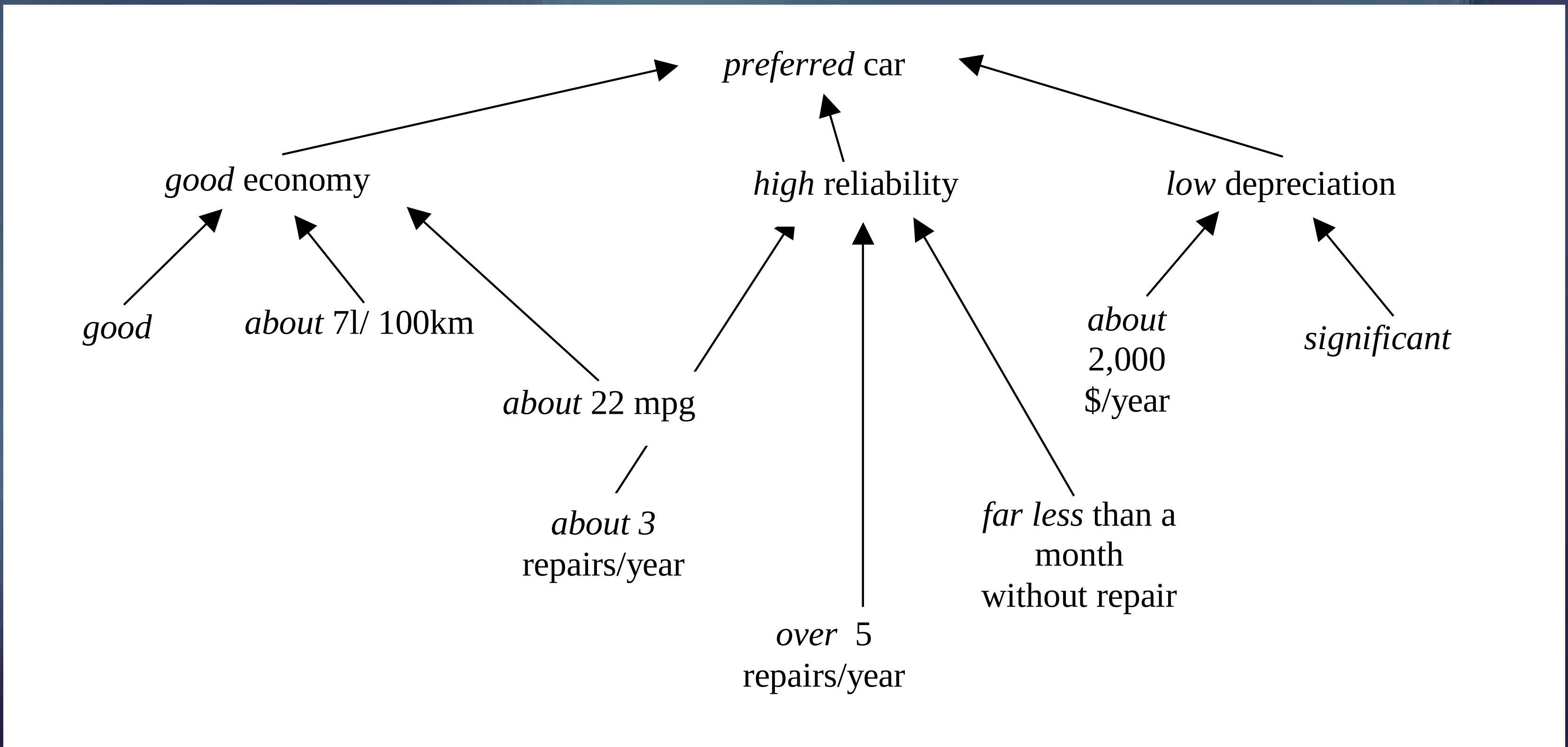
warm,
hot,
cold,
around 15C,

…

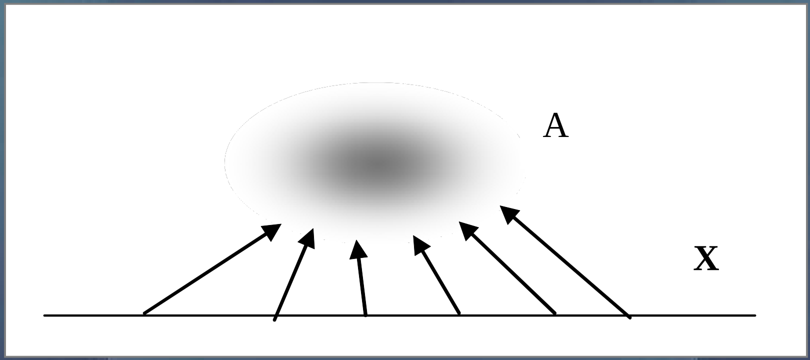
Fuzzy set of order 2

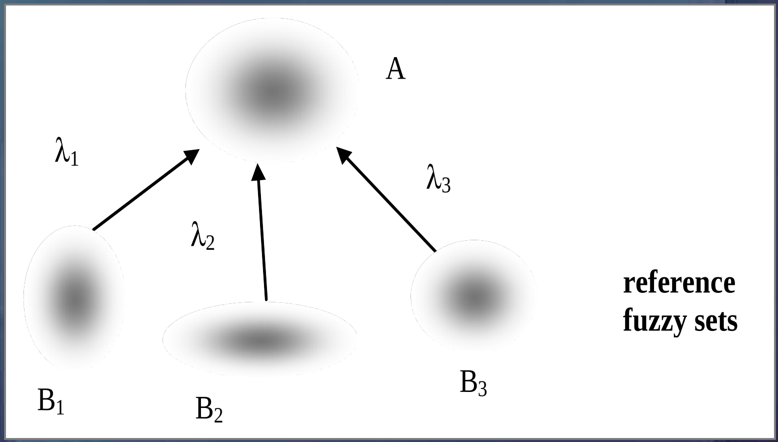


Fuzzy set of order 2



Fuzzy sets of order 2 vs. fuzzy sets: a comparative view





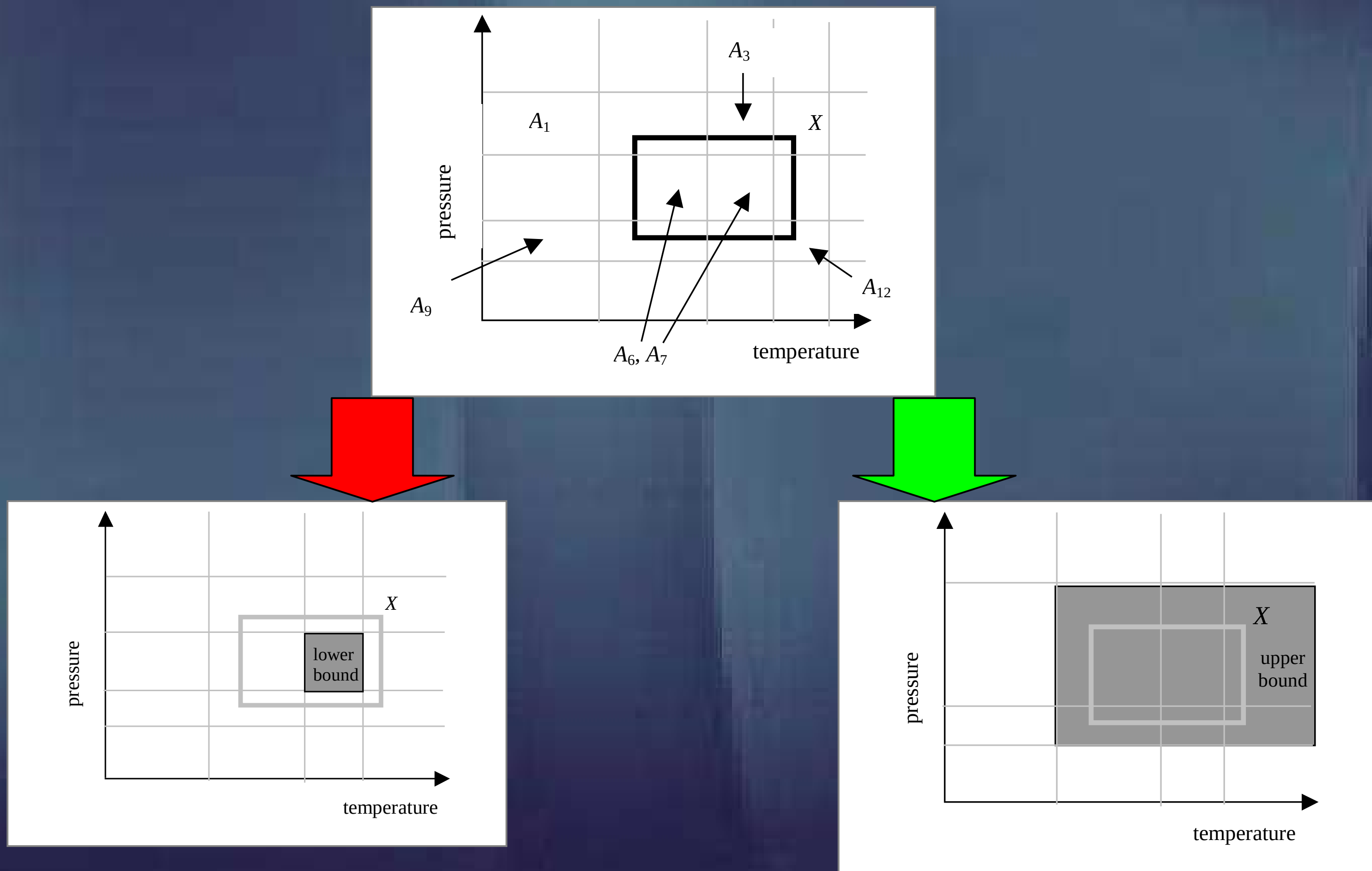
note the fundamental difference in terms of the universes of discourse for fuzzy sets and fuzzy sets of 2nd order

8.2 Rough fuzzy sets and fuzzy rough sets

Fuzzy sets and rough sets

Recall that in rough sets we start with a finite collection of information granules using which we express any given granule in terms of so-called lower and upper bound

Rough sets – an example

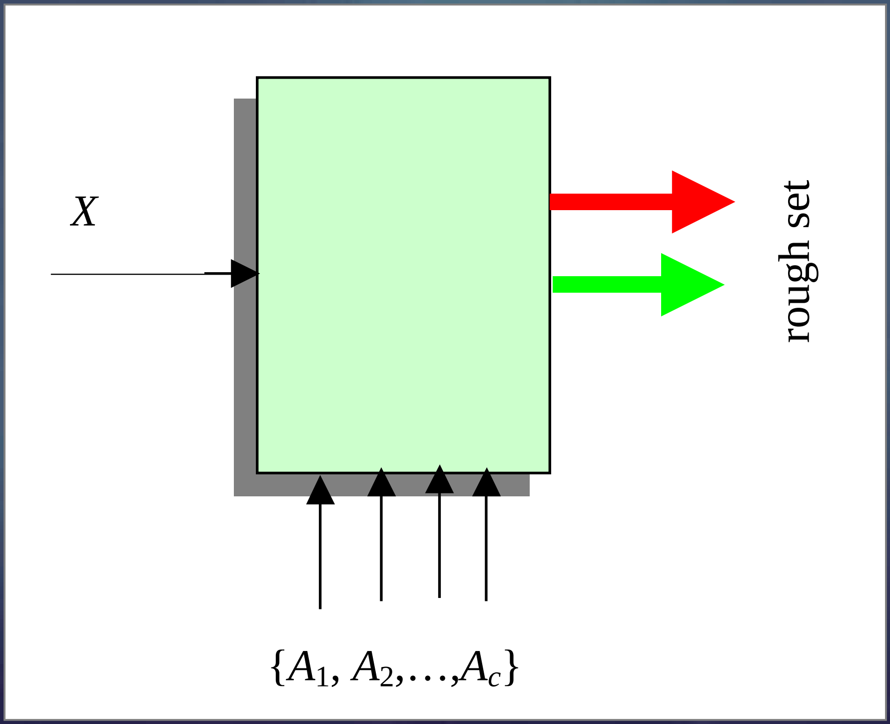


Upper bound $X_+=\{A_i \mid A_i \cap X \neq \varnothing\}$

Lower bound $X_-=\{A_i \mid A_i \subset X\}$

Pedrycz and Gomide, FSE 2007

Rough sets – schematic representation



Fuzzy rough sets and rough fuzzy sets

- In rough sets the vocabulary and incoming object were information granules represented as sets.

- Two useful alternatives could be considered:

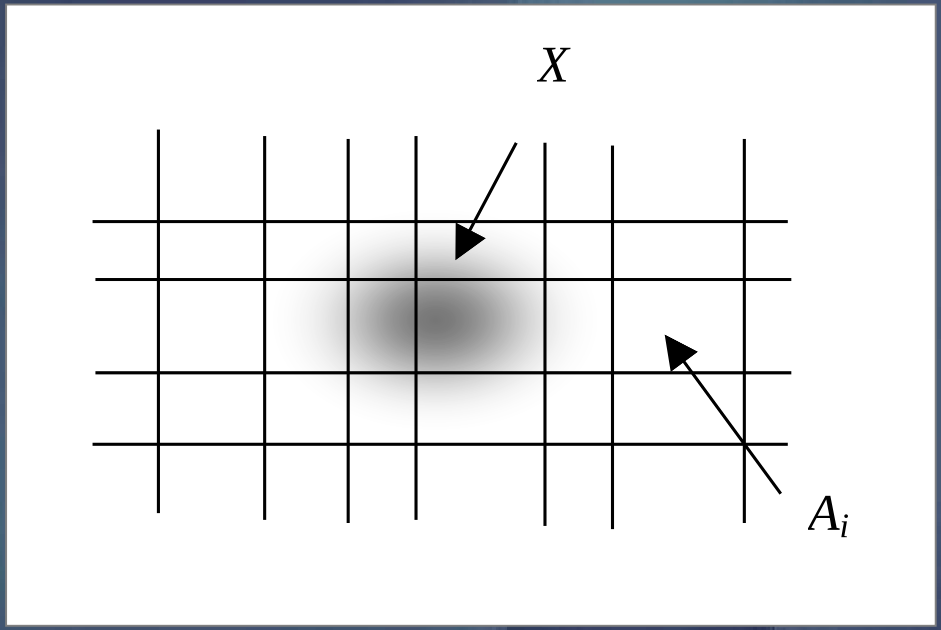
  1-Reference information granules== sets
  Object to be described == fuzzy set

  Fuzzy rough sets

  2-Reference information granules== fuzzy sets
  Object to be described == set

  Rough fuzzy sets

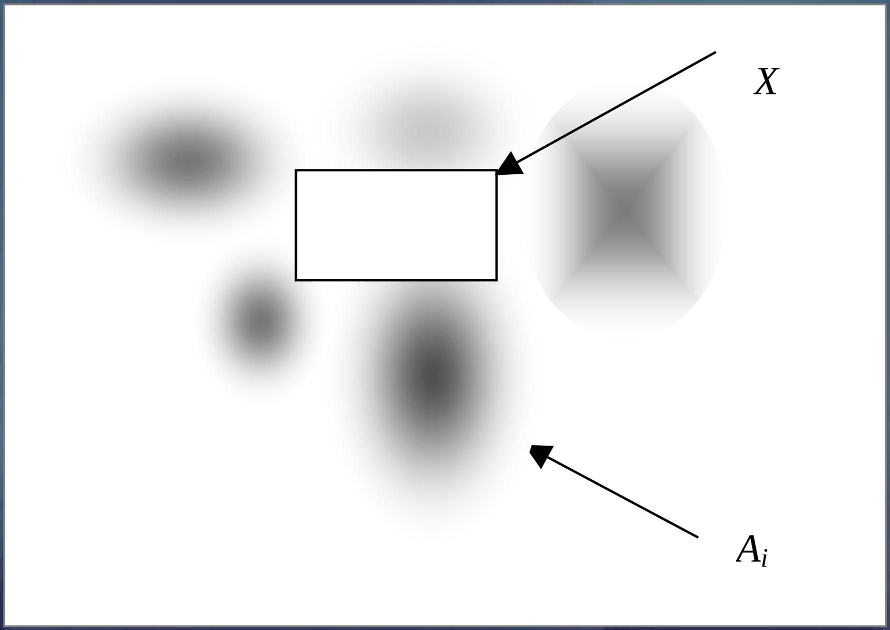
Fuzzy rough sets



$$X_{+}(A_i) = \sup_x[\min(A_i(x), X(x))] = \sup_{x \in \mathrm{Supp}(A_i)} X(x)$$

$$X_{-}(A_i) = \inf_x[\max(1 - X(x), A_i(x))]$$

Rough fuzzy sets

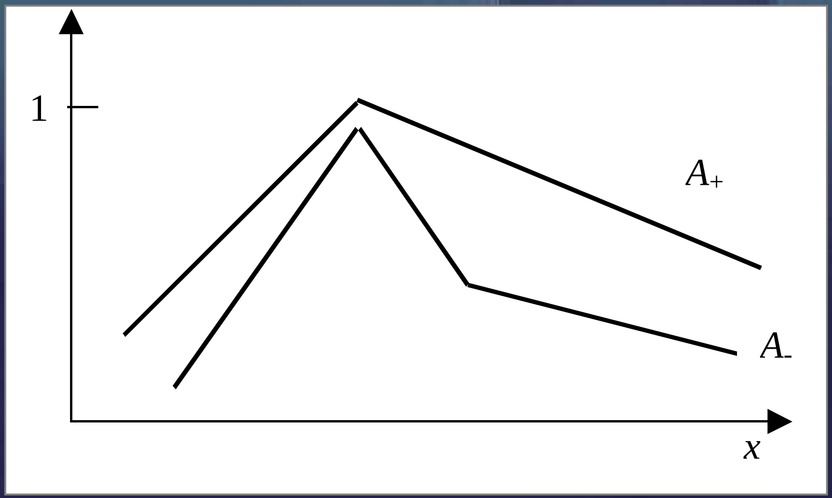


8.3 Interval-valued fuzzy sets

Interval-valued fuzzy sets

- We consider that instead of single membership grades, there are intervals of feasible membership values

- This brings a concept of interval-valued fuzzy sets where the concept of membership is represented in the form of interval



Interval-valued fuzzy sets: operations

- Given $A = (A_-, A^+)$ and $B = (B_-, B_+)$

$$(A \cup B)(x) = (\min(A_+(x), B_+(x)), \max(A_-(x), B_-(x)))$$

Union

$$(A \cap B)(x) = (\max(A_+(x), B_+(x)), \min(A_-(x), B_-(x)))$$

Intersection

$$\overline{A}(x) = (\overline{A}_+(x), A_-(x))$$

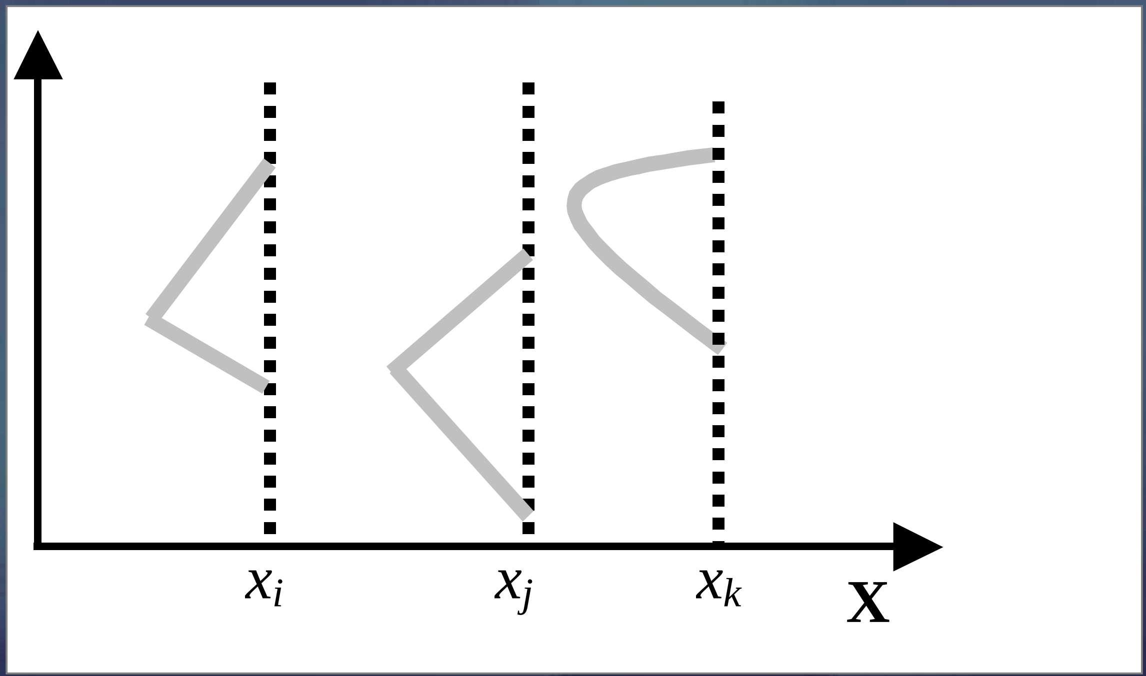
Complement

8.4 Type-2 fuzzy sets

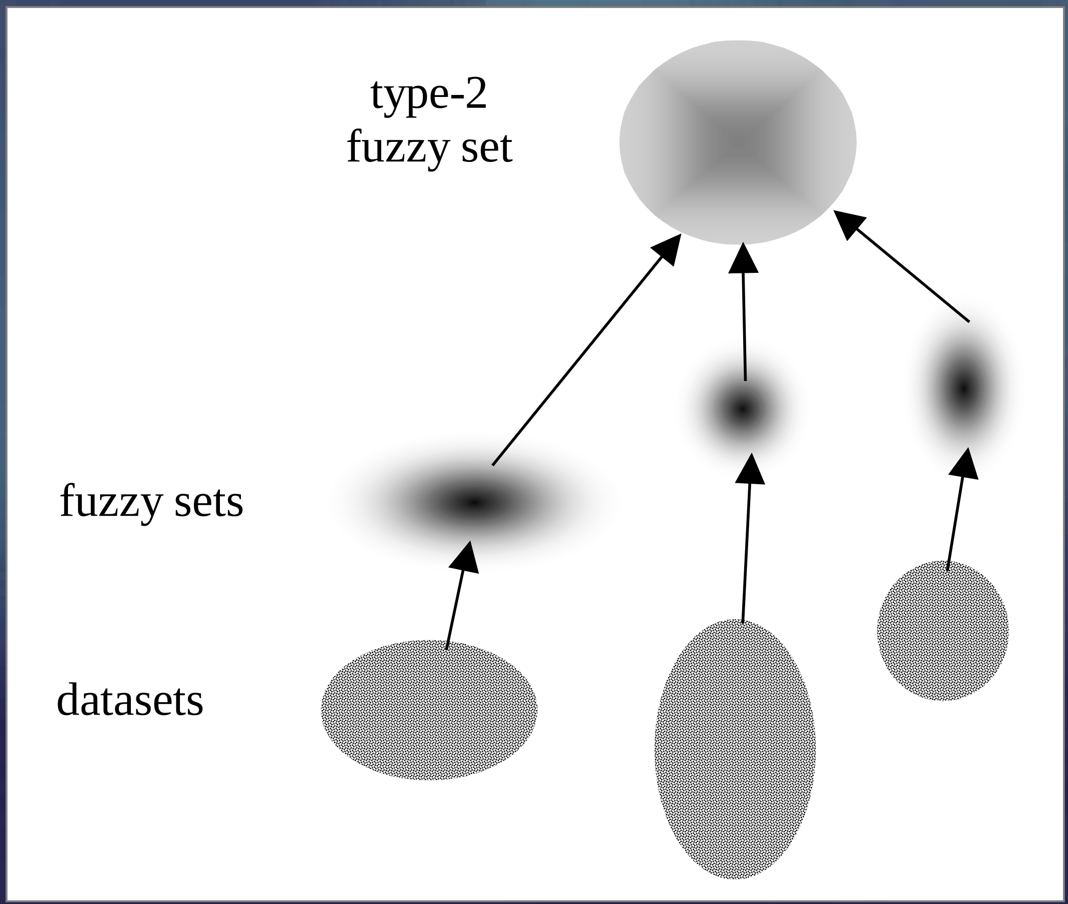
Type-2 fuzzy sets

- Membership degree treated as a single number in [0,1]
- Could the membership itself be a fuzzy set?
- Type-2 fuzzy set: admit membership modeled as fuzzy sets defined in [0,1]

Type-2 fuzzy set: Example



Type-2 fuzzy sets as results of aggregation



Intuitionistic fuzzy set

- Information granule A in which we consider:
  - – degree of membership A^+
  - – degree of non-membership A^-

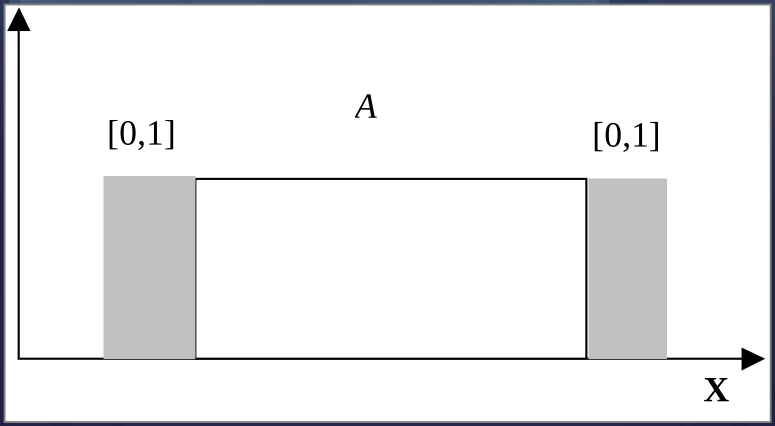
where

$$A^+(x) + A^-(x) \leq 1$$

8.5 Shadowed sets as a three-valued logic characterization of fuzzy sets

Shadowed sets

- Information granule A in which we admit:
- Full membership
- Full exclusion, and
- Shadow – range of $[0,1]$



Shadowed sets: operations

A: $\mathbf{X} \to \{0, 1, [0,1]\}$ $\quad\quad S = [0,1]$

$A \backslash B$	0	S	1
0	0	0	0
S	0	S	S
1	0	S	1

intersection

union

$A \backslash B$	0	S	1
0	0	S	1
S	S	S	1
1	0	1	1

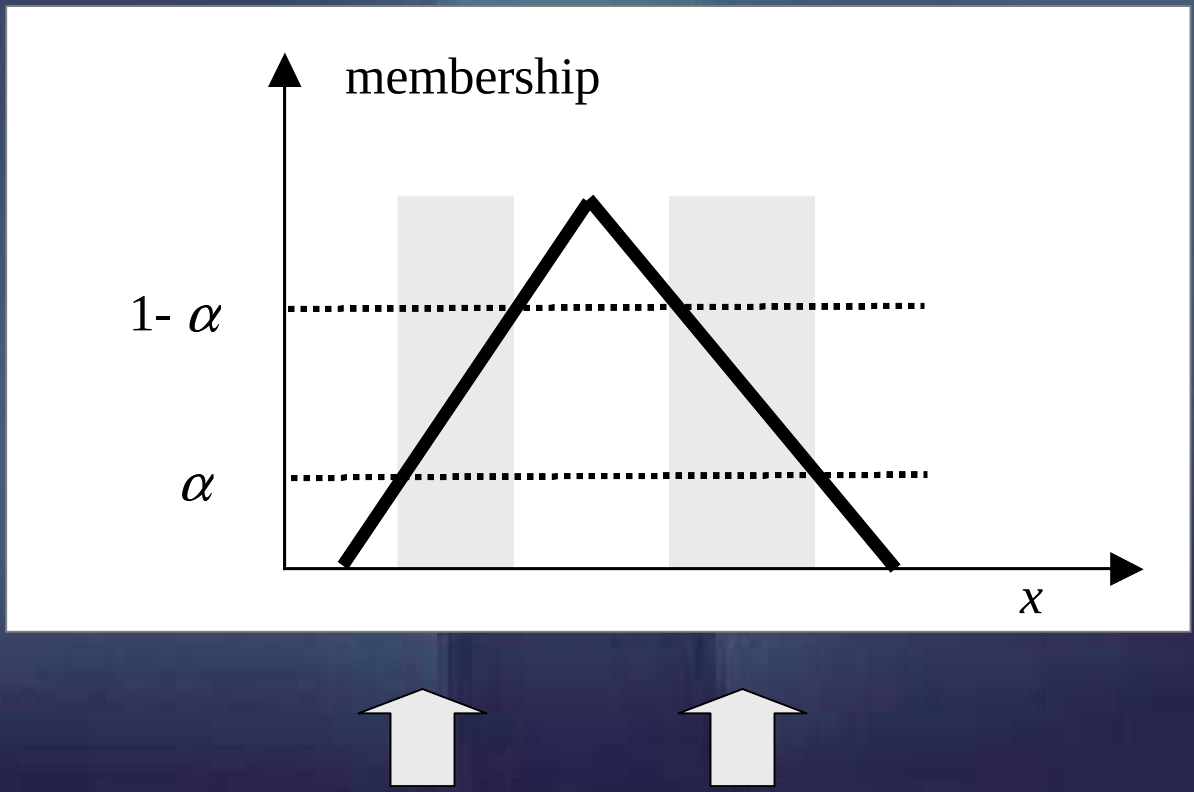
A	$\underline{A}$
0	1
S	S
1	0

complement

Development of shadowed sets

- Shadowed sets are viewed as constructs implied by fuzzy sets:
  - “localization” of membership values by forming shadows and using only 0 - 1 degrees of membership
  - shadowed sets support simpler computing by operating on three logic values

From fuzzy set to shadowed set



reduction of membership + elevation of membership = shadow

From fuzzy set to shadowed set

$$\int_{x:A(x)\le\alpha} A(x)dx$$

membership reduction

$$\int_{x:A(x)\ge 1-\alpha} (1-A(x))dx$$

membership elevation

$$\int_{x:\alpha<A(x)<1-\alpha} dx$$

shadow

$$V(\alpha)=\left|\int_{x:A(x)\le\alpha} A(x)dx+\int_{x:A(x)\ge 1-\alpha} (1-A(x))dx+\int_{x:\alpha<A(x)<1-\alpha} dx\right|$$

performance index

$$\alpha_{opt}=\arg\min_{\alpha} V(\alpha)$$

From fuzzy set to shadowed set

- Triangular membership function: $\alpha = \sqrt{2} - 1$

- Parabolic membership function: $\alpha = 0.405$

From fuzzy set to shadowed set: discrete case

$$V(\alpha) = \left| \sum_{k \in \Omega} u_k + \sum_{k \in \Phi} (1 - u_k) - \mathrm{Card}(\Omega) \right|$$

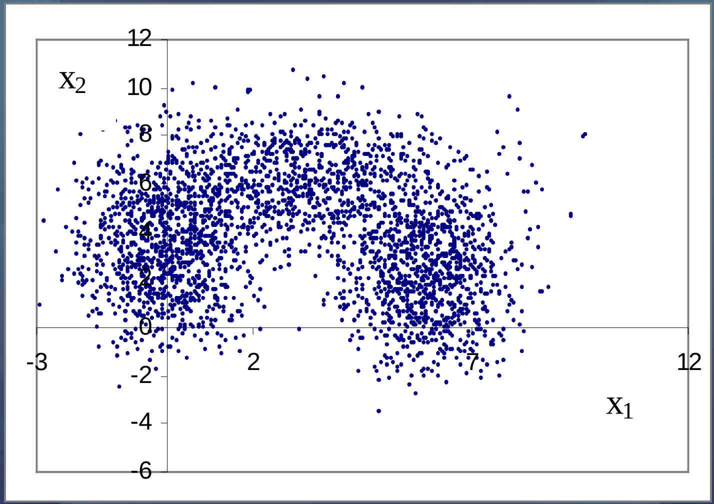
Minimize $V(\alpha)$ w.r.t. α

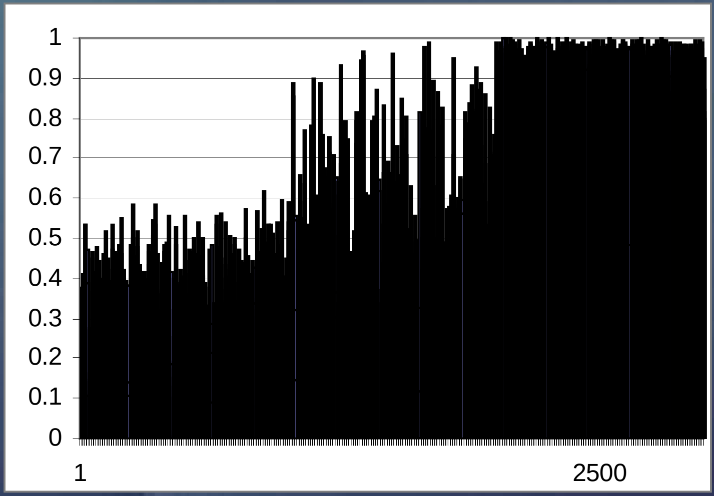
s.t. $u_{\min} \leq \alpha \leq (u_{\min} + u_{\max})/2$

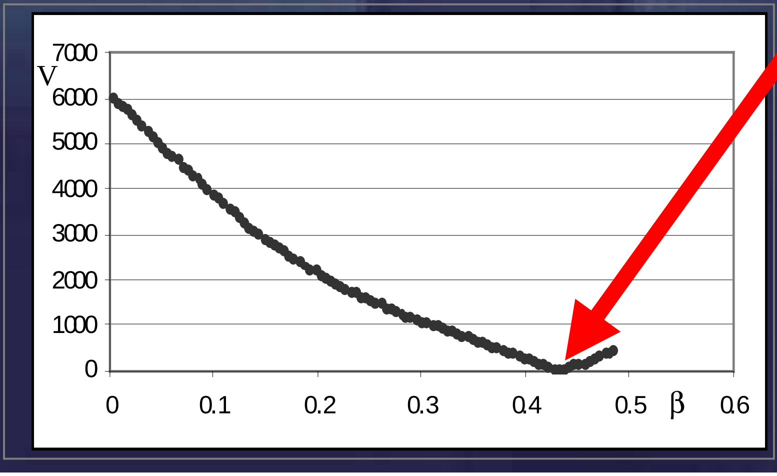
Shadowed sets in fuzzy clustering

- Fuzzy clustering could be conveniently interpreted using shadowed sets
  - elements completely belonging to the cluster
  - elements completely excluded from the cluster
  - elements with uncertainty (*shadow* of the cluster) that are "flagged" in this way and may require further attention

Shadowed sets in fuzzy clustering: Example







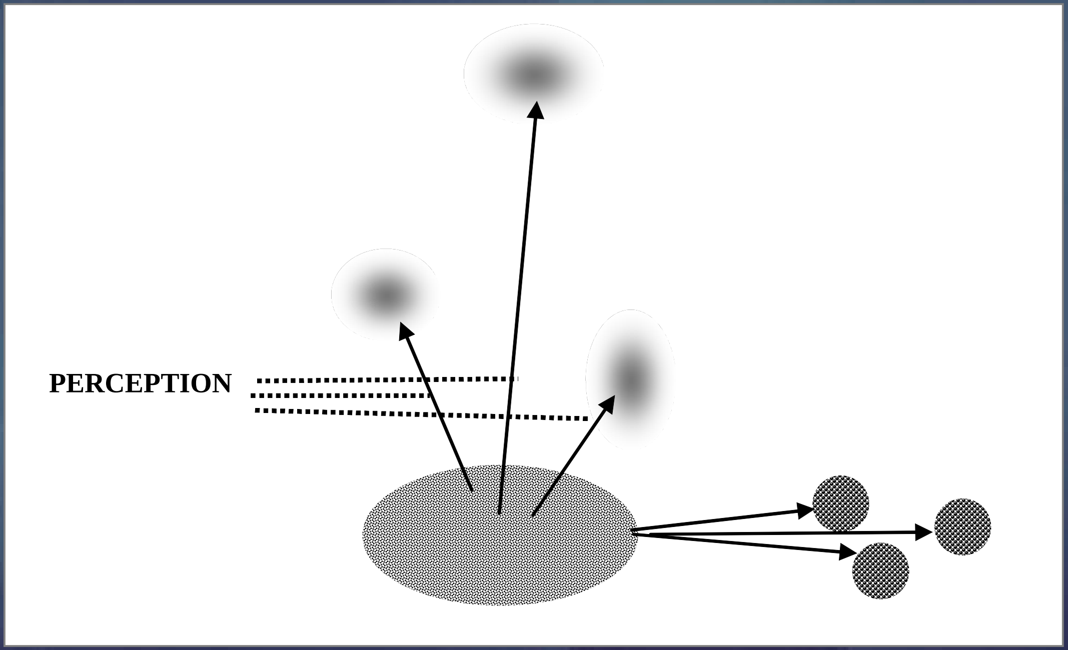
Pedrycz and Gomide, FSE 2007

8.6 Probability and fuzzy sets

Probability and fuzzy sets

- Fuzzy sets and probability are orthogonal concepts:
  - **probability** is concerned with occurrence of Boolean phenomena
  - **fuzzy sets** are concerned with perception of concepts

Probability and fuzzy sets



8.7 Probability of fuzzy events

Probability of fuzzy events

- What is the probability of *low* temperature tomorrow
- What is the probability of *high* inflation in a *short* term
- What is the probability of *small* steady state error of boiler pressure

Probability of fuzzy events

- Underlying probability density function in $\mathbf{X}$: $p(x)$
- Fuzzy event (fuzzy set): A
- Probability of fuzzy event

$$\int_{\mathbf{X}} A(x)dP(x) = \int_{\mathbf{X}} A(x)p(x)dx$$

(assume that the integral does exist)

$\Downarrow$

This is the expected value $E(A)$ of the membership function of A

Probability of fuzzy events

Variance

$$E^2(A) = \int_{\mathbf{X}} [A(x) - E(A)]^2 p(x)dx$$

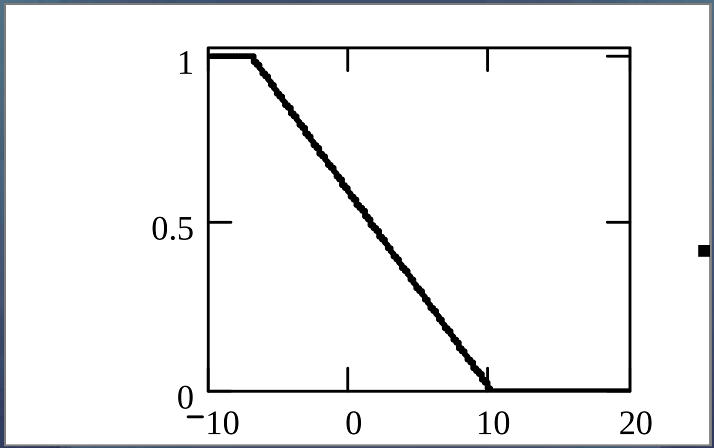
Standard deviation

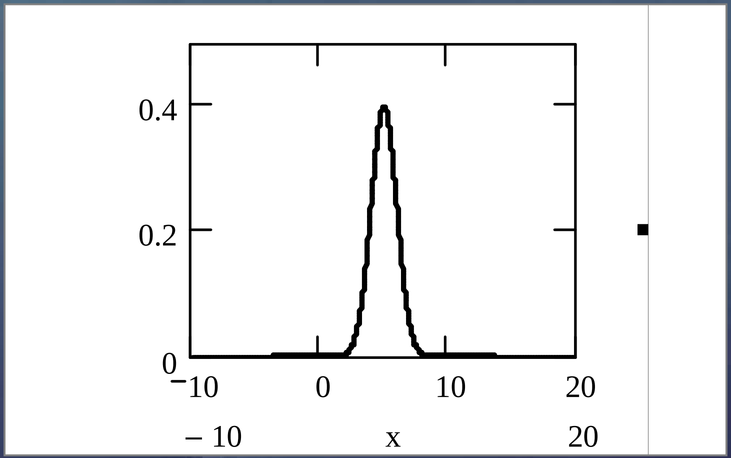
$$\sigma(A) = \sqrt{E^2(A)}$$

High order moments

$$\int_{\mathbf{X}} [A(x) - E(A)]^r p(x)dx \qquad r > 2$$

Probability of fuzzy events: Example





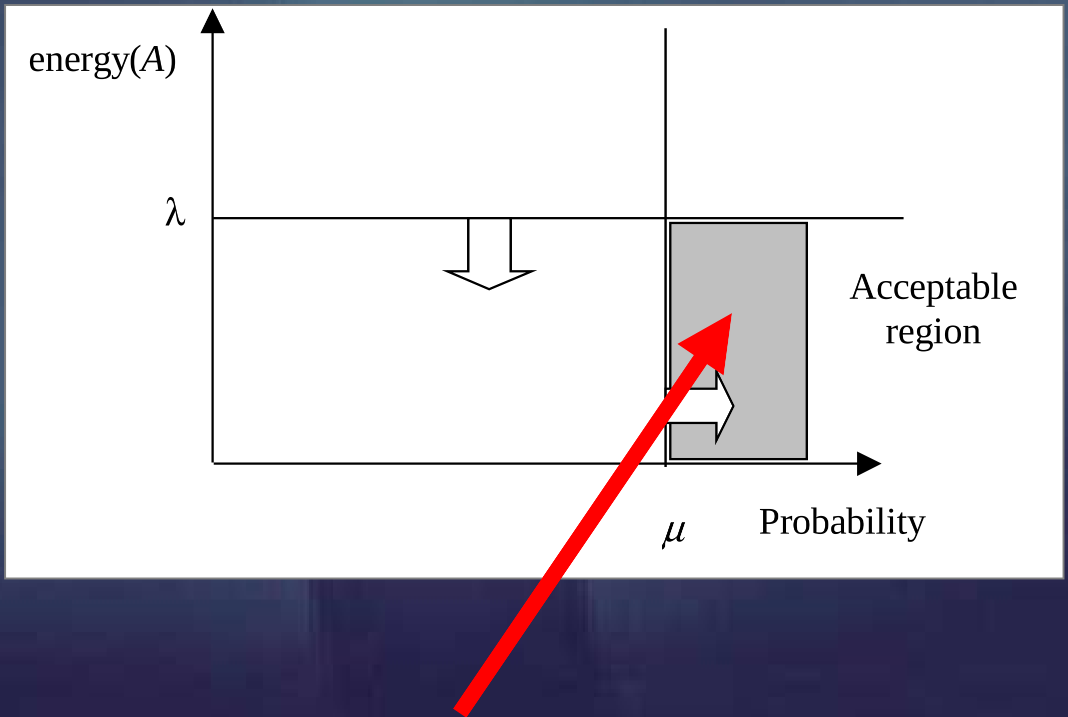
A = *low* temperature

pdf of temperature

$P(A) = 0.294 \quad \sigma(A)\ 3.46\times10^{-3}$

Probability of fuzzy events: orthogonality

Semantic validity

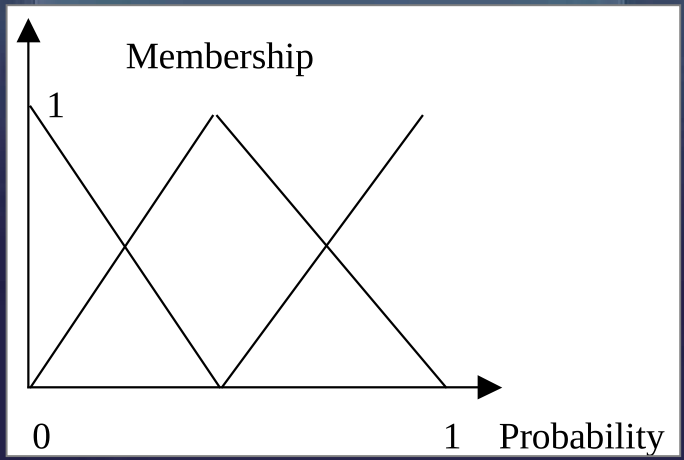


experimental evidence

Linguistically quantified statements

Linguistic probabilities:

low probability, *high* probability, probability *around* 60%...



Linguistically quantified statements: computing

- Random variable a_i with linguistic probabilities P_i

$$Z = \sum_{i=1}^{n} a_i P_i$$

- Extension principle:

$$Z(z) = \sup[\min(P_1(p_1), P_2(p_2), \ldots, P_n(p_n))]$$

$$s.t. \qquad z = \sum_{i=1}^{n} a_i p_i$$

$$\sum_{i=1}^{n} p_i = 1$$

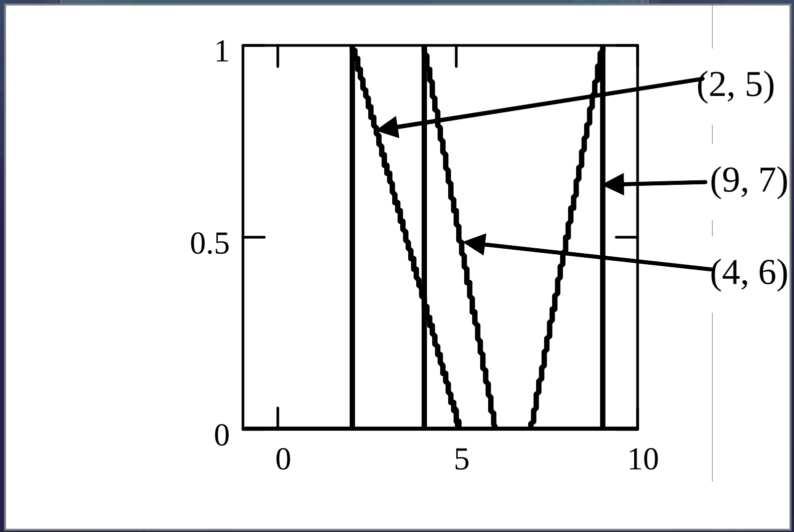
Linguistically quantified statements: Example

$$Z = a_1 \ likely + a_2 \ unlikely$$

$$Z(z) = likely\left(\frac{z-a_2}{a_1-a_2}\right)$$

$likely\ (u) = unlikely\ (1 - u)$

$likely\ (u) = u$



$likely\ (u) = u^2$