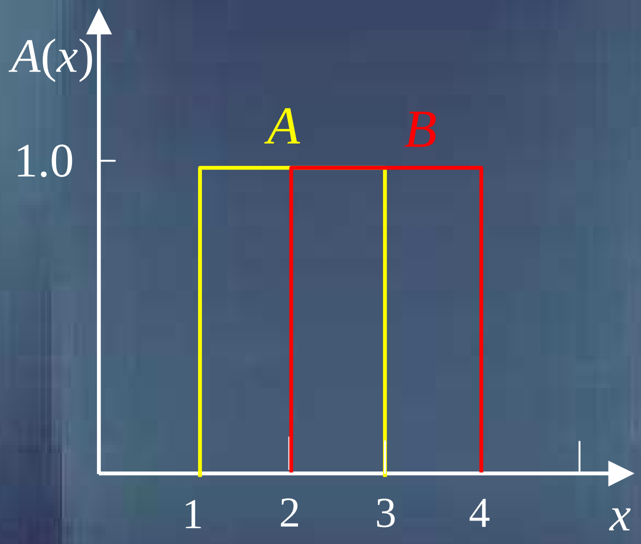
5 Operations and Aggregations of Fuzzy Sets

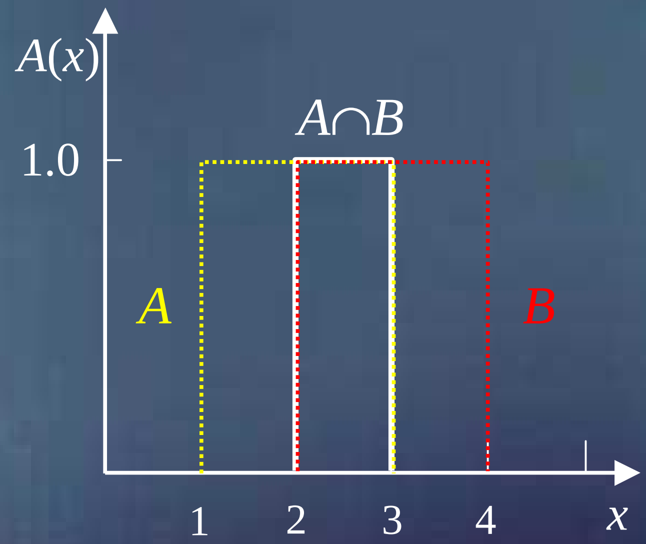
Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

5.1 Standard operations on sets and fuzzy sets

Intersection of sets





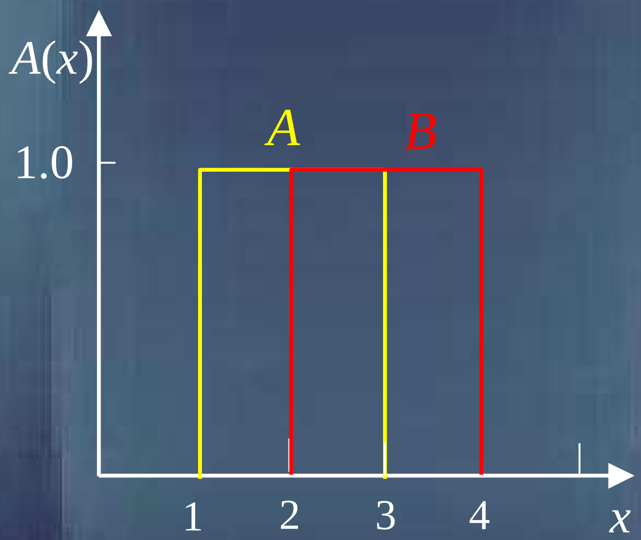
$A = \{x \in \mathbf{R} \mid 1 \le x \le 3\}$

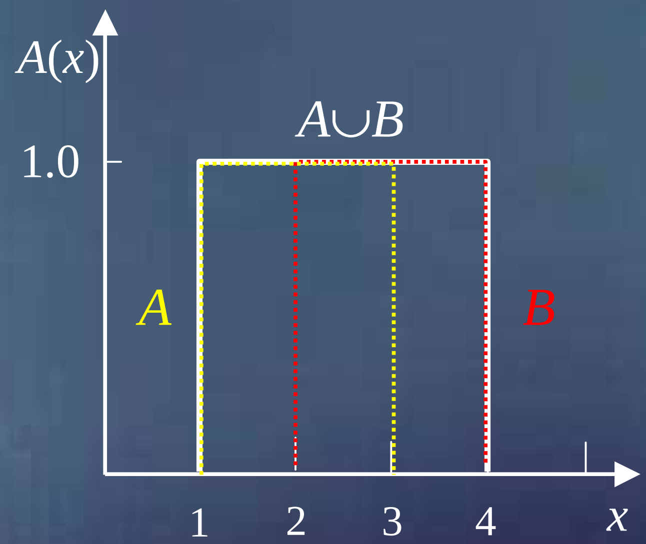
$A \cap B$: $\{x \in \mathbf{R} \mid 2 \le x \le 3\}$

$B = \{x \in \mathbf{R} \mid 2 \le x \le 4\}$

$$(A \cap B)(x) = \min\,[A(x), B(x)]\ \forall x \in \mathbf{X}$$

Union of sets





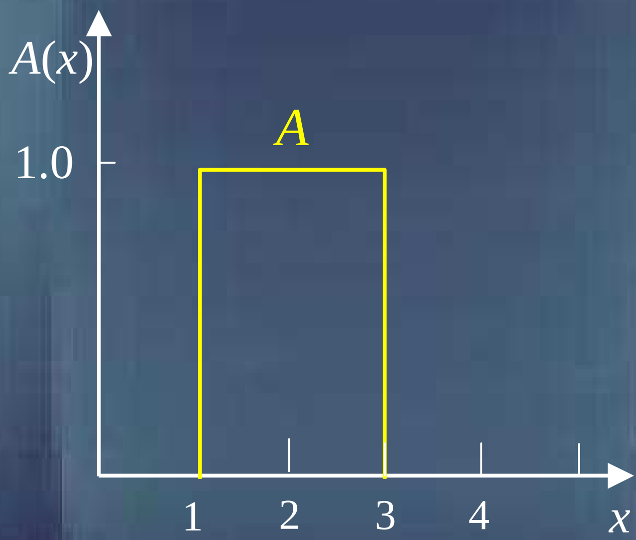
$A = \{x \in \mathbf{R} \mid 1 \le x \le 3\}$

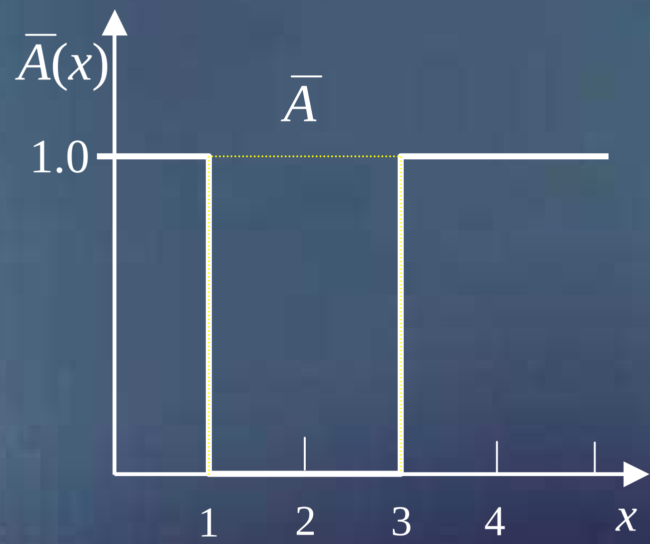
$A \cup B$: $\{x \in \mathbf{R} \mid 1 \le x \le 4\}$

$B = \{x \in \mathbf{R} \mid 2 \le x \le 4\}$

$$\boxed{(A \cup B)(x) = \max\,[A(x), B(x)]\ \forall x \in \mathbf{X}}$$

Complement of sets



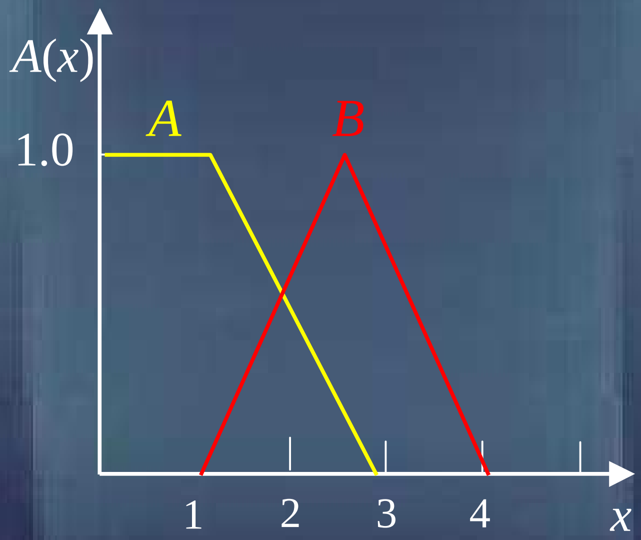


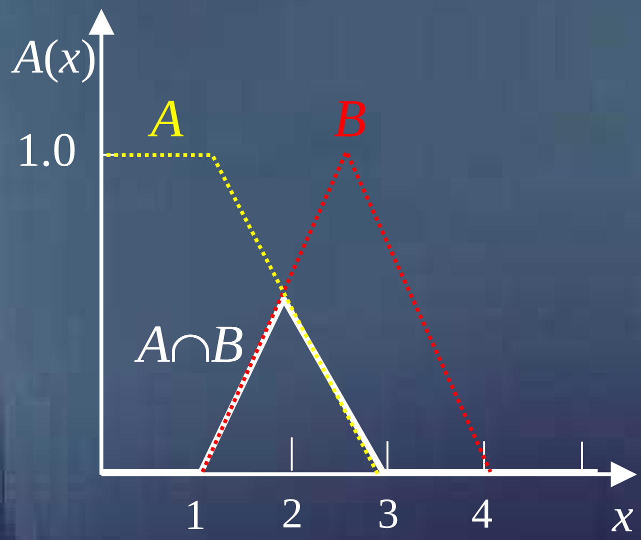
$A = \{x \in \mathbf{R} \mid 1 \le x \le 3\}$

$\bar{A}(x) = \{x \in \mathbf{R} \mid x < 1 \text{ , } x > 3\}$

$$\boxed{\bar{A}(x) = 1 - A(x) \quad \forall x \in \mathbf{X}}$$

Intersection of fuzzy sets

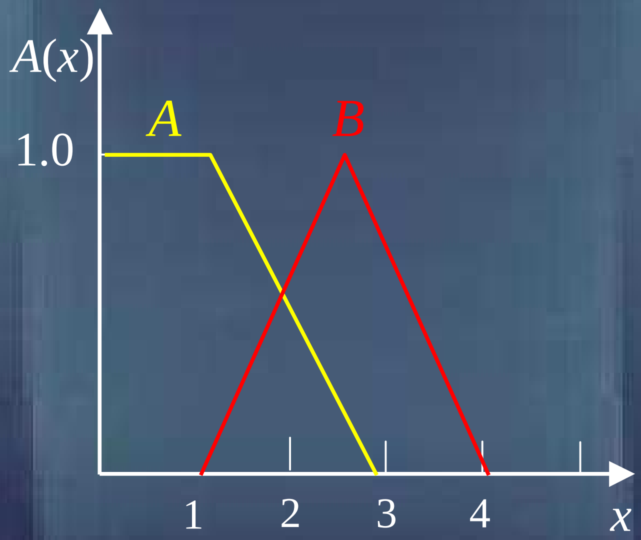


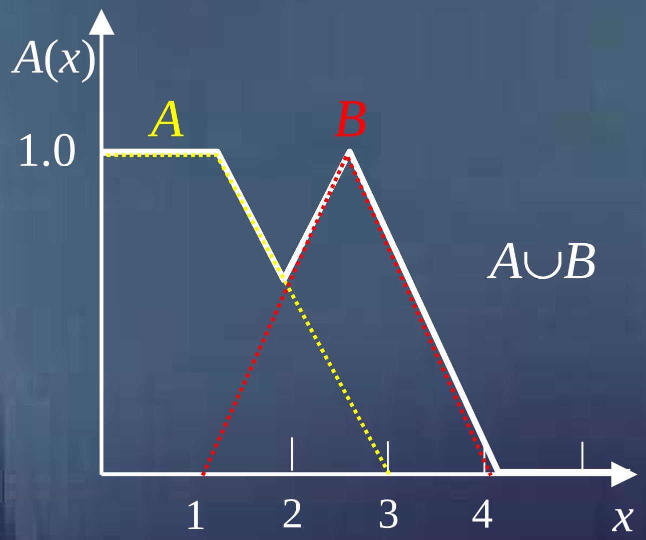


$(A \cap B)(x) = \min\,[A(x), B(x)]\ \forall x \in \mathbf{X}$

Standard intersection

Union of fuzzy sets

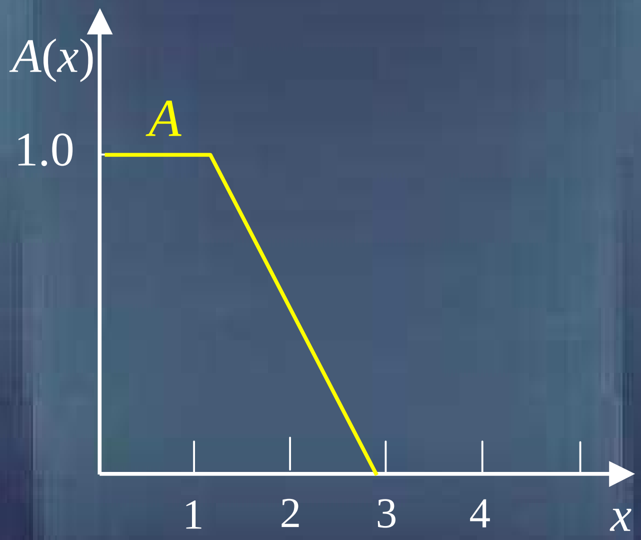


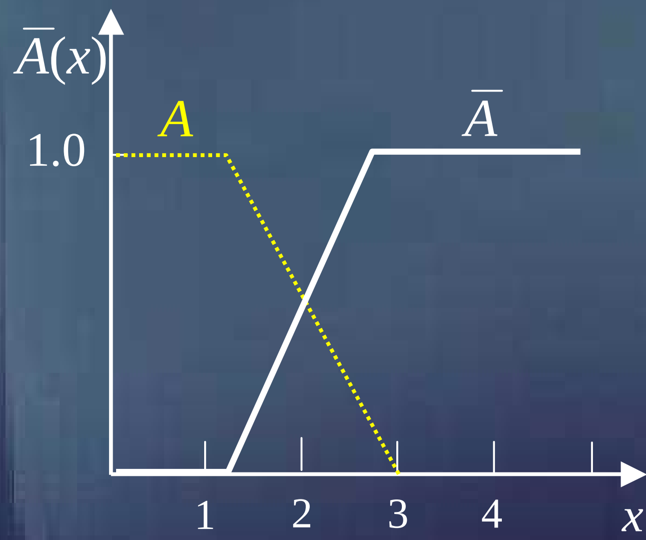


$(A \cup B)(x) = \max\,[A(x), B(x)]\ \forall x \in \mathbf{X}$

Standard union

Complement of fuzzy sets





$$\bar{A}(x) = 1 - A(x) \quad \forall x \in \mathbf{X}$$

Standard complement

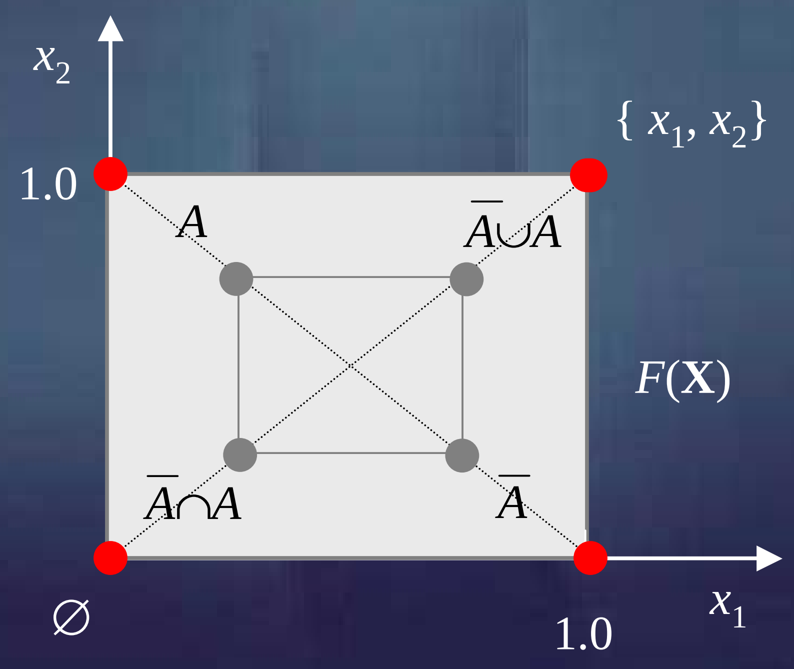
Basic properties of sets and fuzzy sets

1 Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
2 Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$
	$A \cap (B \cap C) = (A \cap B) \cap C$
3 Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4 Idempotency	$A \cup A = A$
	$A \cap A = A$
5 Boundary Conditions	$A \cup \phi = A$ and $A \cup \mathbf{X} = \mathbf{X}$
	$A \cap \phi = \phi$ and $A \cap \mathbf{X} = A$
6 Involution	$\overline{\overline{A}} = A$
7 Transitivity	if $A \subset B$ and $B \subset C$ then $A \subset C$

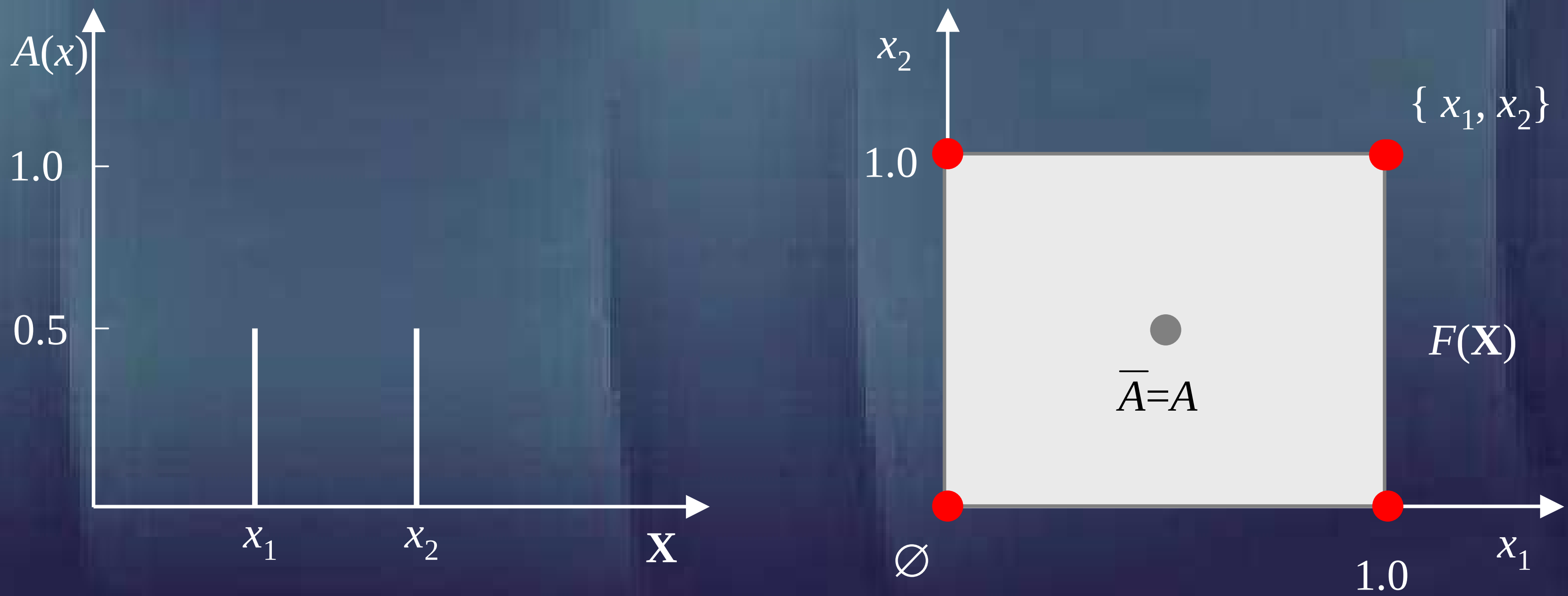
Noncontradiction and excluded middle for standard operations

	Sets	Fuzzy sets
8-Noncontradiction	$\overline{A} \cap A = \phi$	$\overline{A} \cap A \neq \phi$
9-Excluded middle	$\overline{A} \cup A = \mathbf{X}$	$\overline{A} \cup A \neq \mathbf{X}$

Geometric view of standard operations



Example



5.2 Generic requirements for operations on fuzzy sets

Set operation is a binary operator

$$[0,1] \times [0,1] \rightarrow [0,1]$$

- Requirements:
  - commutativity
  - associativity
  - identity
- Identity:
  - its form depends on the operation

5.3 Triangular norms

Definition

$$t : [0,1] \times [0,1] \to [0,1]$$

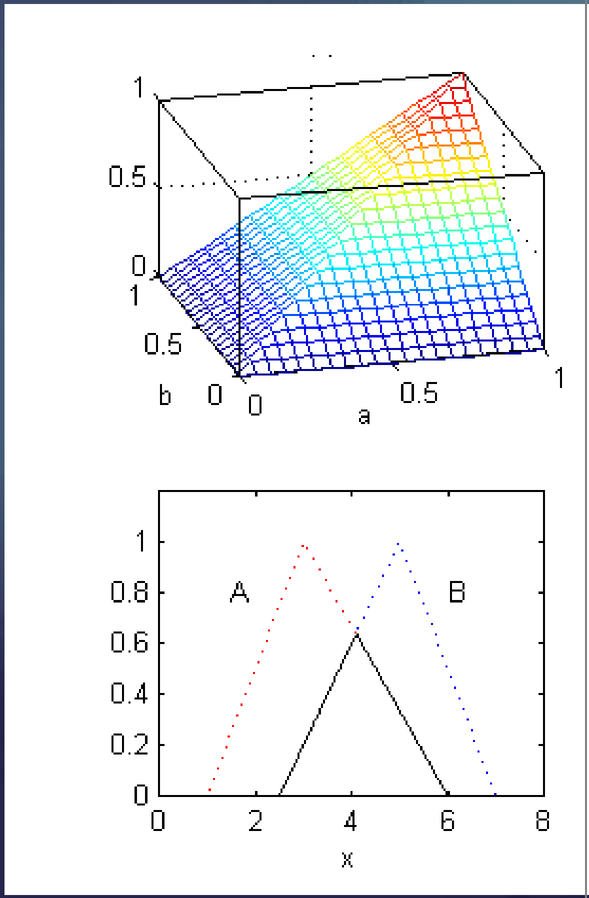
- Commutativity: $a\ t\ b = b\ t\ a$
- Associativity: $a\ t\ (b\ t\ c) = (a\ t\ b)\ t\ c$
- Monotonicity: if $b \le c$ then $a\ t\ b \le a\ t\ c$
- Boundary conditions: $a\ t\ 1 = a$
  $a\ t\ 0 = 0$

Examples

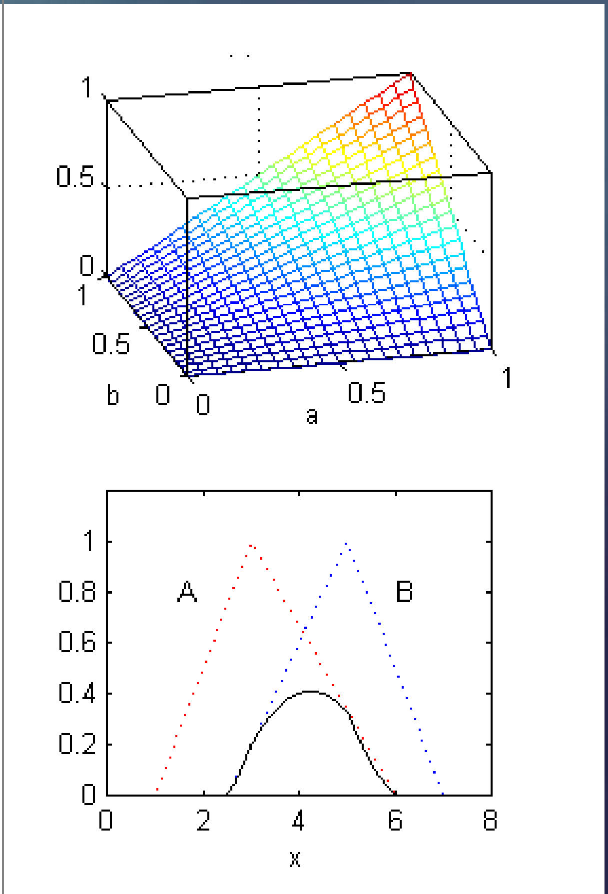
minimum

$a\ t_m\ b = \min(a,b)$

$a\ t_p\ b = ab$



(a)



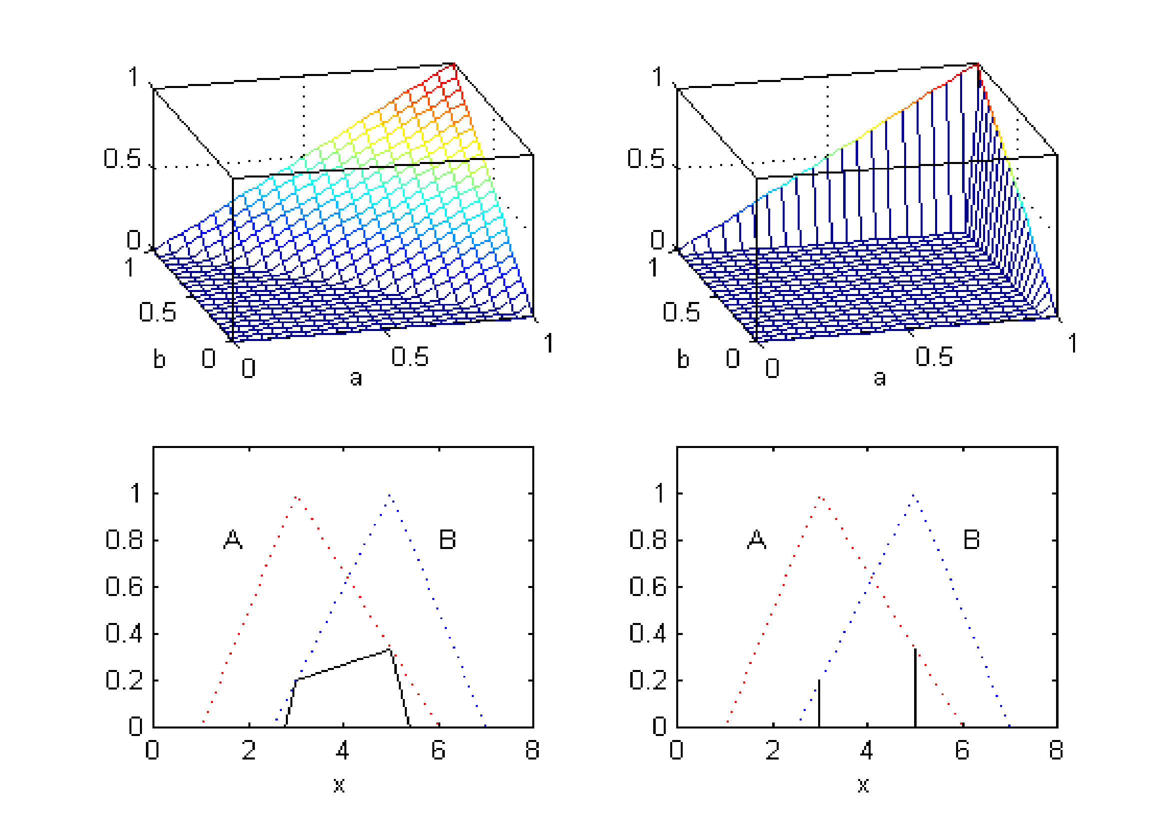
(b)

product

$$a\ t_l\ b = \max(a+b-1, 0)$$

$$at_d b = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

Lukasiewicz



(c) (d)

drastic product

$$a t_d b \le atb \le \min(a, b)$$

lower and upper bounds

$$ata \le a$$

Archimedean (if t is continuous)

$$a^n = 0$$

nilpotent (e.g. Lukasiewicz)

$$a\ t\ a = a^2, \ldots, a^{n-1}\ t\ a = a^n$$

Constructors of t-norms

- Monotonic function transformation
- Additive and multiplicative generators
- Ordinal sums

Monotonic function transformation

- If h: $[0,1] \rightarrow [0,1]$ is a strictly increasing bijection

  then $t_h(a,b) = h^{-1}(t(h(a),h(b)))$ is a t-norm

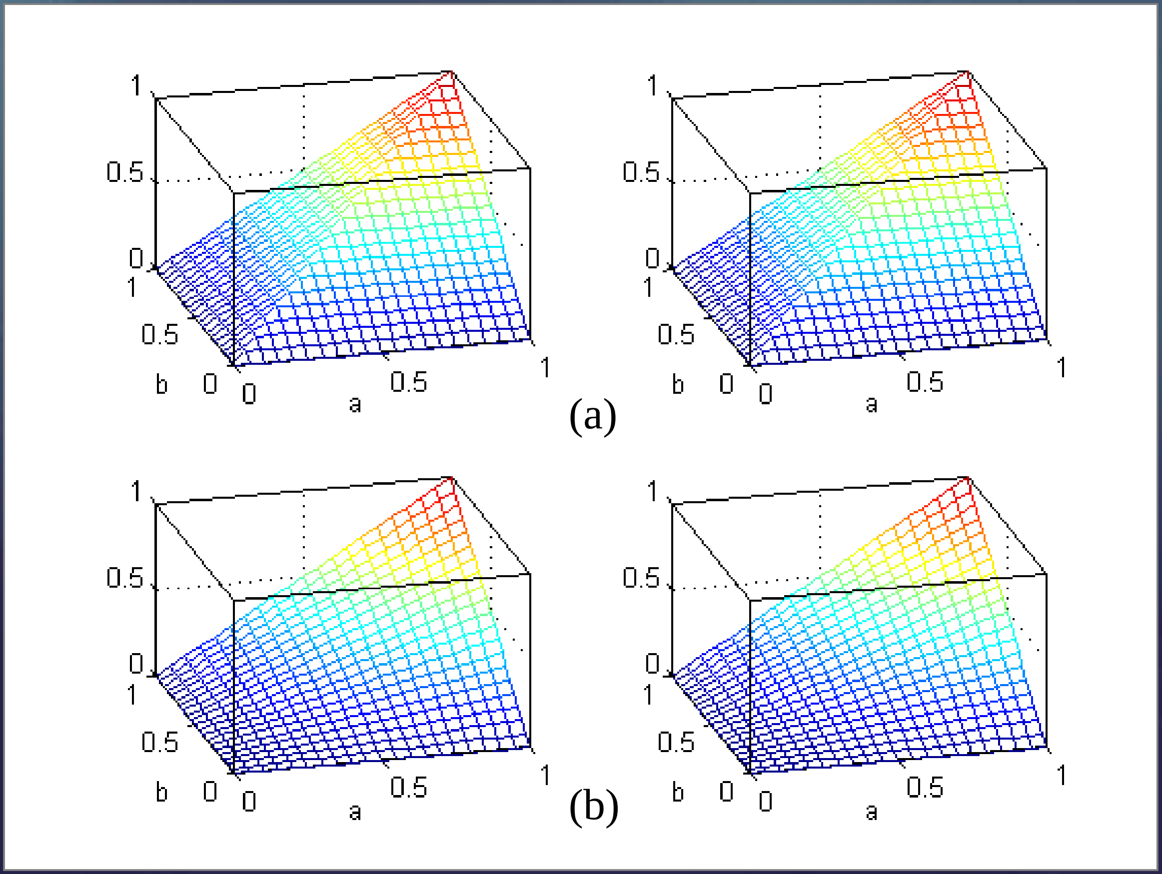
- h is a scaling transformation

Obs: h bijective means both, h injective (one-to-one)
and h surjective (onto)

Examples

$h = x^2, \; x \in [0,1]$

t $\qquad$ t_h

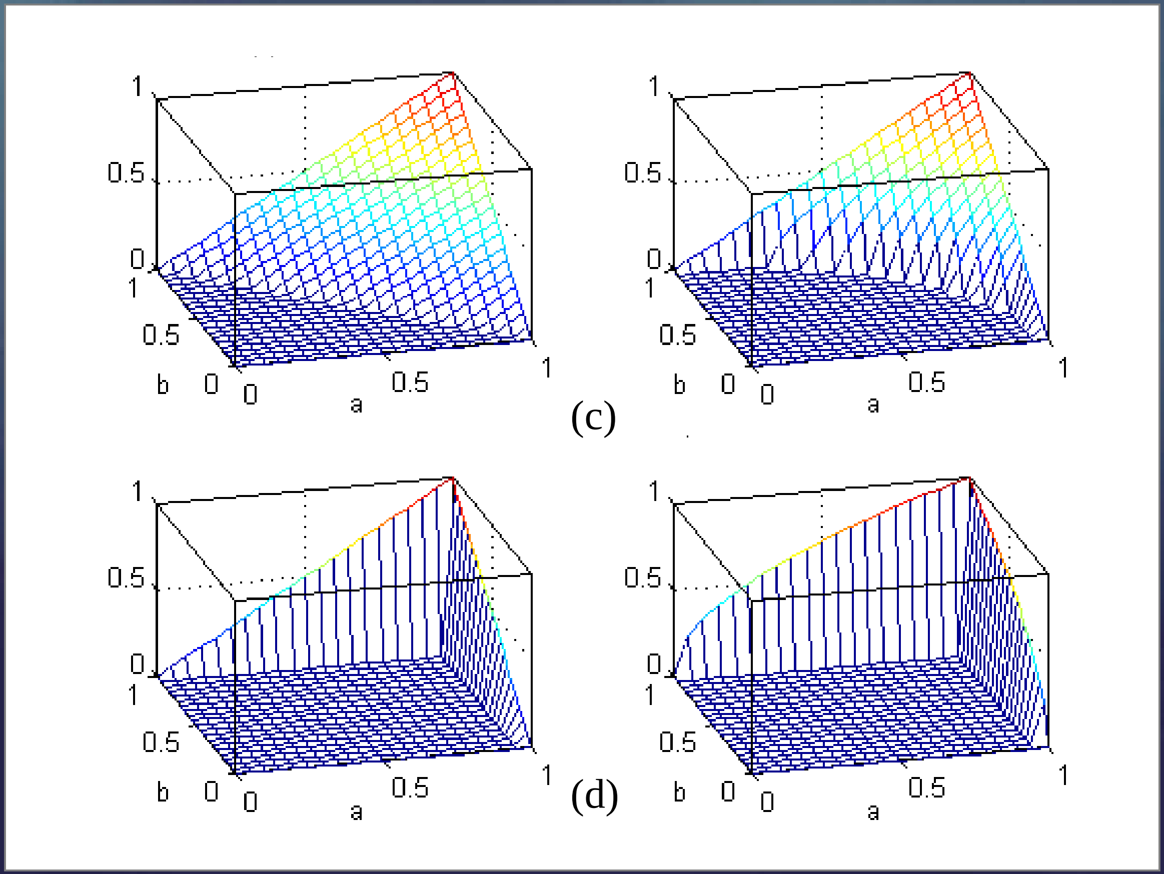


minimum

product

$h = x^2, \ x \in [0,1]$

t t_h



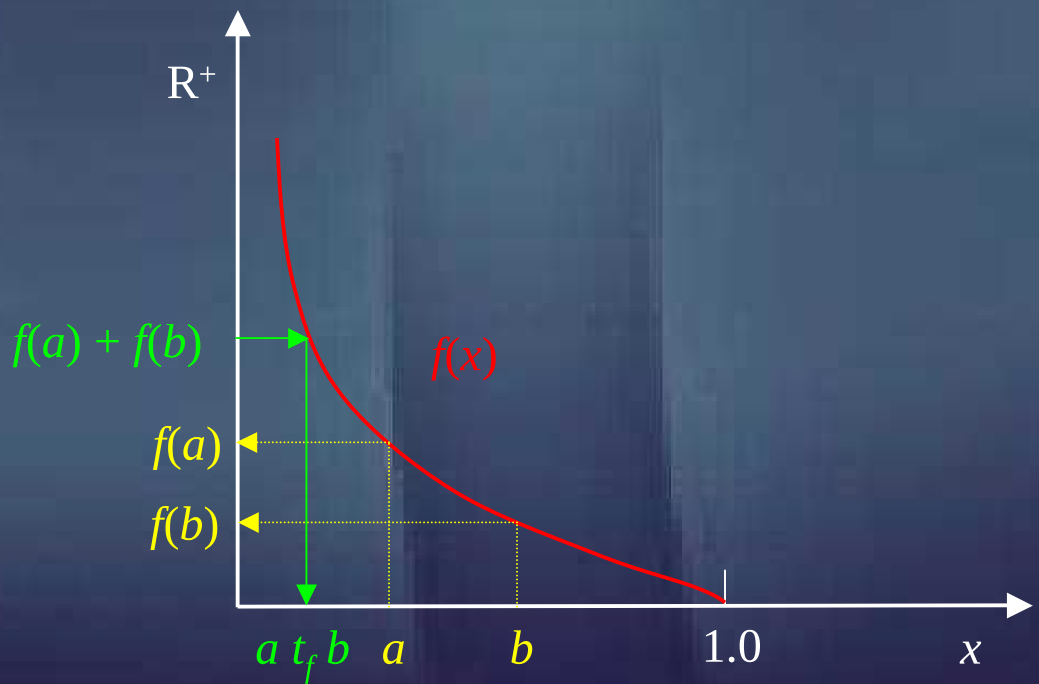
Lukasiewicz

drastic product

Additive generators

- f: $[0,1] \rightarrow [0, \infty)$, $f(1) = 0$
  - continuous
  - strictly decreasing
- $a\ t_f\ b = f^{-1}(f(a) + f(b)) \;\Leftrightarrow\;$ is a Archimedean t-norm

Additive generators of t-norms



$R^+ \equiv [0, \infty)$,

Example

$f = -\log(x)$

$f(a) + f(b) = -\log(a) - \log(b)$

$\quad\quad\quad\quad\quad - (\log(a) + \log(b))$

$\quad\quad\quad\quad\quad - \log(ab)$

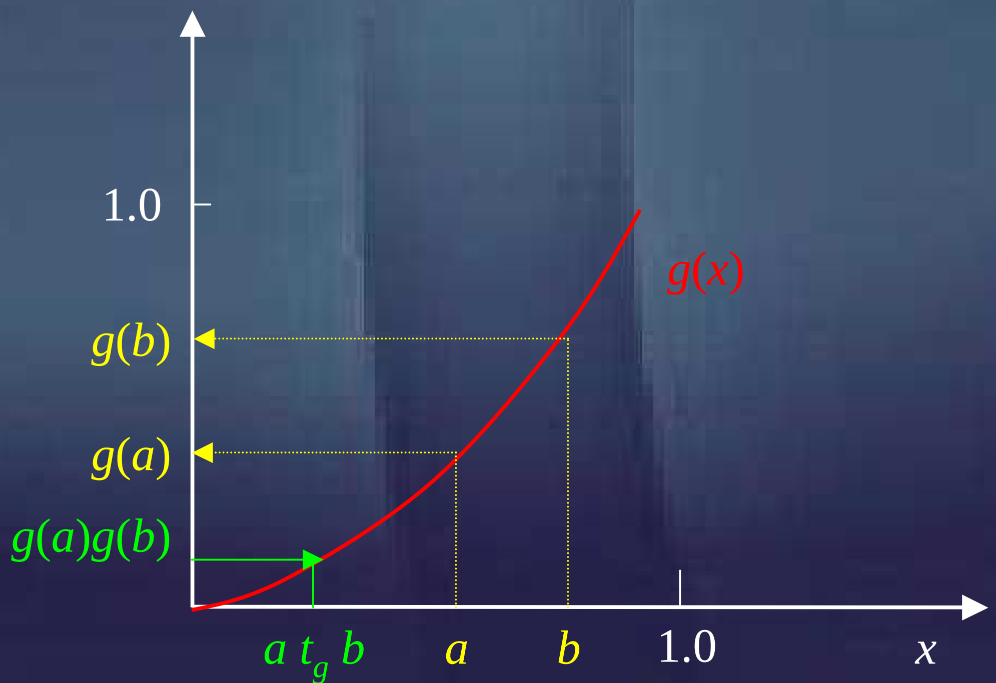
$f^{-1}(f(a) + f(b)) = e^{\log(ab)} = ab$

- $a\ t_f\ b = ab$ (Archimedean t-norm)

Multiplicative generators

- g: $[0,1] \to [0, 1]$, $g(1) = 1$
  - continuous
  - strictly increasing
- $a\ t_g\ b = g^{-1}(g(a)g(b))$ $\Leftrightarrow$ is a Archimedean t-norm

Multiplicative generators of t-norms



Example

- $g = x^2$

$a\ t_f\ b = ab$ (Archimedean t-norm)

- $g = e^{-f(x)}$

multiplicative and additive generators $\rightarrow$ same t-norm

Ordinal sums

$t_o: [0,1] \rightarrow [0, 1]$ denoted $t_o = (<\alpha_k, \beta_k, t_k>, k \in K)$

$$t_o(a,b,I,\tau) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) t_k\left(\dfrac{a-\alpha_k}{\beta_k - \alpha_k}, \dfrac{b-\alpha_k}{\beta_k - \alpha_k}\right) & \text{if } a,b \in [\alpha_k, \beta_k] \\ \min(a,b) & \text{otherwise} \end{cases}$$

$I = \{[\alpha_k, \beta_k], k \in K\}$

- nonempty, countable family
- pairwise disjoint subintervals of [0,1]

$\tau = \{t_k, k \in K\}$

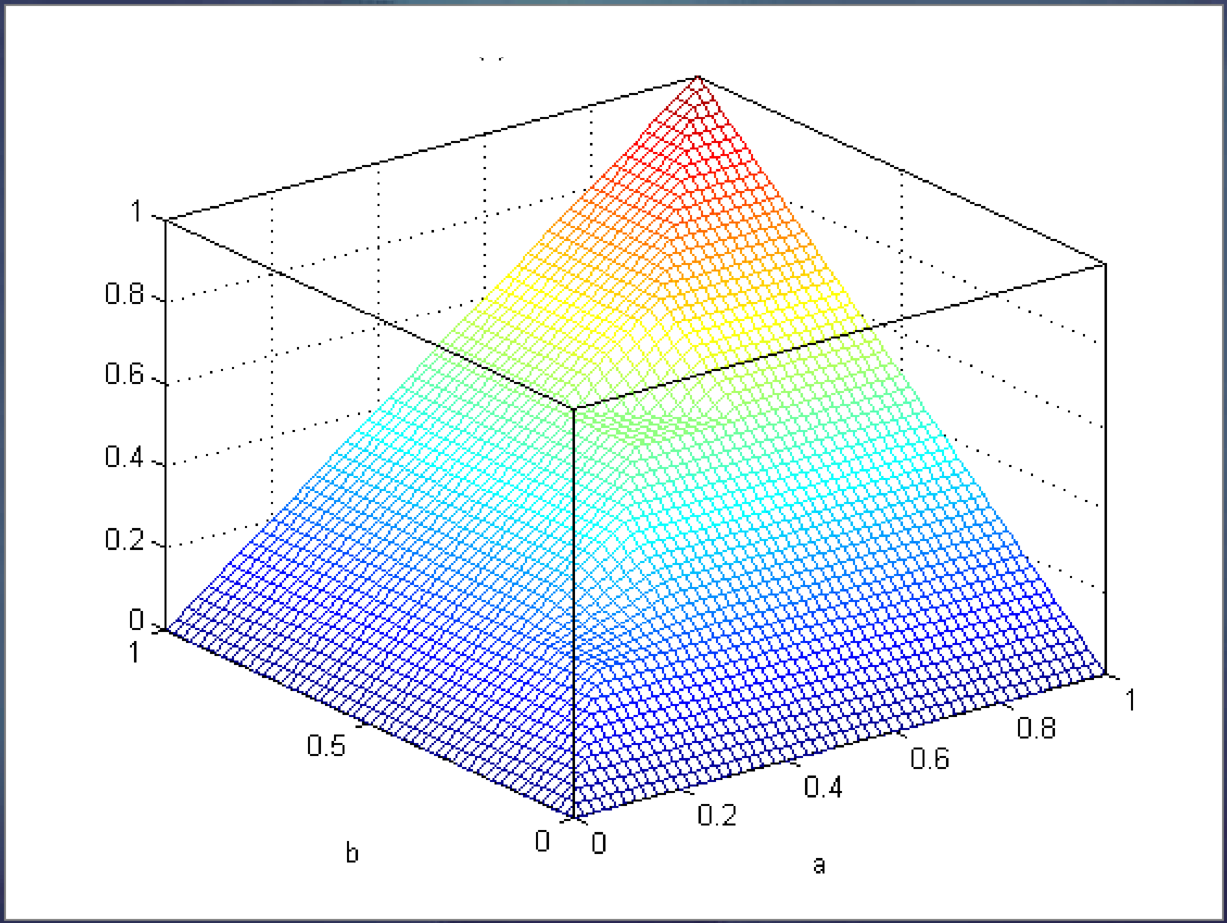
- family of t-norms

Example

$$t_o(a,b,I,\tau) = \begin{cases} 0.2 + 5(a-0.2)(b-0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + \max(a+b-1.2,0) & \text{if } a,b \in [0.5,0.7] \\ \min(a,b) & \text{otherwise} \end{cases}$$

$I = \{[0.2, 0.4], [0.5, 0.7]\}$ $\quad K = \{1,2\}$

$\tau = \{t_p, t_l\}$, $t_1 = t_p$, $t_2 = t_l$



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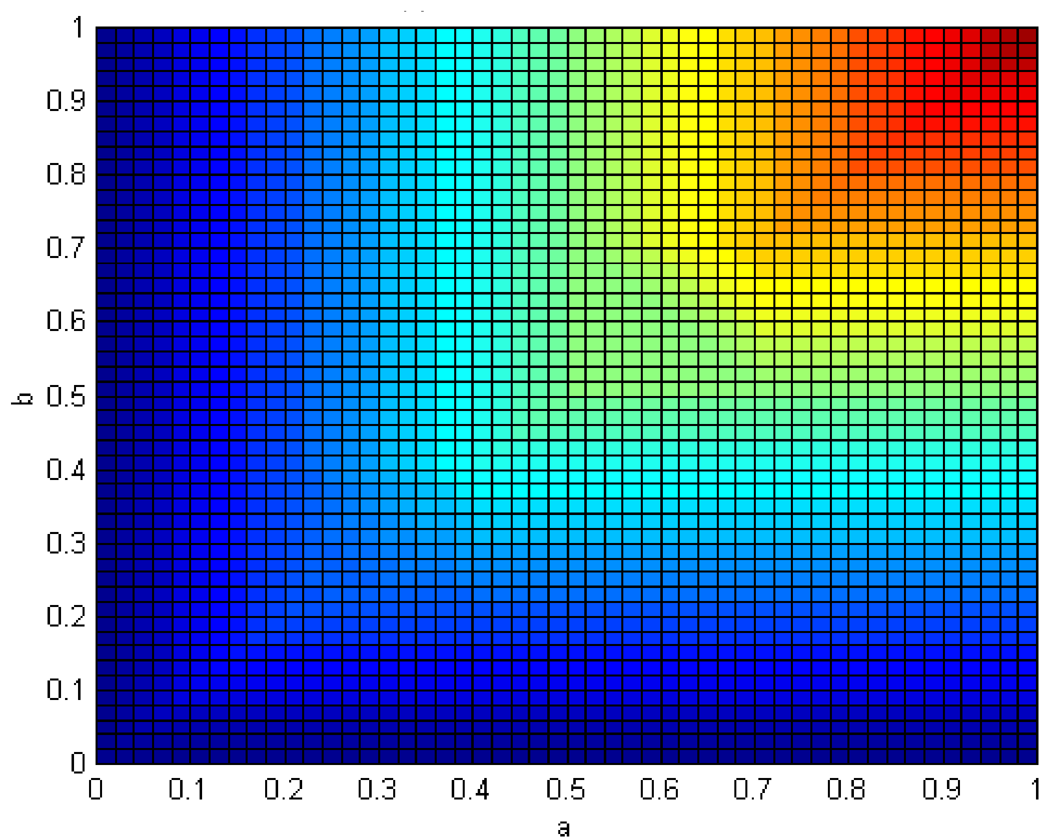


$$t_o(a,b,I,\tau) = \begin{cases} 0.2 + 5(a-0.2)(b-0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + \max(a+b-1.2,0) & \text{if } a,b \in [0.5,0.7] \\ \min(a,b) & \text{otherwise} \end{cases}$$

5.4 Triangular conorms

Definition

$s : [0,1] \times [0,1] \rightarrow [0,1]$

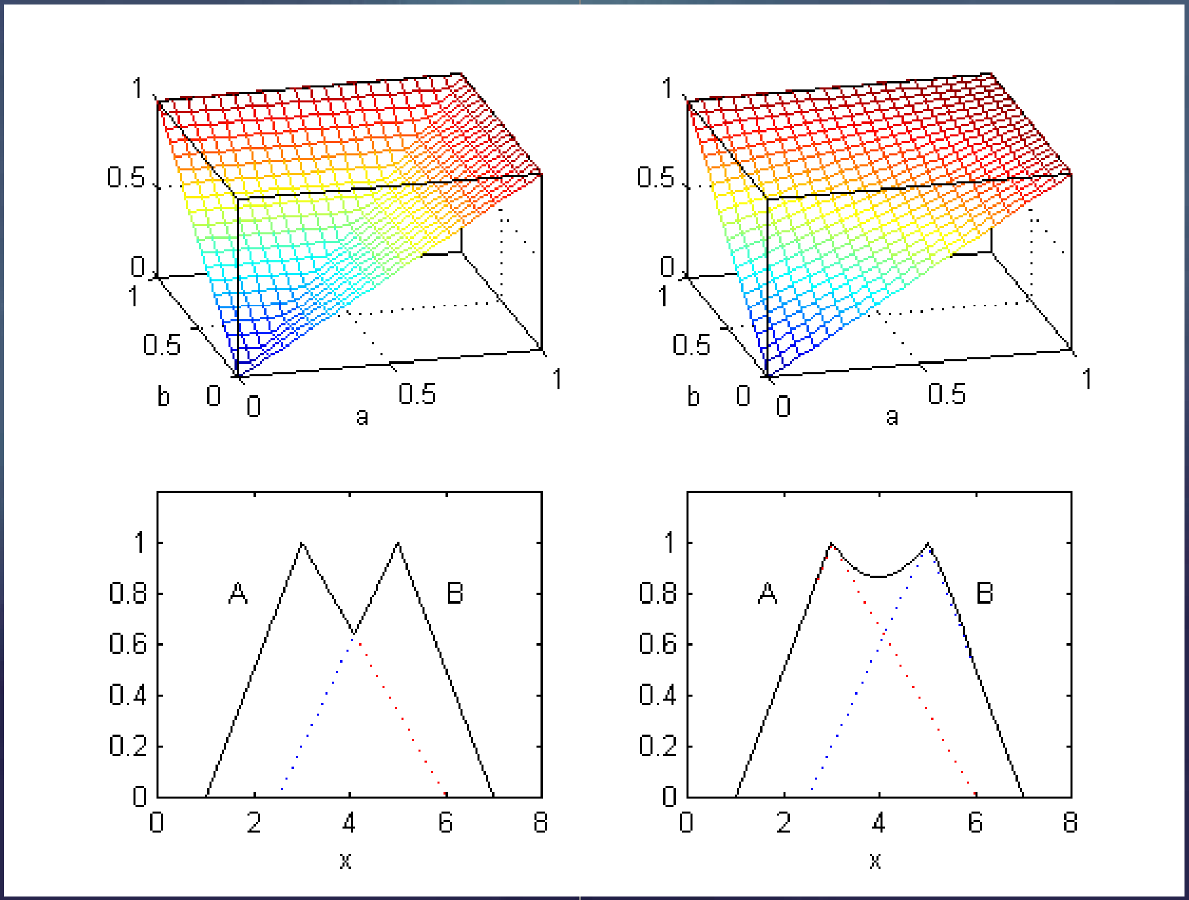
- Commutativity: $a \; s \; b = b \; s \; a$
- Associativity: $a \; s \; (b \; s \; c) = (a \; s \; b) \; s \; c$
- Monotonicity: if $b \leq c$ then $a \; s \; b \leq a \; s \; c$
- Boundary conditions: $a \; s \; 1 = 1$
  $a \; s \; 0 = a$

Examples

$a \; s_m \; b = \max(a,b)$

$a \; s_p \; b = a + b - ab$

maximum



(a)

(b)

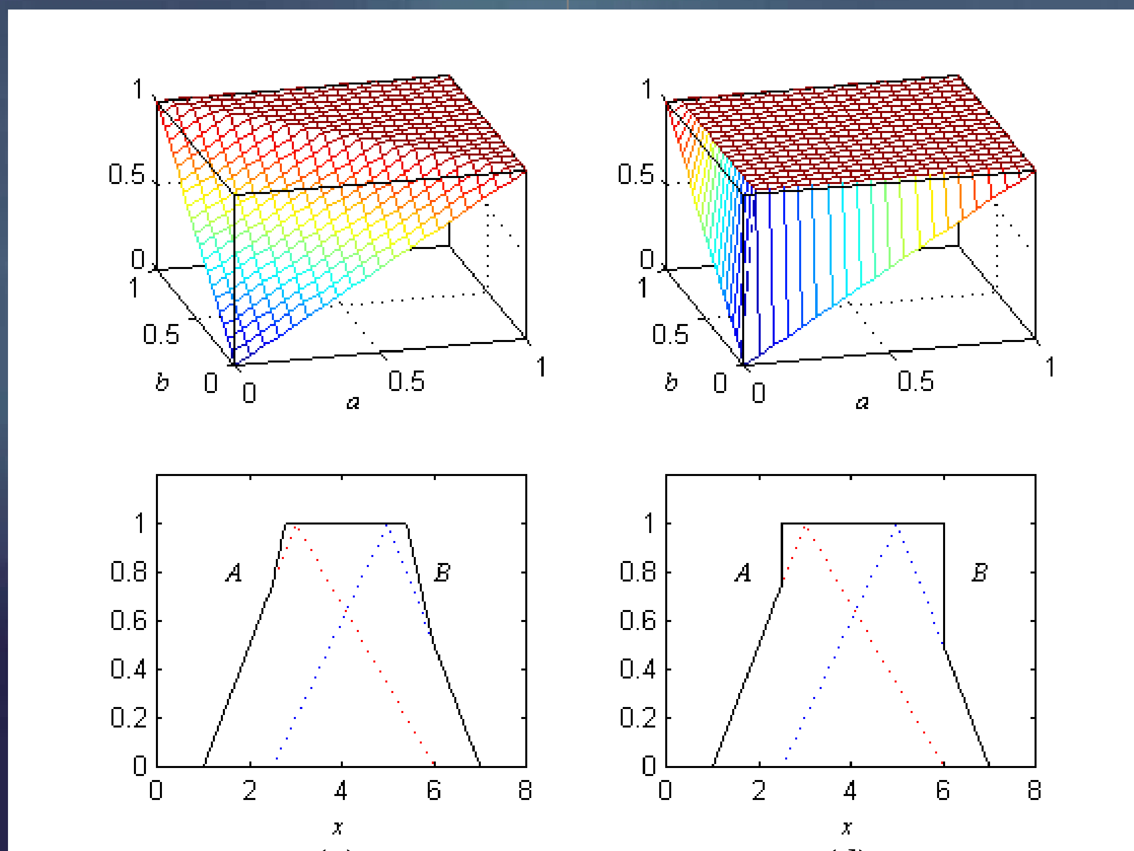
probabilistic sum

Lukasiewicz

$$a\ s_l\ b = \max(a+b,\ 1)$$

drastic sum

$$a s_d b = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$



(c) (d)

$\max(a,b) \le asb \le as_d b$ lower and upper bounds

$asa \ge a$ Archimedean (if t is continuous)

$a^n = 1$ nilpotent (e.g. Lukasiewicz)

$a\ s\ a = a^2, \ldots, a^{n-1}\ s\ a = a^n$

Dual norms and De Morgan laws

$$a \; s \; b = 1 - (1-a) \; t \; (1-b)$$

$$a \; t \; b = 1 - (1-a) \; s \; (1-b)$$

Dual triangular norms

$$(1-a) \; t \; (1-b) = 1 - a \; s \; b$$

$$(1-a) \; s \; (1-b) = 1 - a \; t \; b$$

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

$$\overline{A} \cup \overline{B} = \overline{A \cap B}$$

De Morgan

Constructors of s-norms

- Monotonic function transformation
- Additive and multiplicative generators
- Ordinal sums

Monotonic function transformation

- If h: $[0,1] \to [0,1]$ is a strictly increasing bijection

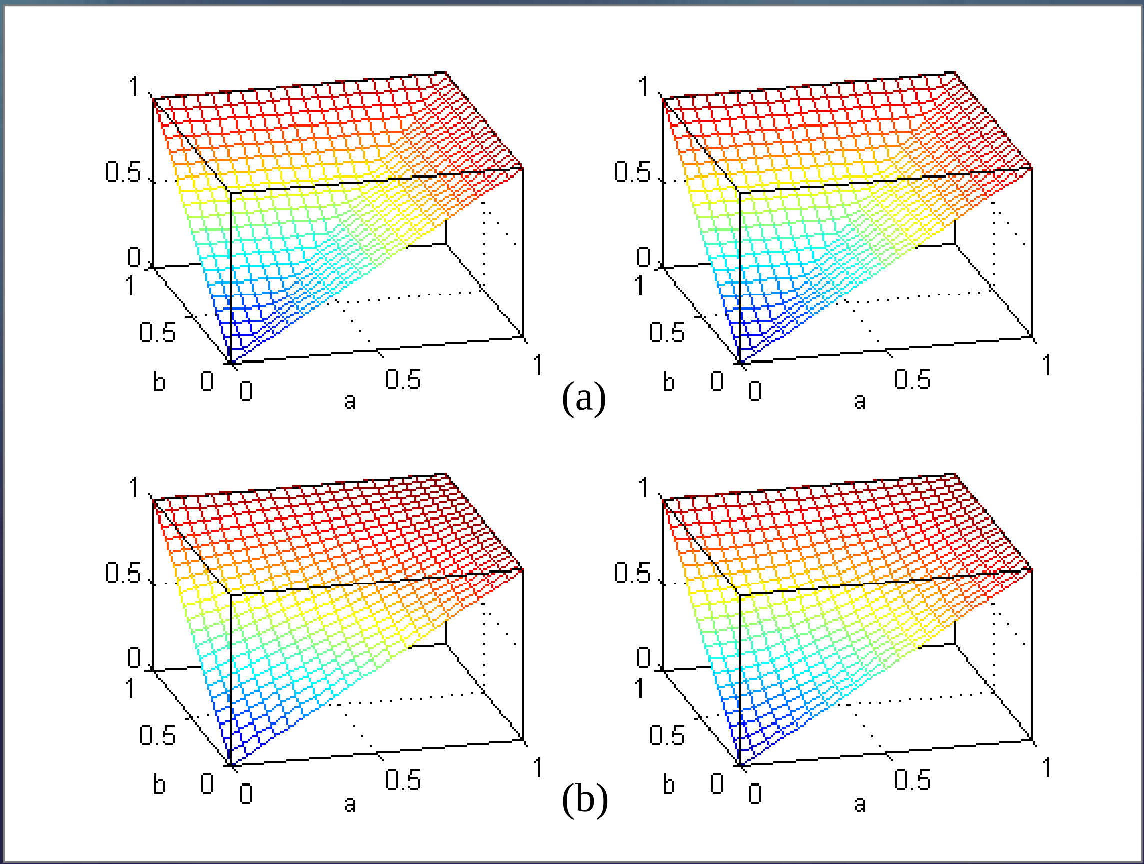
  then $s_{\mathrm{h}}(a,b) = h^{-1}(s(h(a),s(b)))$ is a t-norm

- h is a scaling transformation

Examples

$h = x^2, \; x \in [0,1]$

s $\quad$ s_h

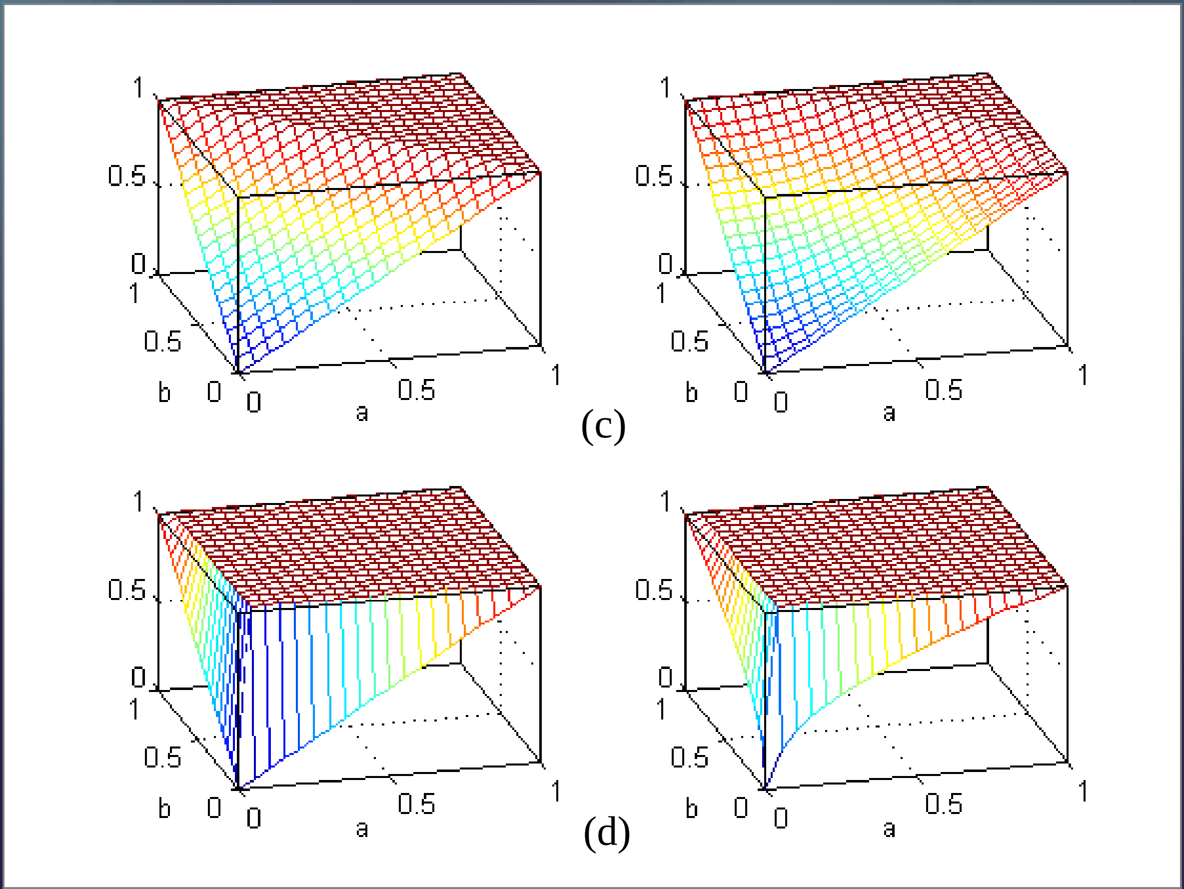


maximum

probabilistic sum

$h = x^2, \ \ x \in [0,1]$

S $\quad$ S_h



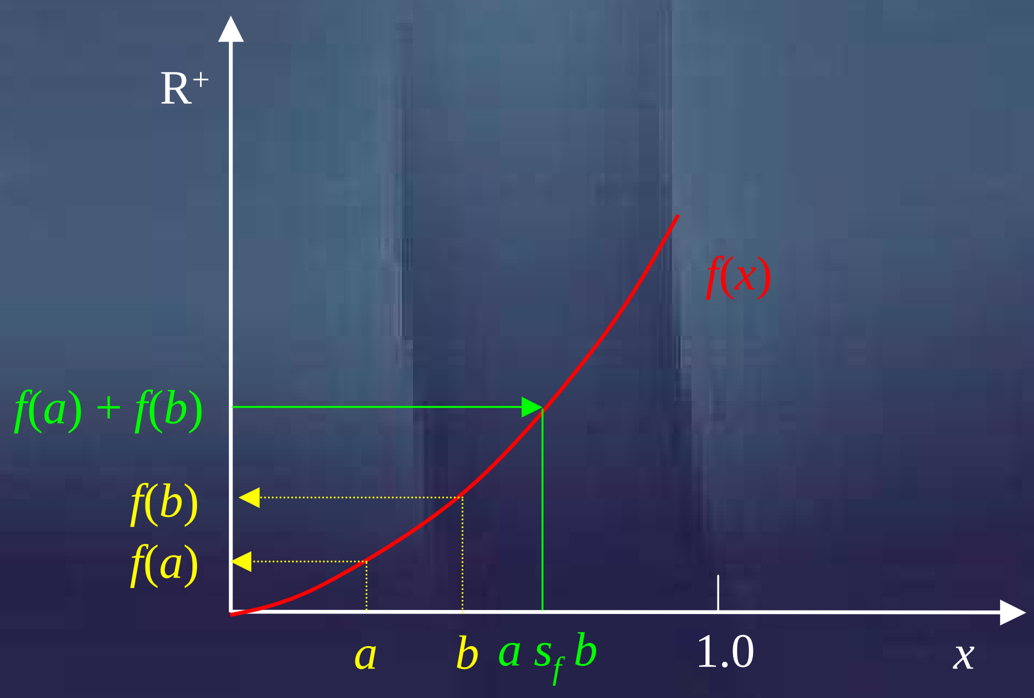
Lukasiewicz

drastic sum

Additive generators

- f: $[0,1] \rightarrow [0, \infty)$, $f(0) = 0$
  - continuous
  - strictly increasing
- $a \; s_f \; b = f^{-1}(f(a) + f(b))$ $\Leftrightarrow$ is a Archimedean t-conorm

Additive generators of t-conorms



Example

$f = -\log(1 - x)$

$f(a) + f(b) = -\log(1 - a) - \log(1 - b)$

$-(\log(1 - a) + \log(1 - b))$

$-\log(1 - a)(1 - b)$

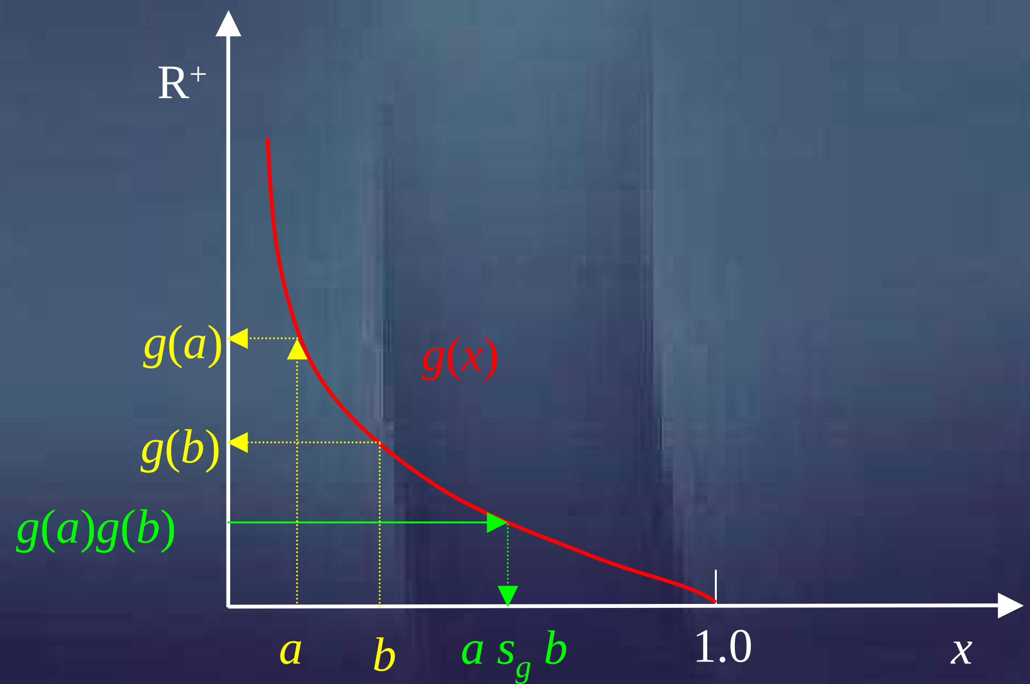
$f^{-1}(f(a) + f(b)) = 1 - e^{\log(1 - a)(1 - b)} = a + b - ab$

- $a\ s_f\ b = a + b - ab$ (Archimedean t-conorm)

Multiplicative generators

- g: $[0,1] \rightarrow [0, 1]$, $g(0) = 1$
  - continuous
  - strictly decreasing
- $a\ s_g\ b = g^{-1}(g(a)g(b)) \Leftrightarrow$ is a Archimedean t-norm

Multiplicative generators of t-conorms



$R^+ \equiv [0, \infty)$,

Example

- $g = 1 - x$

  $a \; s_g \; b = a + b - ab$ (Archimedean t-norm)

- $g = e^{-f(x)}$

  multiplicative and additive generators $\rightarrow$ same t-conorm

Ordinal sums

s_o: $[0,1] \to [0, 1]$ denoted $s_o = (<\alpha_k,\beta_k, s_k>, k\in K)$

$$s_o(a,b,I,\sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)s_k\left(\dfrac{a-\alpha_k}{\beta_k - \alpha_k}, \dfrac{b-\alpha_k}{\beta_k - \alpha_k}\right) & \text{if } a,b \in [\alpha_k, \beta_k] \\ \max(a,b) & \text{otherwise} \end{cases}$$

$I = \{[\alpha_k,\beta_k], k\in K \}$

- nonempty, countable family
- pairwise disjoint subintervals of $[0,1]$

$\sigma = \{s_k, k\in K\}$

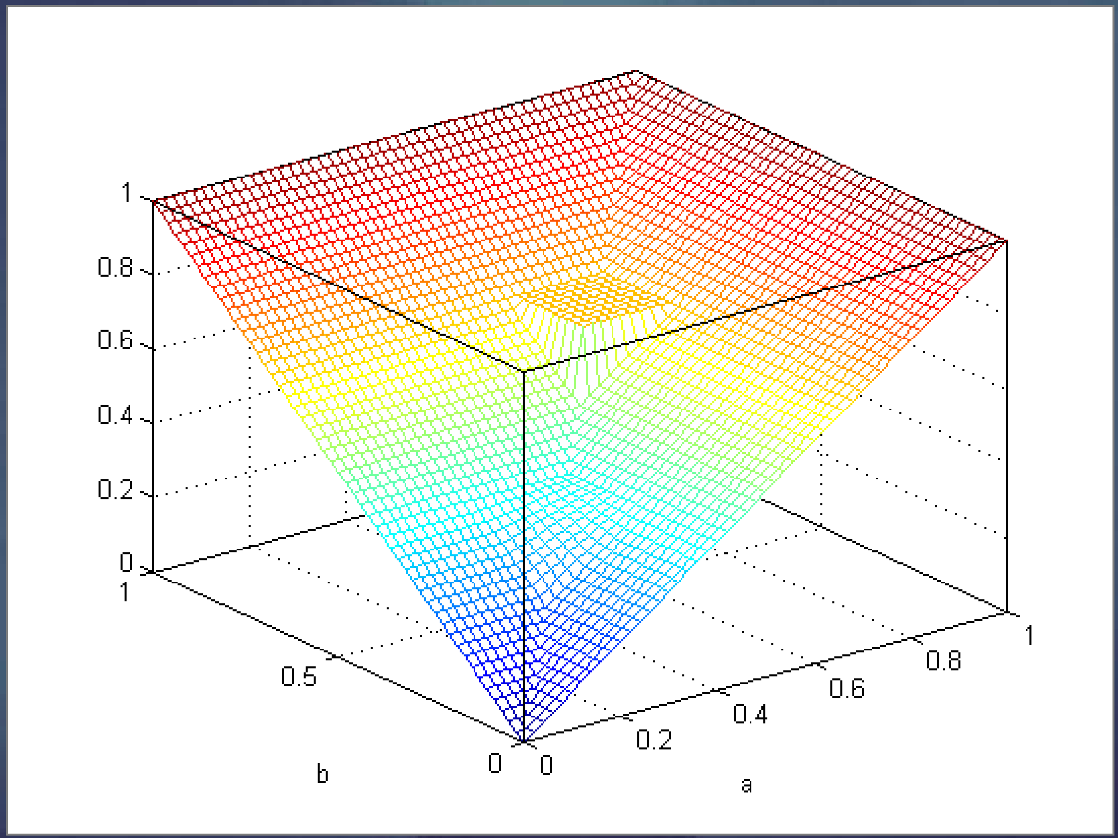
- family of t-conorms

Example

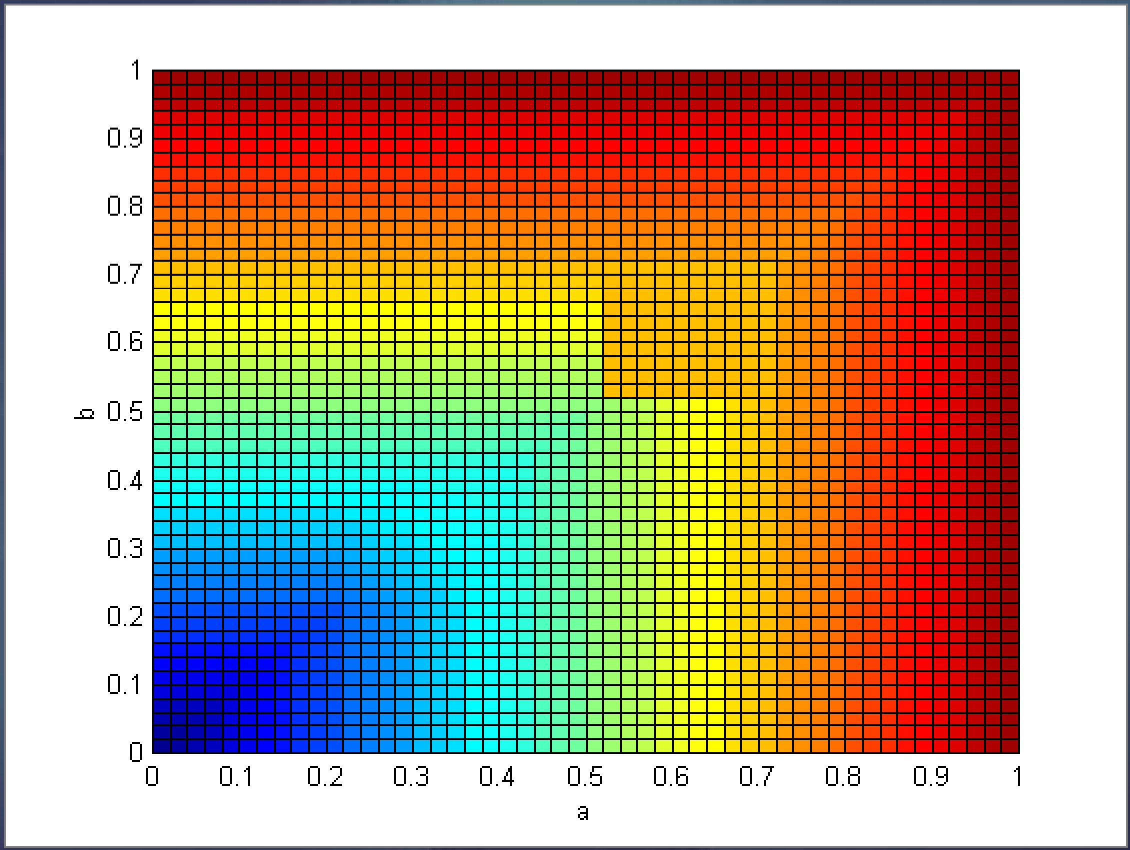
$$s_o(a,b,I,\sigma)=\begin{cases}0.2+(a-0.2)+(b-0.2)\text{-}5(a\text{-}0.2)(b\text{-}0.2) & \text{if } a,b\in[0.2,0.4]\\ 0.5+0.2\min(5(a-0.2)+5(b-0.2,0),1) & \text{if } a,b\in[0.5,0.7]\\ \max(a,b) & \text{otherwise}\end{cases}$$

$I=\{[0.2,\ 0.4],\ [0.5,\ 0.7]\}$ $K=\{1,2\}$

$\sigma=\{s_p,\ s_l\},\ s_1=s_p,\ s_2=s_l$



$$s_o(a,b,I,\sigma)=\begin{cases}0.2+(a-0.2)+(b-0.2)\text{-}5(a\text{-}0.2)(b\text{-}0.2) & \text{if } a,b\in[0.2,0.4]\\ 0.5+0.2\min(5(a-0.2)+5(b-0.2,0),1) & \text{if } a,b\in[0.5,0.7]\\ \max(a,b) & \text{otherwise}\end{cases}$$



$$s_o(a,b,I,\sigma) = \begin{cases} 0.2 + (a-0.2) + (b-0.2) \text{-} 5(a\text{-}0.2)(b\text{-}0.2) & \text{if } a,b \in [0.2, 0.4] \\ 0.5 + 0.2\min(5(a-0.2) + 5(b-0.2, 0), 1) & \text{if } a,b \in [0.5, 0.7] \\ \max(a,b) & \text{otherwise} \end{cases}$$

5.5 Triangular norms as general category of logical operators

Motivation

- **Fuzzy propositions involves linguistic statements:**
  - temperature is low *and* humidity is high
  - velocity is high *or* noise level is low
- **Logical operations:**
  - and ($\wedge$)
  - or ($\vee$)

Truth value assignment

$L = \{P, Q, ...\}$ $P, Q, ...$ atomic statements

truth: L $\rightarrow$ [0, 1] $p, q, ... \in [0, 1]$

truth (P and Q) = truth ($P \wedge Q$) $\rightarrow p \wedge q = p\ t\ q$

truth (P or Q) = truth ($P \vee Q$) $\rightarrow p \vee q = p\ s\ q$

Examples

p	q	$\min(p, q)$	$\max(p, q)$	pq	$p + q - pq$
1	1	1	1	1	1
1	0	0	1	0	1
0	1	0	1	0	1
0	0	0	0	0	0
0.2	0.5	0.2	0.5	0.1	0.6
0.5	0.8	0.5	0.8	0.4	0.9
0.8	0.7	0.7	0.8	0.56	0.94

Implication induced by a t-norm

$$a \ \varphi\, b \equiv a \Rightarrow b$$

$$a \ \varphi\, b = \sup \{ \, c \in [0, 1] \mid a \ t \ c \leq b \} \quad \forall a,b \in [0,1]$$

residuation

φ operator or Boolean values of its arguments

a	b	$a \Rightarrow b$	$a \ \varphi \ b$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

5.6 Aggregation operations

Definition

$g : [0,1]^n \rightarrow [0,1]$

1. Monotonicity:

   $g\,(x_1, x_2,..., x_n) \geq g\,(y_1, y_2,.., y_n)$ if $x_i \geq y_i,\ i = 1,.., n$

2. Boundary conditions:

   $g\,(0, 0,..., 0) = 0$
   $g\,(1, 1,..., 1) = 1$

1. Neutral element (e):

$$g\,(x_1, x_2,\ldots, x_{\mathrm{i}-1}, e, x_{i+1},\ldots, x_n) = g\,(x_1, x_2,\ldots, x_{\mathrm{i}-1}, e, x_{i+1},\ldots, x_n)\ n \geq 2$$

2. Annihilator (l):

$$g\,(x_1, x_2,\ldots, x_{\mathrm{i}-1}, l, x_{i+1},\ldots, x_n) = l$$

Observation: Annihilator ≡ absorbing element

Averaging operations

$$g(x_1, x_2, \ldots, x_n) = \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}(x_i)^p}, \; p \in \mathbf{R}, \; p \neq 0$$

$p = 1$ $\quad g(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ — arithmetic mean

$p \to 0$ $\quad g(x_1, x_2, \ldots, x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i}$ — geometric mean

$p = -1$ $\quad g(x_1, x_2, \ldots, x_n) = \dfrac{n}{\sum_{i=1}^{n} 1/x_i}$ — harmonic mean

$p \rightarrow -\propto \qquad g(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n)$

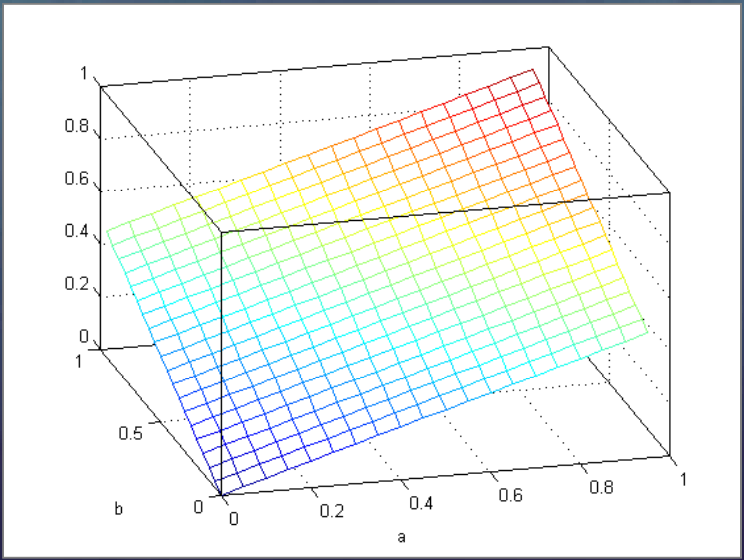
$p \rightarrow \propto \qquad g(x_1, x_2, \ldots, x_n) = \max(x_1, x_2, \ldots, x_n)$

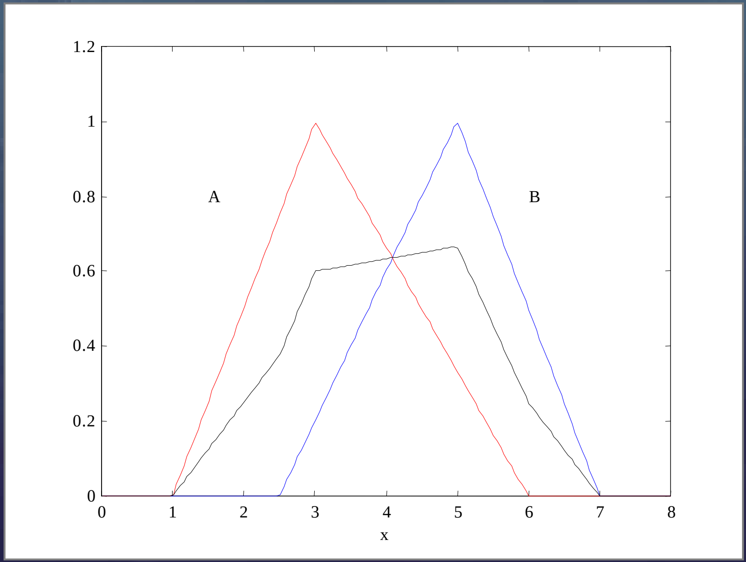
Bounds

$$\min(x_1, x_2, \ldots, x_n) \leq g(x_1, x_2, \ldots, x_n) \leq \max(x_1, x_2, \ldots, x_n)$$

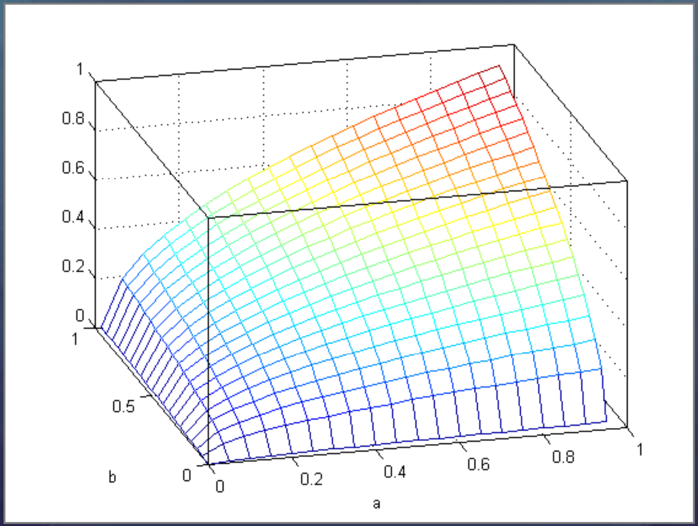
Examples

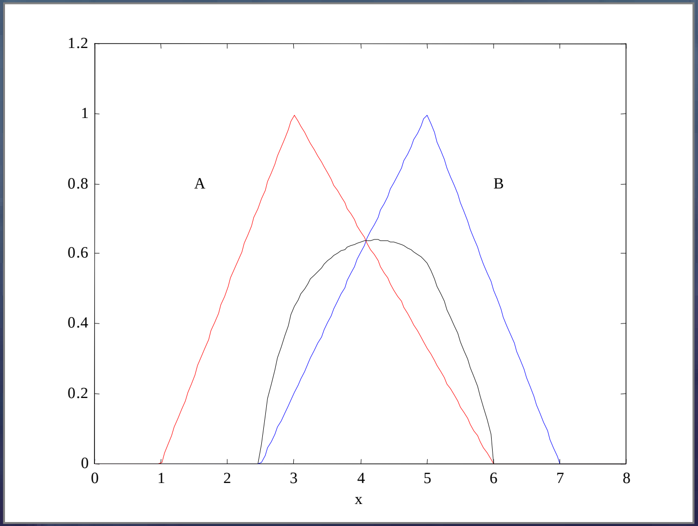
Arithmetic mean



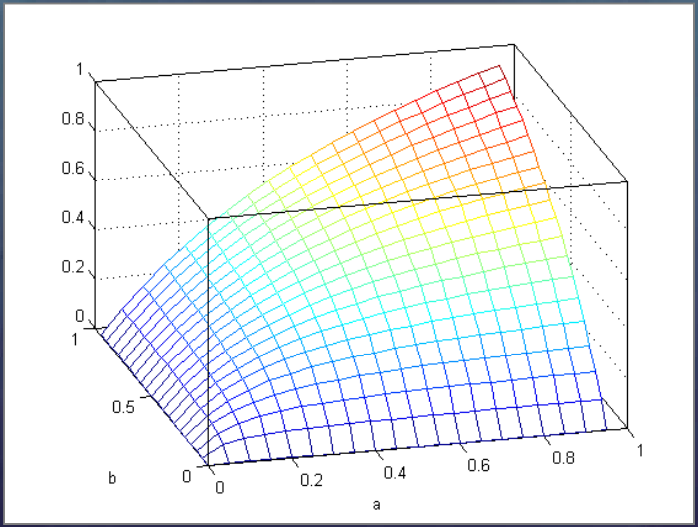


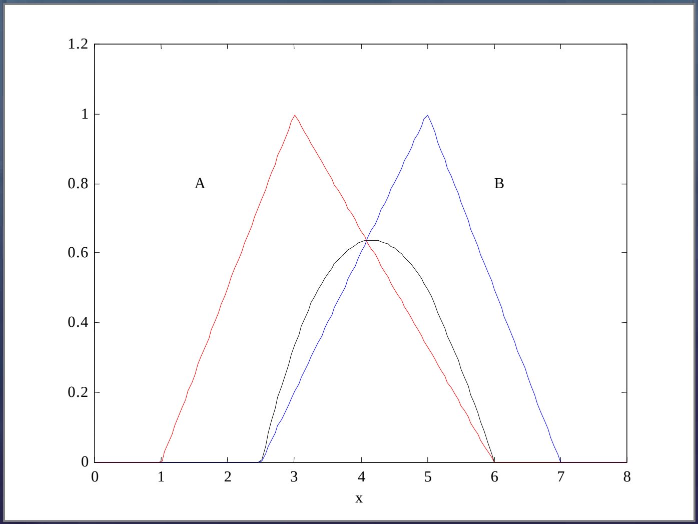
Geometric mean





Harmonic mean





Ordered Weighted Averaging (OWA)

$$\mathrm{OWA}(A, \mathbf{w}) = \sum_{i=1}^{n} w_i A(x_i)$$

$$\Sigma w_i = 1, \quad w_i \in [0, 1]$$

$$A(x_1) \le A(x_2) \le \ldots \le A(x_n)$$

Examples

1. $\mathbf{w} = [1, 0,...,0]$ OWA(A,$\mathbf{w}$) = min $(A(x_1), A(x_2),..., A(x_n))$

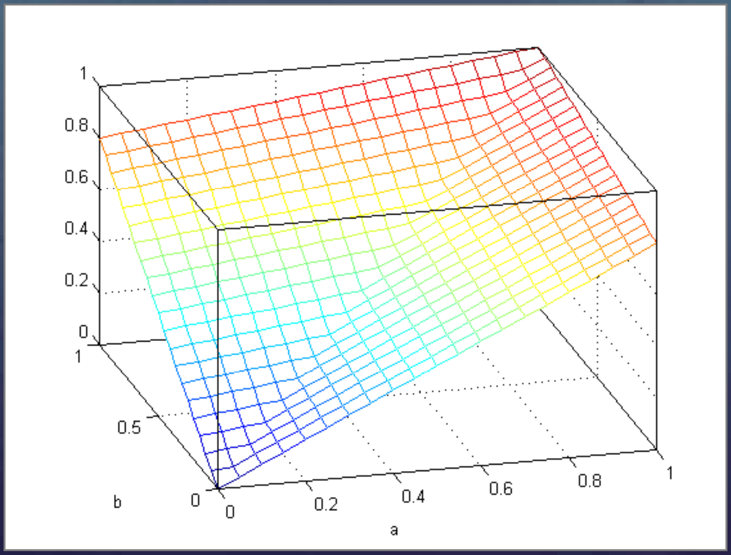
2. $\mathbf{w} = [0, 0,...,1]$ OWA(A,$\mathbf{w}$) = max $(A(x_1), A(x_2),..., A(x_n))$

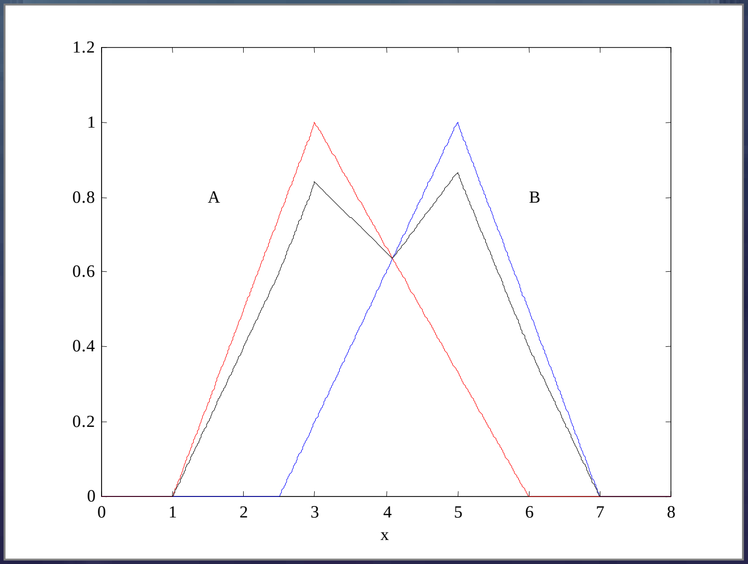
3. $\mathbf{w} = [1/n, 1/n,...,1/n]$ OWA(A,$\mathbf{w}$) = arithmetic mean

$$\boxed{\min(A(x_1), A(x_2),..., A(x_n)) \leq \mathrm{OWA}(A,\mathbf{w}) \leq \max(A(x_1), A(x_2),..., A(x_n))}$$

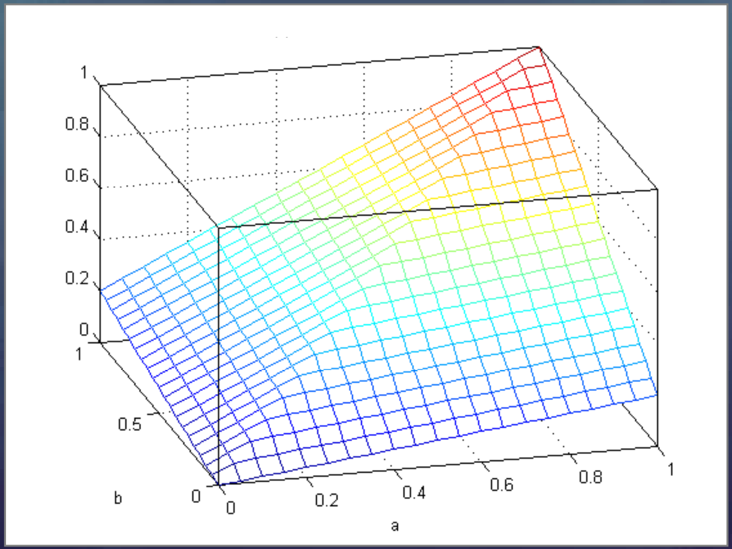
Examples

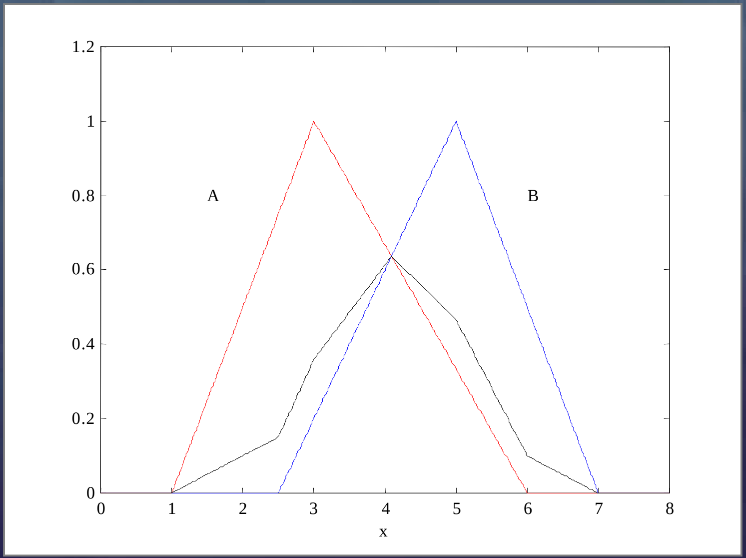
$\mathbf{w} = [0.8, 0.2]$





w = [0.2, 0.8]





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Uninorms

$$u : [0,1] \times [0,1] \to [0,1]$$

- Commutativity: $a\ u\ b = b\ u\ a$
- Associativity: $a\ u\ (b\ u\ c) = (a\ u\ b)\ u\ c$
- Monotonicity: if $b \le c$ then $a\ u\ b \le a\ u\ c$
- Identity: $a\ u\ e = a \quad \forall a \in [0, 1]$

$e \in [0, 1]$, $\quad e = 1\ \ u$ is a t-norm, $\quad e = 0\ \ u$ is a t-conorm

Results on uninorms

1. t_u, s_u: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $\forall e \in [0, 1]$

$$a t_u b = \frac{(ea)u(eb)}{e}$$

$$a s_u b = \frac{(e+(1-e)a)u(e+(1-e)b)-e}{1-e}$$

t_u and s_u are t-norm and t-conorm

2. If $a \le e \le b$ or $a \ge e \ge b$ then

If $a \le e \le b$ or $a \ge e \ge b$ then
$$\min(a,b) \le aub \le \max(a,b)$$

3. For any u with $e \in [0, 1]$

$$a u_w b \leq a u b \leq a u_s b$$

$$a u_w b = \begin{cases} 0 & \text{if } 0 \leq a, b \leq e \\ \max(a, b) & \text{if } e \leq a, b \leq 1 \\ \min(a, b) & \text{otherwise} \end{cases}$$

$$a u_s b = \begin{cases} \min(a, b) & \text{if } 0 \leq a, b \leq e \\ 1 & \text{if } e \leq a, b \leq 1 \\ \max(a, b) & \text{otherwise} \end{cases}$$

4. Conjunctive and disjunctive uninorm

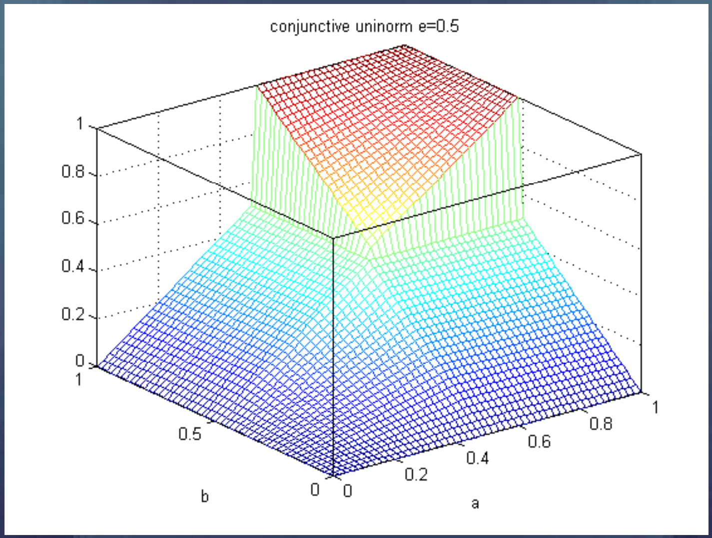
if $(0u1) = 0$ then

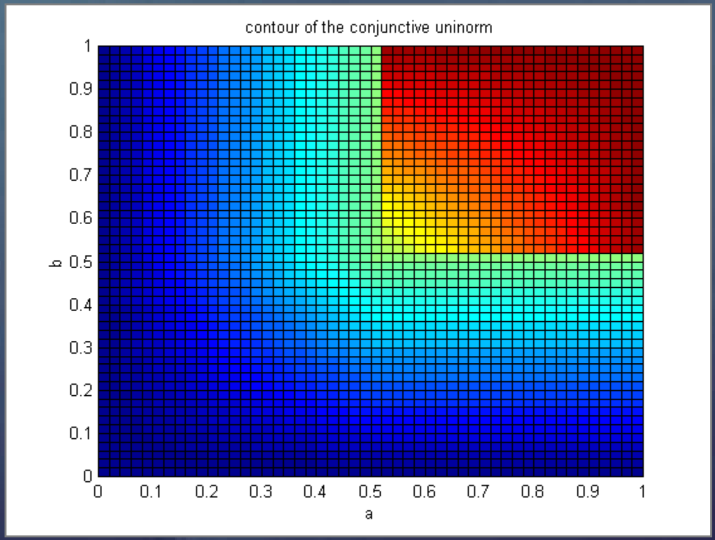
$$a u_c b = \begin{cases} e(\frac{a}{e})t(\frac{b}{e}) & \text{if } 0 \le a,b \le e \\ e + (1-e)(\frac{a-e}{1-e})s(\frac{b-e}{1-e}) & \text{if } e \le a,b \le 1 \\ \min(a,b) & \text{otherwise} \end{cases}$$

if $(0u1) = 1$ then

$$a u_d b = \begin{cases} e(\frac{a}{e})t(\frac{b}{e}) & \text{if } 0 \le a,b \le e \\ e + (1-e)(\frac{a-e}{1-e})s(\frac{b-e}{1-e}) & \text{if } e \le a,b \le 1 \\ \max(a,b) & \text{otherwise} \end{cases}$$

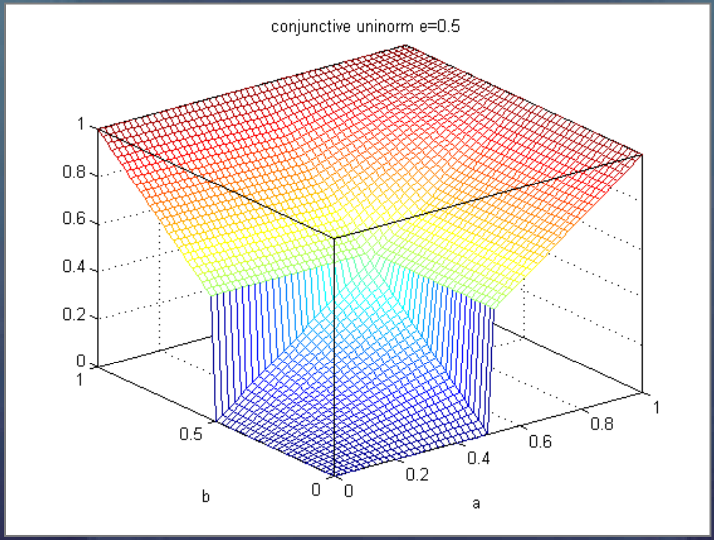
Conjunctive uninorm

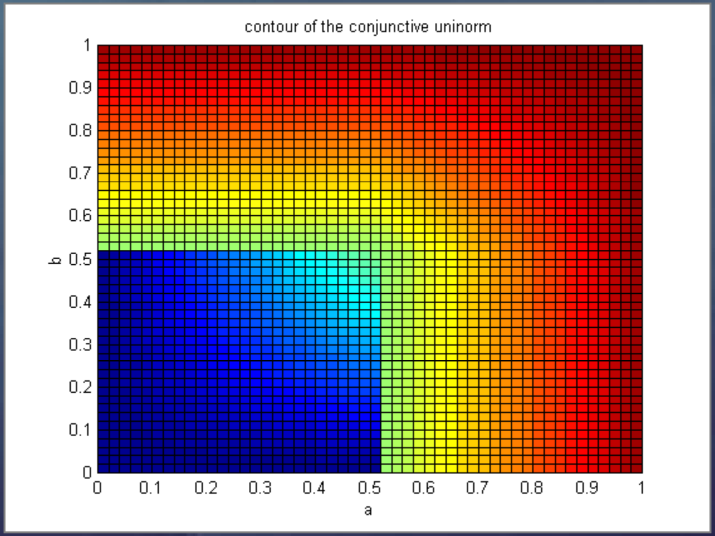




$e = 0.5$

Disjunctive uninorm





$e = 0.5$

5. Almost continuous Archimedian uninorms

$a \, u \, a < a$ for $0 < a < e$

$a \, u \, a > a$ for $e < a < 1$

6. Additive and multiplicative generators of almost continuous uninorms

$a \, u_f \, b = f^{-1}(f(a) + f(b))$

$a \, u_g \, b = g^{-1}(g(a)\mathrm{g}(b))$

$g(x) = \mathrm{e}^{-f(x)}$

f strictly increasing

g strictly decreasing

7. Ordinal sum

$$u_{co}(a,b,\iota,\tau,\sigma)=\begin{cases}\alpha_k+(\beta_k-\alpha_k)t_k\left(\dfrac{a-\alpha_k}{\beta_k-\alpha_k},\dfrac{b-\alpha_k}{\beta_k-\alpha_k}\right) & \text{if } [a,b]\in \iota_1\\ \alpha_k+(\beta_k-\alpha_k)s_k\left(\dfrac{a-\alpha_k}{\beta_k-\alpha_k},\dfrac{b-\alpha_k}{\beta_k-\alpha_k}\right) & \text{if } [a,b]\in \iota_2\\ \max(a,b) & \text{if } a,b\notin[\alpha_k,\beta_k] \text{ and } a,b\geq e\\ \min(a,b) & \text{otherwise}\end{cases}$$

$l=\{[\alpha_k,\beta_k], k\in K\}$
$l_1=\{[\alpha_k,\beta_k]\in l \mid \beta_k\leq e\}$
$l_2=\{[\alpha_k,\beta_k]\in l \mid \alpha_k\geq e\}$
$\tau=\{t_k, k\in K\},\quad \sigma=\{s_k, k\in K\}$

$$
u_{co}(a,b,\iota,\tau,\sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)\, t_k\left(\dfrac{a-\alpha_k}{\beta_k-\alpha_k}, \dfrac{b-\alpha_k}{\beta_k-\alpha_k}\right) & \text{if } [a,b] \in \iota_1 \\ \alpha_k + (\beta_k - \alpha_k)\, s_k\left(\dfrac{a-\alpha_k}{\beta_k-\alpha_k}, \dfrac{b-\alpha_k}{\beta_k-\alpha_k}\right) & \text{if } [a,b] \in \iota_2 \\ \min(a,b) & \text{if } a,b \notin [\alpha_k, \beta_k] \text{ and } a,b \le e \\ \max(a,b) & \text{otherwise} \end{cases}
$$

Nullnorms

$v : [0,1] \times [0,1] \rightarrow [0,1]$

- Commutativity: $a \; v \; b = b \; v \; a$
- Associativity: $a \; v \; (b \; v \; c) = (a \; v \; b) \; v \; c$
- Monotonicity: if $b \leq c$ then $a \; v \; b \leq a \; v \; c$
- Absorbing element: $a \; v \; e = e \quad \forall a \in [0, 1]$
- Boundary conditions: $a \; v \; 0 = \mathrm{a} \quad \forall a \in [0, e]$
  $a \; v \; 1 = \mathrm{a} \quad \forall a \in [e, 1]$

$$a s_v b = \frac{(ea)v(eb)}{e}$$

$$a t_v b = \frac{(e+(1-e)a)v(e+(1-e)b)-e}{1-e}$$

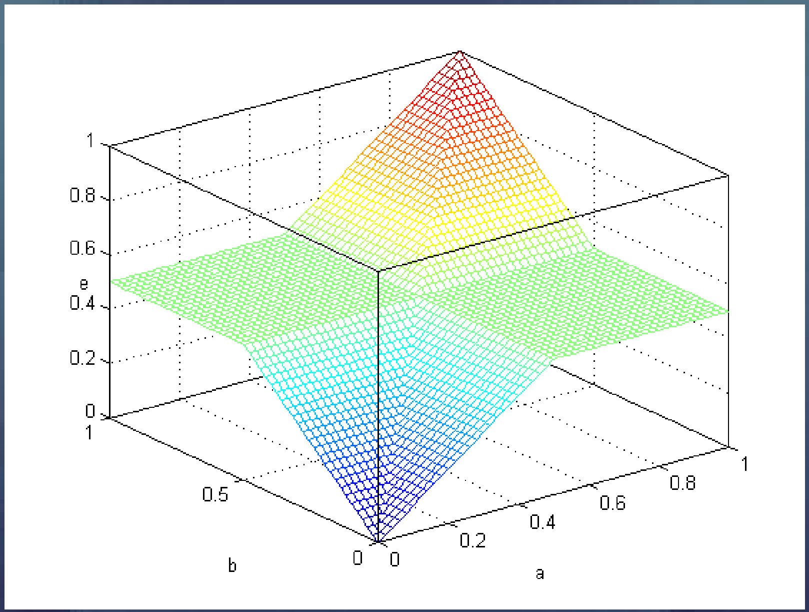
$e \in [0, 1]$

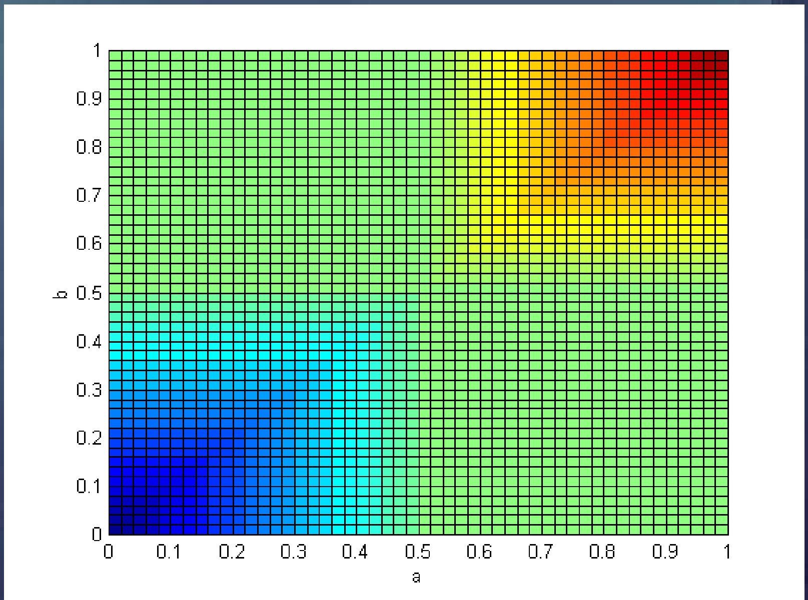
v behaves as a t-norm in $[0, e]\times[0, e]$

v behaves as a t-conorm in $[e, 1]\times[e, 1]$

$v = e$ in the rest of the unit square

Example





$e = 0.5$, t-norm = min, t-conorm = max

Symmetric sums

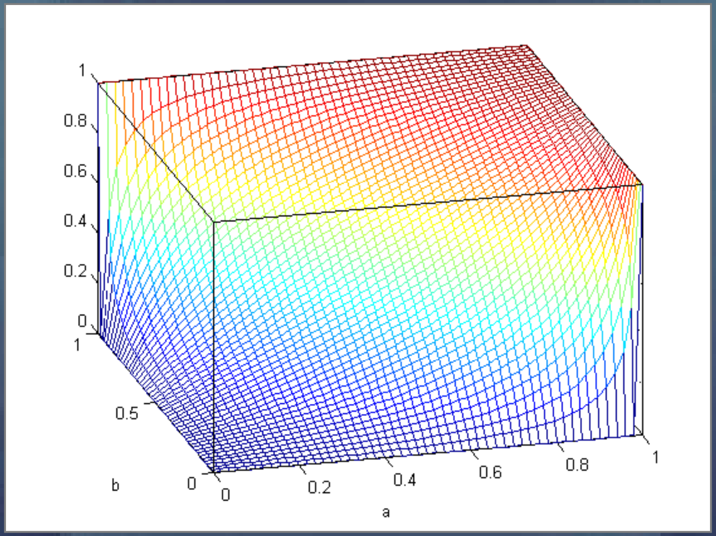
$\sigma_s(a_1, a_2,...., a_n) = 1 - \sigma_s(1 - a_1, 1 - a_2,...., 1 - a_n)$

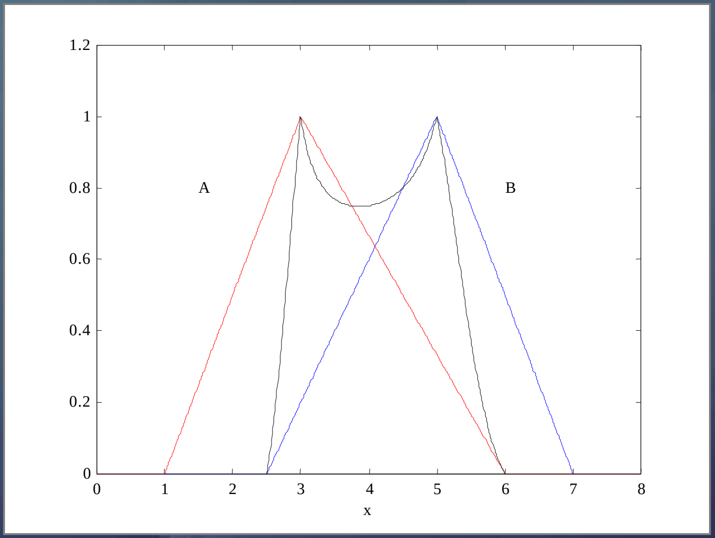
$$\sigma_s(a_1, a_2, \ldots, a_n) = \left[1 + \frac{f(1-a_1, 1-a_2, \ldots, a_n)}{f(a_1, a_2, \ldots, a_n)}\right]^{-1}$$

f increasing, continuous

$f(0,0,...,0) = 0$

Example





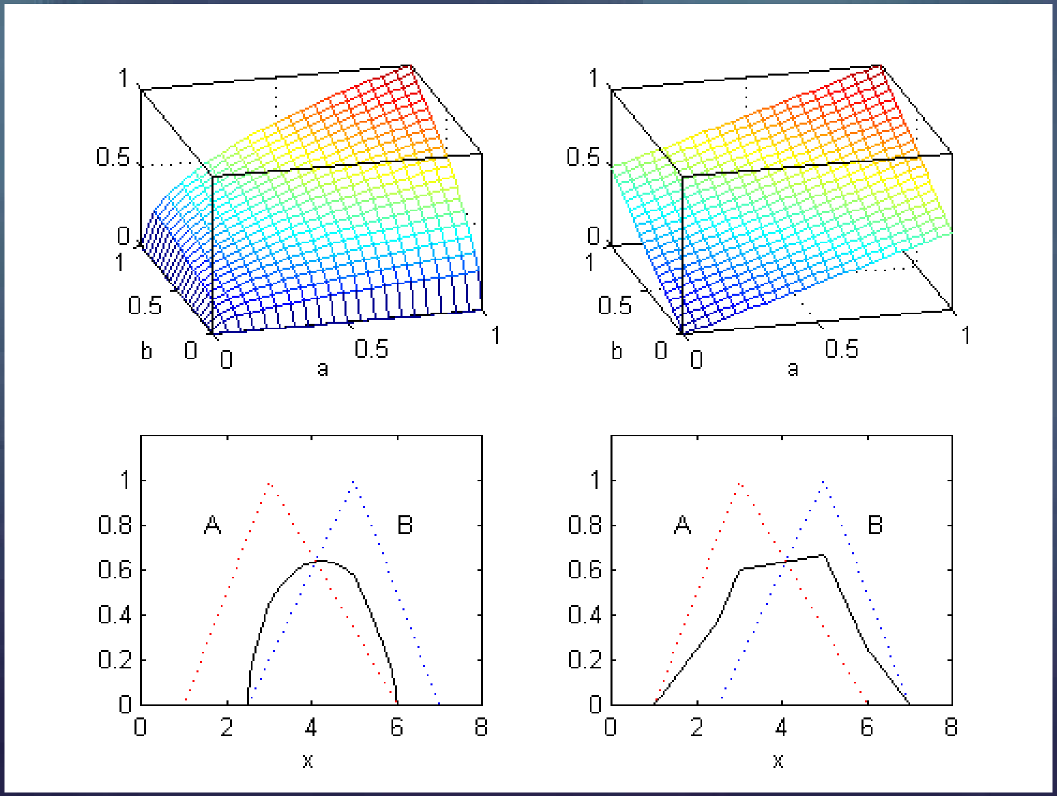
$$f(a,b) = a^2 + b^2$$

Compensatory operations

$a \odot b = (a\ t\ b)^{1-\gamma}\ (a\ s\ b)^{\gamma}$ compensatory product

$a \boxplus b = (1 - \gamma)(a\ t\ b) + \gamma(a\ t\ b)$ compensatory sum

Example



(a) (b)

$\gamma = 0.5$, t-norm = min, t-conorm = max

5.7 Fuzzy measure and integral

Fuzzy measure

$g : \Omega \rightarrow [0,1]$

- Boundary conditions: $g(\emptyset) = 0$
  $g(\mathbf{X}) = 1$

- Monotonicity: if $A \subset B$ then $g(A) \leq g(B)$

λ–fuzzy measure

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)\, g(B), \qquad \lambda > -1$$

- $\lambda = 0$ $\quad g(A \cup B) = g(A) + g(B)$ $\quad$ additive
- $\lambda > 0$ $\quad g(A \cup B) \geq g(A) + g(B)$ $\quad$ super-additive
- $\lambda < 0$ $\quad g(A \cup B) \leq g(A) + g(B)$ $\quad$ sub-additive

Fuzzy integral

$h : \mathbf{X} \rightarrow [0,1]$ Ω measurable

fuzzy integral of h with respect to g over A

$$\int_A h(x) \circ g(\,) = \sup_{\alpha \in [0,1]} \{\min[\alpha, g(A \cap H_\alpha)]\}$$

$$H_\alpha = \{x \mid h(x) \geq \alpha\}$$

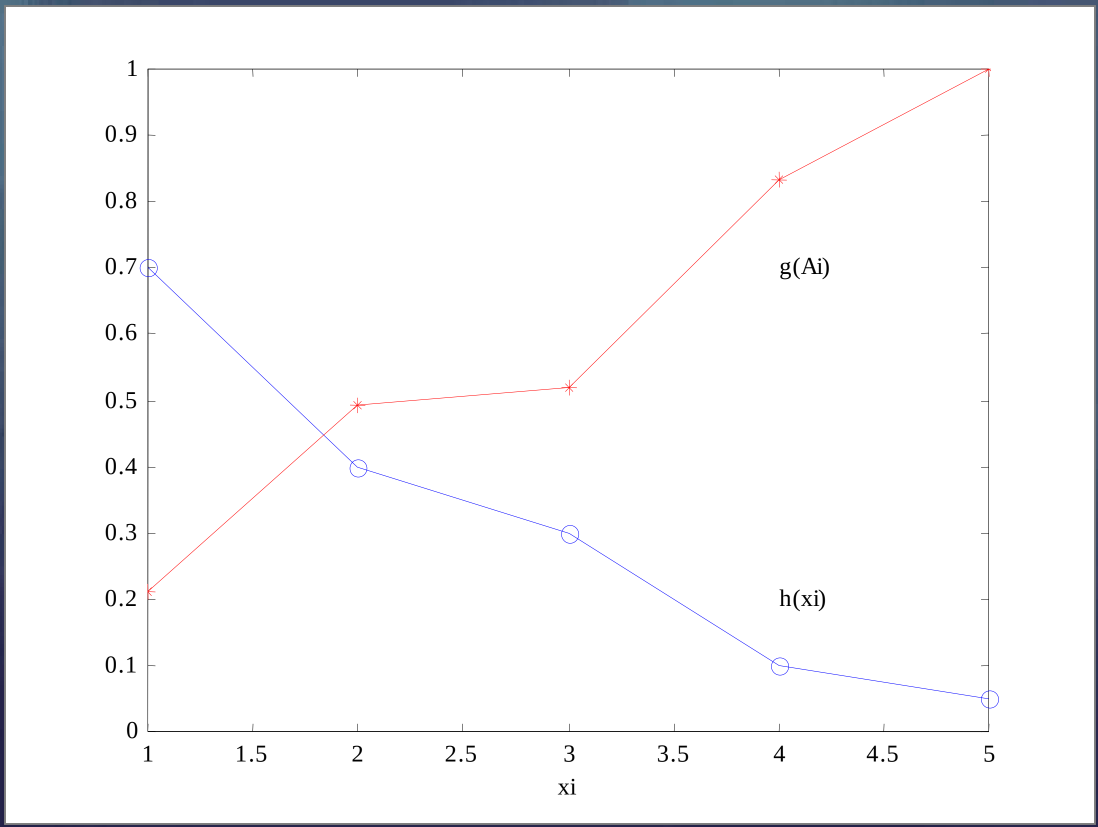
$\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$

$h(x_1) \geq h(x_2) \geq \ldots \geq h(x_n)$

$A_1 = \{x_1\},\ A_2 = \{x_1, x_2\}, \ldots,\ A_n = \{x_1, x_2, \ldots, x_n\} = \mathbf{X}$

$$\int_A h(x) \circ g(\,) = \max_{i=1,\ldots,n} \{\min[h(x_i), g(A_i)]\}$$

Example



Choquet integral

$$(Ch)\int f \circ g = \sum_{i=1}^{n}[h(x_i) - h(x_{i+1})]g(A_i)$$

$$h(x_{n+1}) = 0$$

5.8 Negations

Definition

$N : [0,1] \rightarrow [0,1]$

1. Monotonicity:

   N is nonincreasing

2. Boundary conditions:

   $N(0) = 1$
   $N(1) = 0$

3. Continuity:

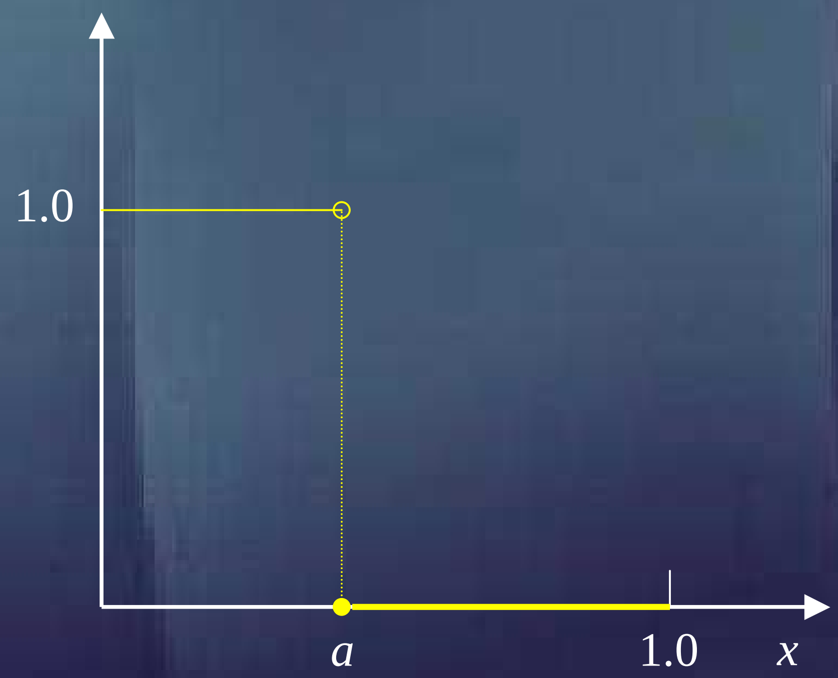
N is a continuous function

4. Involution:

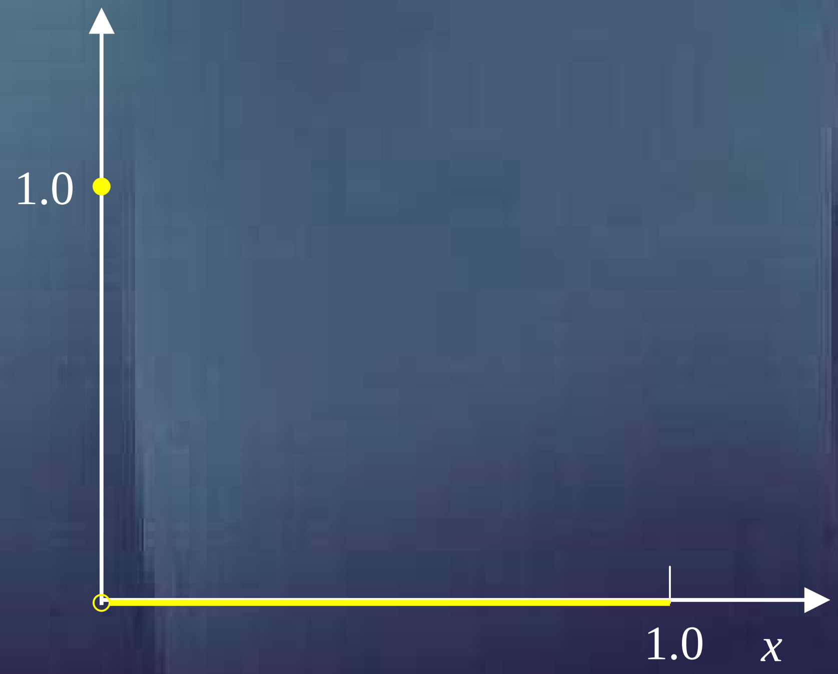
$$N(N(x)) = x \quad \forall x \in [0, 1]$$

Examples

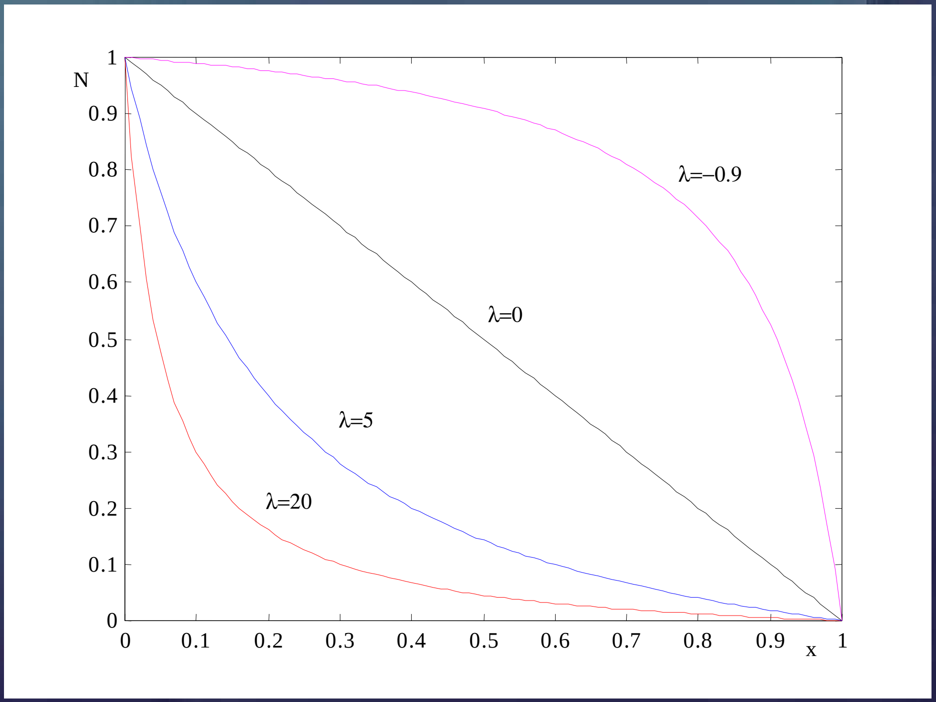
$$N(x) = \begin{cases} 1 & \text{if} \quad x < a \\ 0 & \text{if} \quad x \geq a \end{cases}$$



$$N(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x = 1 \end{cases}$$



$$N(x)=\frac{1-x}{1+\lambda x} \qquad \lambda \in (-1,\infty)$$



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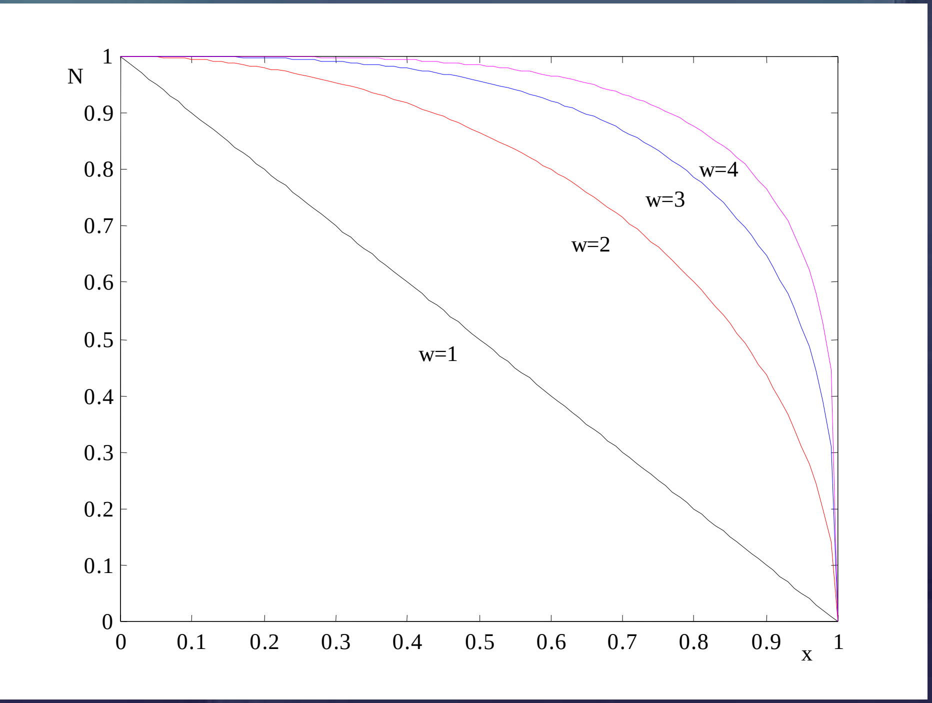
$$N(x) = \sqrt[w]{1 - x^w} \quad w \in (0, \infty)$$



(t, s, N) system

$$x\ s\ y = N\,(N(x)\ t\ N(y))$$

$$x\ t\ y = N\,(N(x)\ s\ N(y))$$

$$\forall x, y \in [0, 1]$$

Examples

$$xty = \max(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda})$$

$$xsy = \min(1, x + y - 1 + \lambda xy)$$

$$N(x) = \frac{1 - x}{1 + \lambda x} \qquad \lambda \in (-1, \infty)$$

$$xty = \min(x, y)$$

$$xsy = \max(x, y)$$

$$N(x) = 1 - x$$