4 Design of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

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4.1 Semantics of fuzzy sets: General observations

Semantics of fuzzy sets

- Generic constructs/building conceptual blocks to describe systems in a meaningful way
- Each fuzzy set comes with a well-delineated semantics (meaning)
  - Example: *small – medium – large* error
- Limited number of fuzzy sets
  - "magic" number of 7 +/- 2 (*Miller*, *1956*) (short–term memory)

- Fuzzy sets require calibration
  – determination of their membership functions
- Two main approaches to the problem:
  – Expert –driven (designer, user, decision-maker…)
  – Data driven (from data to fuzzy sets)

4.2 Fuzzy sets as a descriptor of feasible solutions

Fuzzy sets as descriptor of feasible solutions (1)

Consider some function $f(x)$ defined in $\mathbf{\Omega}$,

$$f: \mathbf{\Omega} \rightarrow \mathbf{R}. \text{ where } \mathbf{\Omega} \subset \mathbf{R}$$

Determine its maximum

$$x^{\text{opt}} = \arg\max_x f(x).$$

Fuzzy set A of *optimal* solutions $\equiv$ a collection of feasible solutions that could be labeled as *optimal* with some degrees of membership.

$$A(x) = \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

Fuzzy sets as descriptor of feasible solutions (2)

Consider some function $f(x)$ defined in $\mathbf{\Omega}$,

$$f: \mathbf{\Omega} \rightarrow \mathbf{R}. \text{ where } \mathbf{\Omega} \subset \mathbf{R}$$

Determine its minimum

$$x^{opt} = \arg\max_x f(x).$$

Fuzzy set A of *optimal* solutions $\equiv$ a collection of feasible solutions that could be labeled as *optimal* with some degrees of membership.

$$A(x) = 1 - \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

Fuzzy sets as descriptors of feasible solutions

Example

Linearization error

Linearize function $y = g(x) = \exp(-x)$ around $x_0=1$ and assess the quality of this linearization in the range [–1, 7].

Linearization formula: $y - y_0 = g'(x_0)(x - x_0)$

$y_0 = g(x_0)$ and $g'(x_0)$ is the derivative of $g(x)$ at x_0.

Linearized version of the function $\exp(-1)(2 - x)$.

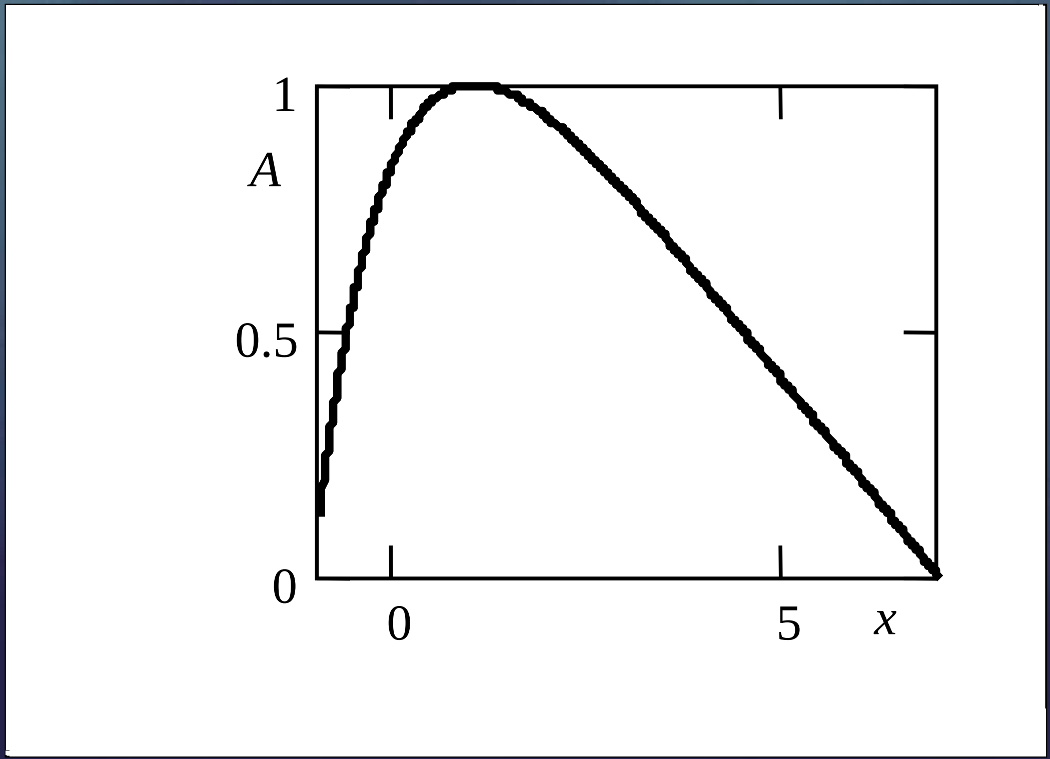
Quality of linearization $f(x) = |g(x) - \exp(-1)(2 - x)|$.

$$A(x) = 1 - \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

$f_{\max} = f(7) = 1.84$ and $f_{\min} = 0.0$

Fuzzy sets as descriptors of feasible solutions

Example

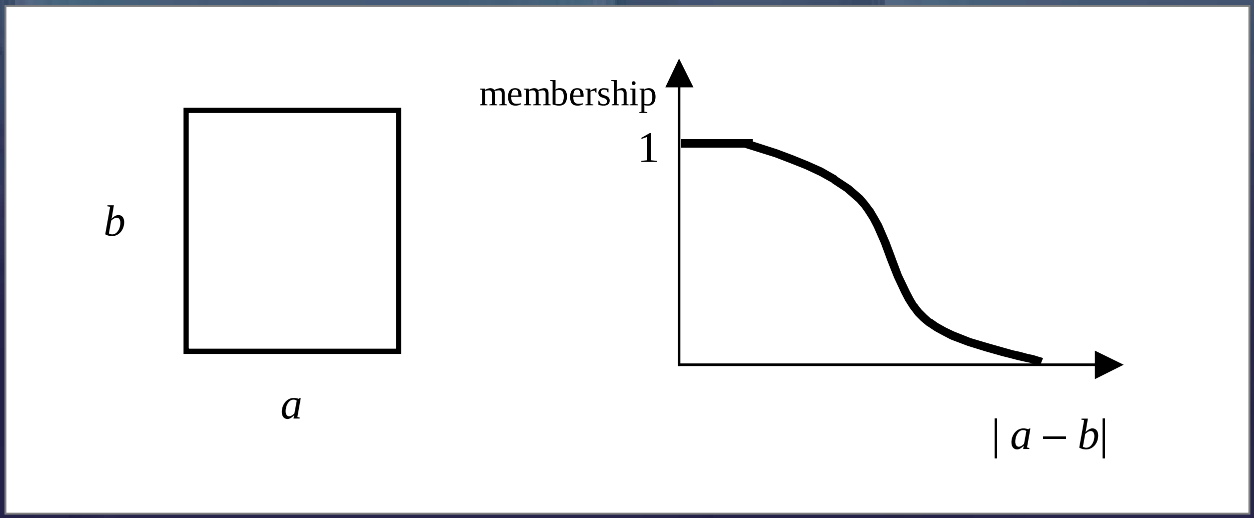


$$A(x) = 1 - \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

4.3 Fuzzy sets as a descriptor of the notion of typicality

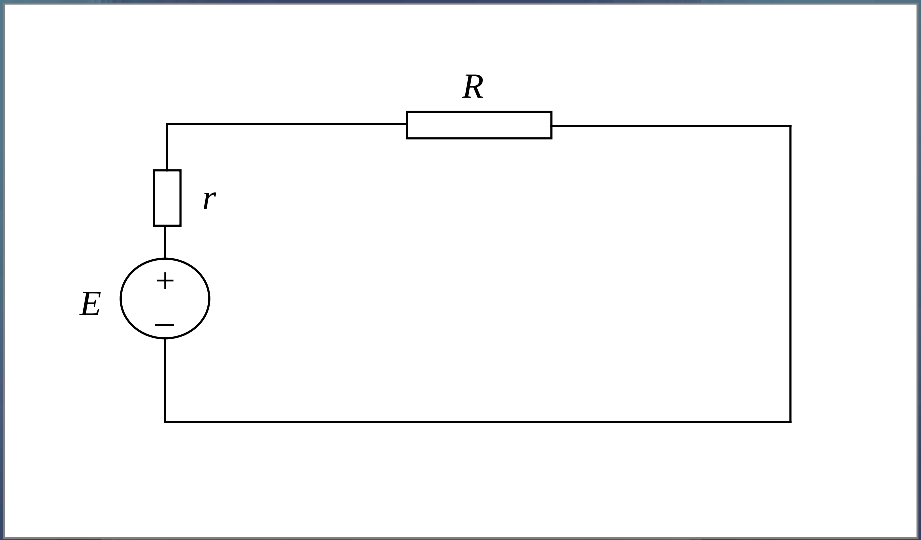
Fuzzy sets as notions of typicality

- Fuzzy set as collection of elements of varying degrees of typicality
- Geometric figures : squares, circles….

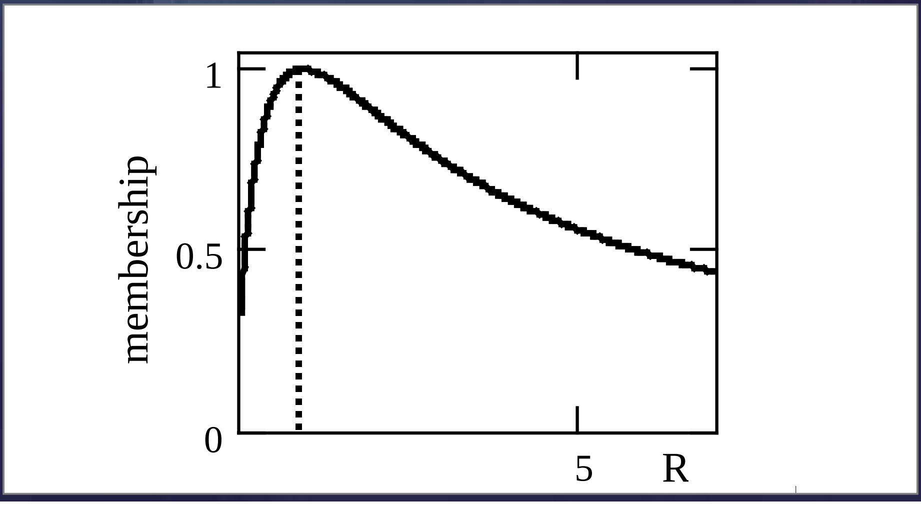


4.4 Membership functions in the visualization of preferences solutions

Fuzzy sets in visualization of preferences of solutions



$$P = i^2 R = \left(\frac{E}{R+r}\right)^2 R$$



4.5 Nonlinear transformations of fuzzy sets

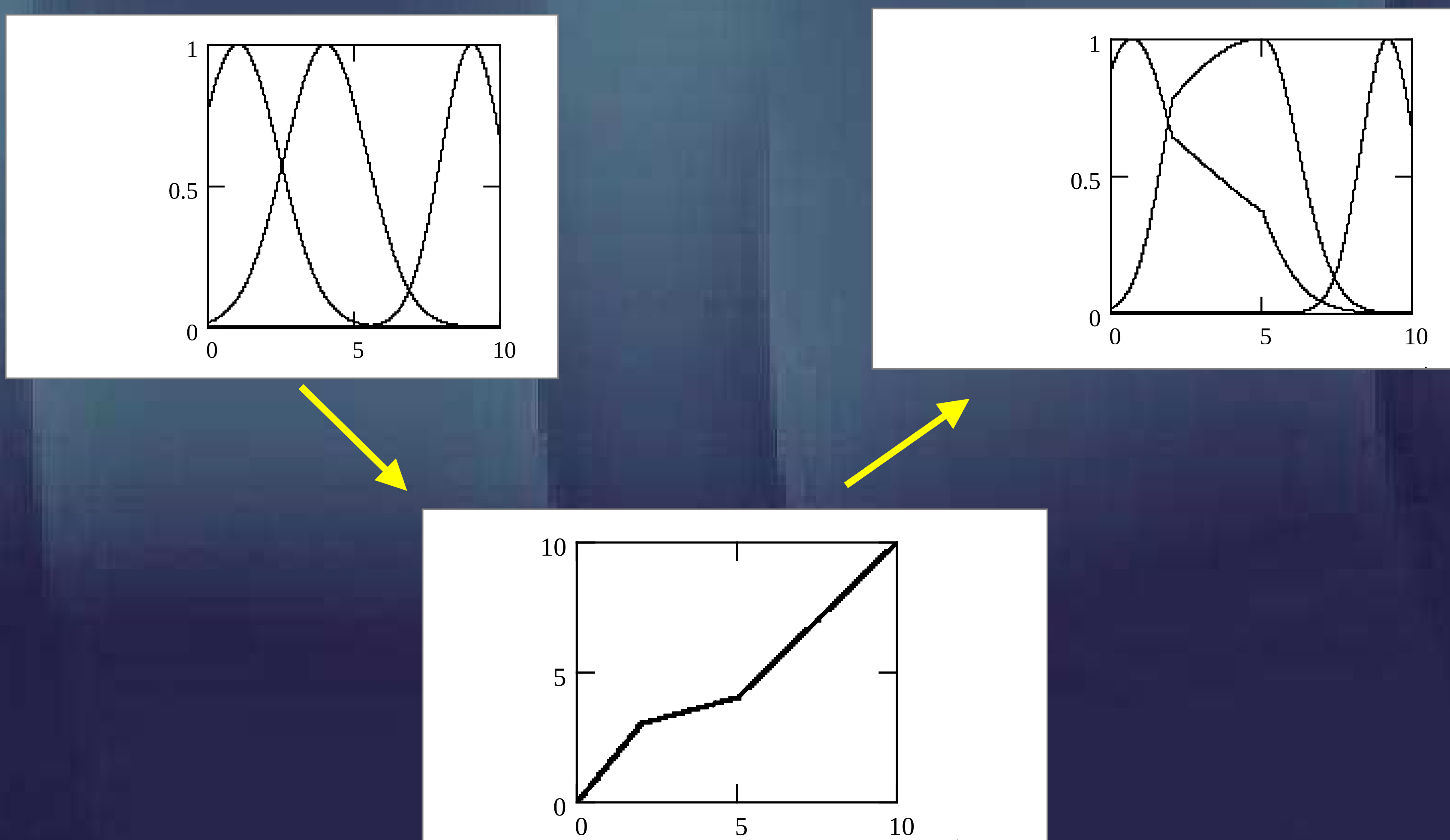
- Experimental data

x_1 — $\mu_1(1)$, $\mu_2(1)$,..., $\mu_c(1)$
x_2 — $\mu_1(2)$, $\mu_2(2)$,..., $\mu_c(2)$
…………

x_1 — $\mu_1(N)$, $\mu_2(N)$,..., $\mu_c(N)$

$$\min \rightarrow \sum_{i=1}^{c}(A_i(\Phi(x_1,\mathbf{p})-\mu_i(1))^2+\cdots+\sum_{i=1}^{c}(A_i(\Phi(x_N,\mathbf{p})-\mu_i(N))^2$$

Example



4.6 Vertical and horizontal schemes of membership estimation

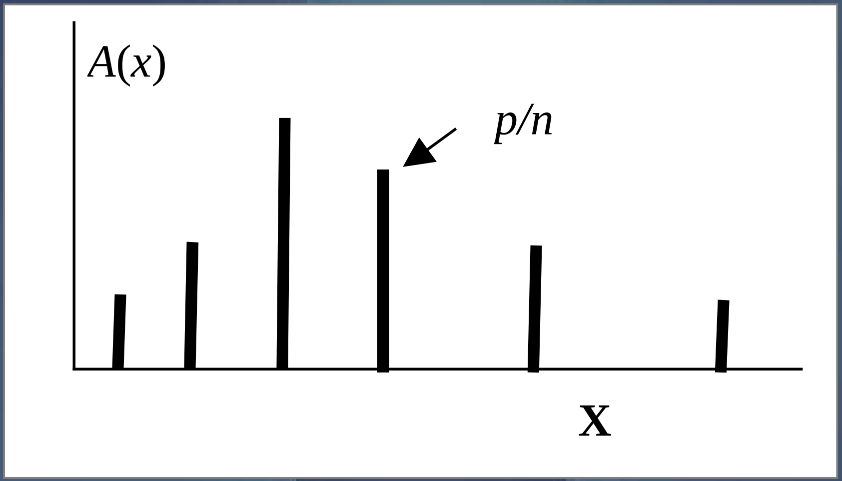
Horizontal scheme of membership estimation

- Finite elements of the universe of discourse **X**
- Question of the form

  -does x belong to concept A?

- Accepted are binary answers (yes-no)
- "n" experts – count of positive (yes) answers: p/n

Horizontal scheme of membership estimation



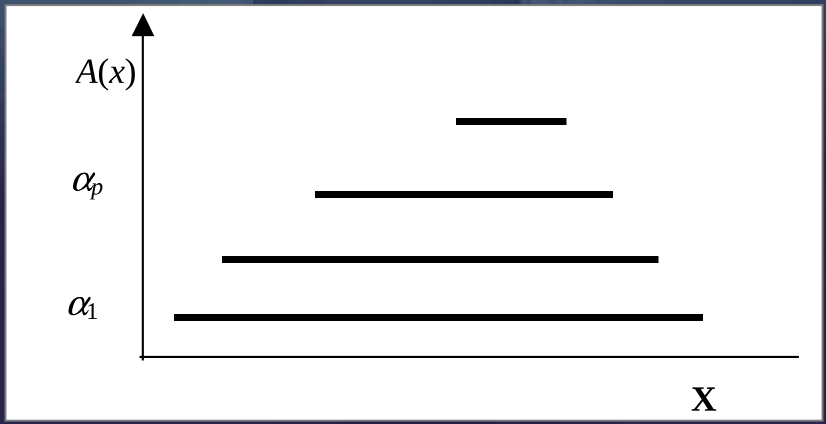
- Binary replies follow binomial distribution
- We can determine confidence interval

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

Vertical scheme of membership estimation

- Estimation of membership function by determining α-cuts and aggregating them (see representation theorem)

- What are the elements of $\mathbf{X}$ which belong to fuzzy set A at degree not lower than α?



Horizontal and vertical schemes of membership estimation

- Simple and intuitively appealing
- Reflective of domain knowledge
- Lack of continuity – elements of **X** considered independently

4.7 Saaty´s priority method of pairwise membership function estimation

Saaty's priority method of pairwise comparison

- Collection of elements $x_1, x_2, \ldots, x_n$
- Membership degrees are given $A(x_1), A(x_2) \ldots . A(x_n)$
- Reciprocal matrix R

$$R=[r_{ij}]=\begin{bmatrix} \frac{A(x_1)}{A(x_1)} & \frac{A(x_1)}{A(x_2)} & \ldots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & \frac{A(x_2)}{A(x_2)} & \ldots & \frac{A(x_2)}{A(x_1)} \\ & \ldots & \ldots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \ldots & \frac{A(x_n)}{A(x_n)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{A(x_1)}{A(x_2)} & \ldots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & 1 & \ldots & \frac{A(x_2)}{A(x_1)} \\ & & \ldots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \ldots & 1 \end{bmatrix}$$

Saaty's priority method of pairwise comparison

$$R = [r_{ij}] = \begin{bmatrix} \frac{A(x_1)}{A(x_1)} & \frac{A(x_1)}{A(x_2)} & \dots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & \frac{A(x_2)}{A(x_2)} & \dots & \frac{A(x_2)}{A(x_1)} \\ & \dots & \dots & \\ \frac{A(\mathrm{x}_\mathrm{n})}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \dots & \frac{A(x_n)}{A(x_n)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{A(x_1)}{A(x_2)} & \dots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & 1 & \dots & \frac{A(x_2)}{A(x_1)} \\ & & \dots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \dots & 1 \end{bmatrix}$$

- Reciprocal matrix R – main properties:
  (a) reflexivity
  (b) reciprocality
  (c) transitivity

Saaty's priority method of pairwise comparison: computing

$$\begin{bmatrix} \frac{A(x_i)}{A(x_1)} & \frac{A(x_i)}{A(x_2)} \cdots \frac{A(x_i)}{A(x_n)} \end{bmatrix} \begin{bmatrix} A(x_1) \\ A(x_2) \\ \cdots \\ A(x_n) \end{bmatrix}$$

i-th row of R

$$RA = nA$$

n-th largest eigenvalue of R

Saaty's priority method of pairwise comparison

- Estimation of reciprocal matrix:

- Scale (typically 1-7 range, could be larger, 1-9)
  - strong preference: high values on the scale (7-9)
  - preference: 4-7
  - weak preference or no preference 1-3

- Solving the eigenvalue problem for R, max eigenvalue, λ_{max}

Saaty's priority method : consistency of results

- $\nu = (\lambda_{\text{max}} - n)/(n - 1)$
- lack of consistency $\nu > 0.1$

Saaty’s priority method : Example

high temperature

Universe of discourse: 10, 20, 30, 40, 45

Scale 1-5

$$R = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/5 \\ 2 & 1 & 1/3 & 1/4 \\ 4 & 3 & 1 & 1/3 \\ 5 & 4 & 3 & 1 \end{bmatrix}$$

max eigenvalue = 4.114
eigenvector [0.122 0.195 0.438 0.869]
after normalization [0.14 0.22 0.50 1.00].

4.8 Fuzzy sets as granular representatives of granular data

Fuzzy sets as granular representation of numeric data

- **The principle of justifiable granularity**

- Experiment-driven and intuitively appealing rationale:

  (a) we expect that A reflects (or matches) the available experimental data to the highest extent, and

  (b) second, the fuzzy set is kept specific enough so that it comes with a well-defined semantics.

The principle of justifiable granularity

(a) we expect that A reflects (or matches) the available experimental data to the highest extent, and

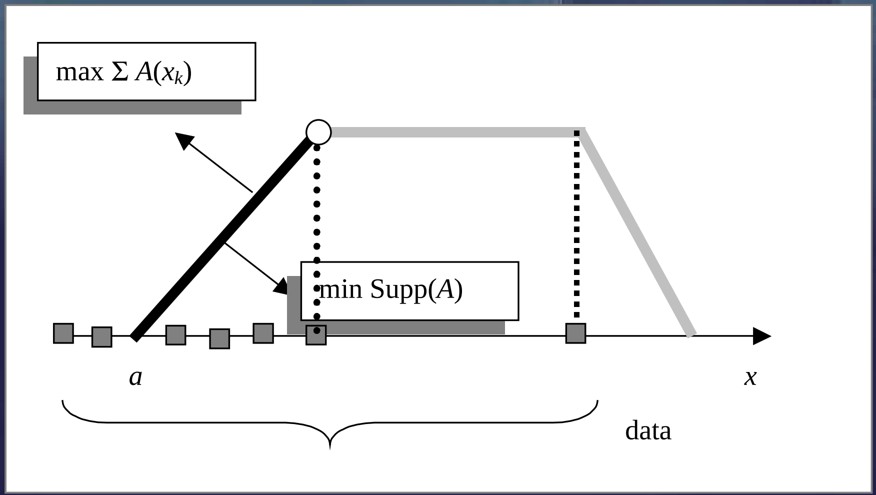
Maximize "coverage" of data

(b) second, the fuzzy set is kept specific enough so that it comes with a well-defined semantics.

Minimize spread of fuzzy set

The principle of justifiable granularity: unimodal fuzzy set

- Numeric data $x_1, x_2, \ldots, x_n$
- Determine its "modified" median
- Consider separately data to the left and right from the median

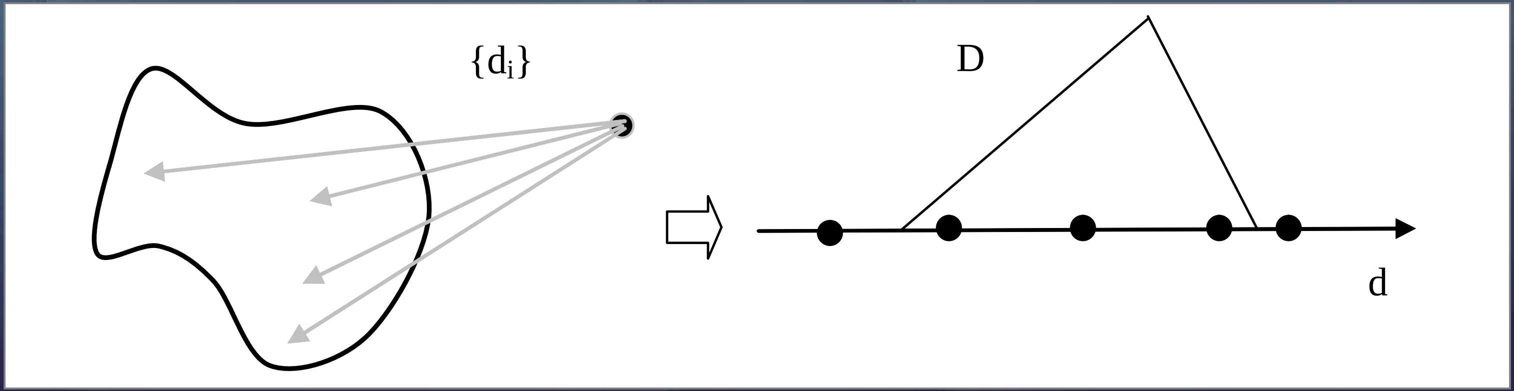


$$\max_{a \neq m} \frac{\sum_k A(x_k)}{|m-a|}$$

Pedrycz and Gomide, FSE 2007

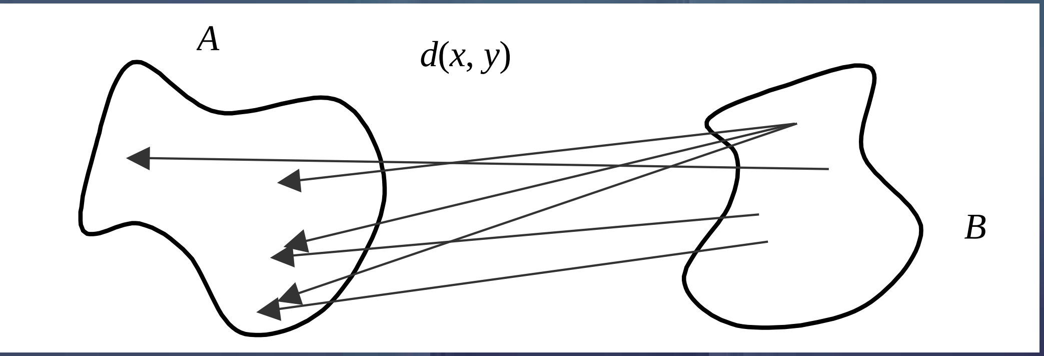
Principle of justifiable granularity: examples

Distance of point from geometric figure



Principle of justifiable granularity: examples

Distance between two geometric figures A and B



$$d_H(A, B) = \max\{\sup_{x\in A}[\min_{y\in B} d(x, y)], \sup_{y\in B}[\min_{x\in A} d(x, y)]\}$$

Clustering: Fuzzy C-Means (FCM)

- Given a n-dimensional data set $\{\mathbf{x}_k\}$, $k = 1, ..., N$
- Determine a structure with c clusters

$$\min Q = \sum_{i=1}^{c}\sum_{k=1}^{N} u_{ik}^{m}(\mathbf{x}_k - \mathbf{v}_i)^2$$

Fuzzy clustering: structure representation

Partition matrix U

Prototypes $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c$

$$\sum_{i=1}^{c} u_{ik} = 1, \quad k = 1,2,\ldots, N$$

$$0 < \sum_{k=1}^{N} u_{ik} < N, \quad i = 1,2,\ldots, c$$

FCM – optimization procedure

Optimization with respect to

- partition matrix U
- prototypes $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c$

Optimization: partition matrix

- Use of Lagrange multipliers

$$V = \sum_{i=1}^{c} u_{ik}^{m} d_{ik}^{2} + \lambda\left(\sum_{i=1}^{c} u_{ik} - 1\right)$$

$$\frac{\partial V}{\partial u_{st}} = 0 \qquad \frac{\partial V}{\partial \lambda} = 0$$

Optimization: partition matrix

$$\frac{\partial V}{\partial u_{st}} = m u_{st}^{m-1} d_{st}^2 + \lambda$$

$$\Rightarrow \quad u_{st} = -\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} d_{st}^{\frac{2}{m-1}}$$

$$\Rightarrow \quad -\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} \sum_{j=1}^{c} d_{jt}^{\frac{2}{m-1}} = 1$$

$$-\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} = \frac{1}{\sum_{j=1}^{c} d_{jt}^{\frac{2}{m-1}}}$$

$$\Rightarrow \quad u_{st} = \frac{1}{\sum_{j=1}^{c} \left(\frac{d_{st}^2}{d_{jt}^2}\right)^{\frac{1}{m-1}}}$$

Optimization: prototypes

$$Q = \sum_{i=1}^{c} \sum_{k=1}^{N} u_{ik}^{m} \sum_{j=1}^{n} (x_{kj} - v_{ij})^2$$

Gradient of Q w.r.t. prototype $\mathbf{v}_s$

$$\sum_{k=1}^{N} u_{ik}^{m}(x_{kt} - v_{st}) = 0$$

$$\mathbf{v}_{st} = \frac{\sum_{k=1}^{N} u_{ik}^{m} \mathbf{x}_{kt}}{\sum_{k=1}^{N} u_{ik}^{m}}$$

FCM: Overview of the algorithm

procedure FCM-CLUSTERING (**x**) **returns** prototypes and partition matrix
input : data $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$
local: fuzzification parameter: *m*
threshold: ε
norm: ||.||

INITIALIZE-PARTITION-MATRIX
$t \leftarrow 0$
repeat
for i =1:c **do**

$$\mathbf{v}_i(t) \leftarrow \frac{\sum_{k=1}^{N} u_{ik}^{m}(t)\mathbf{x}_k}{\sum_{k=1}^{N} u_{ik}^{m}(t)}$$

compute prototypes

for i = 1:c **do**
for k = 1:N **do**

$$u_{ik}(t+1) = \frac{1}{\sum_{j=1}^{c}\left(\frac{||\mathbf{x}_k - \mathbf{v}_i(t)||}{||\mathbf{x}_k - \mathbf{v}_j(t)||}\right)^{2/(m-1)}}$$

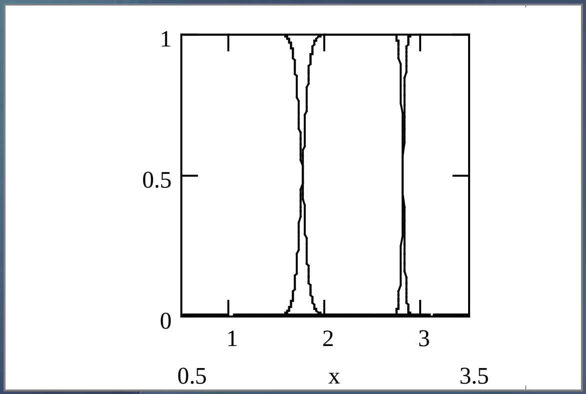
update partition matrix

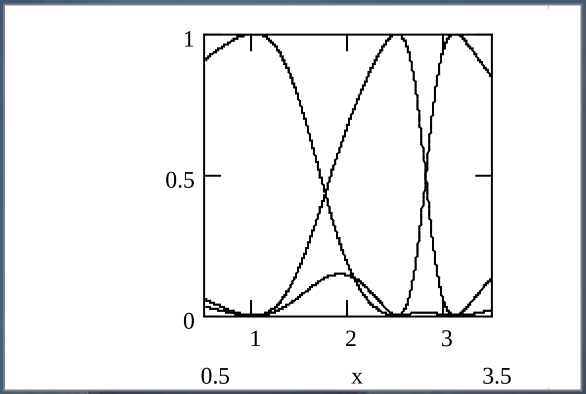
$t \leftarrow t + 1$
until $||U(t+1)\text{-}U(t)|| \le \varepsilon$
return *U*, *V*

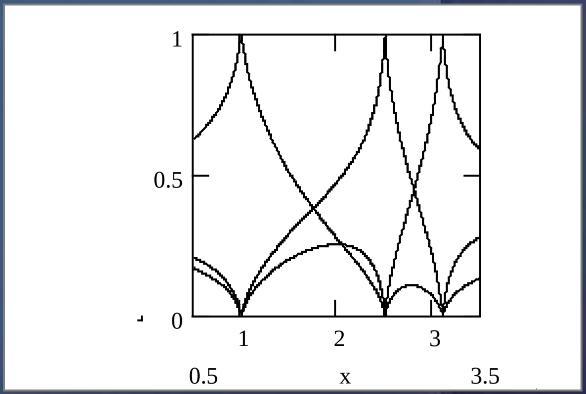
FCM and its parameters

- Number of clusters (c)
- Objective function Q
- Distance function ||.||
- Fuzzification coefficient (m)
- Termination criterion

Geometry of clusters and fuzzification coefficient







$m = 1.2$

$m = 2.0$

$m = 3.5$

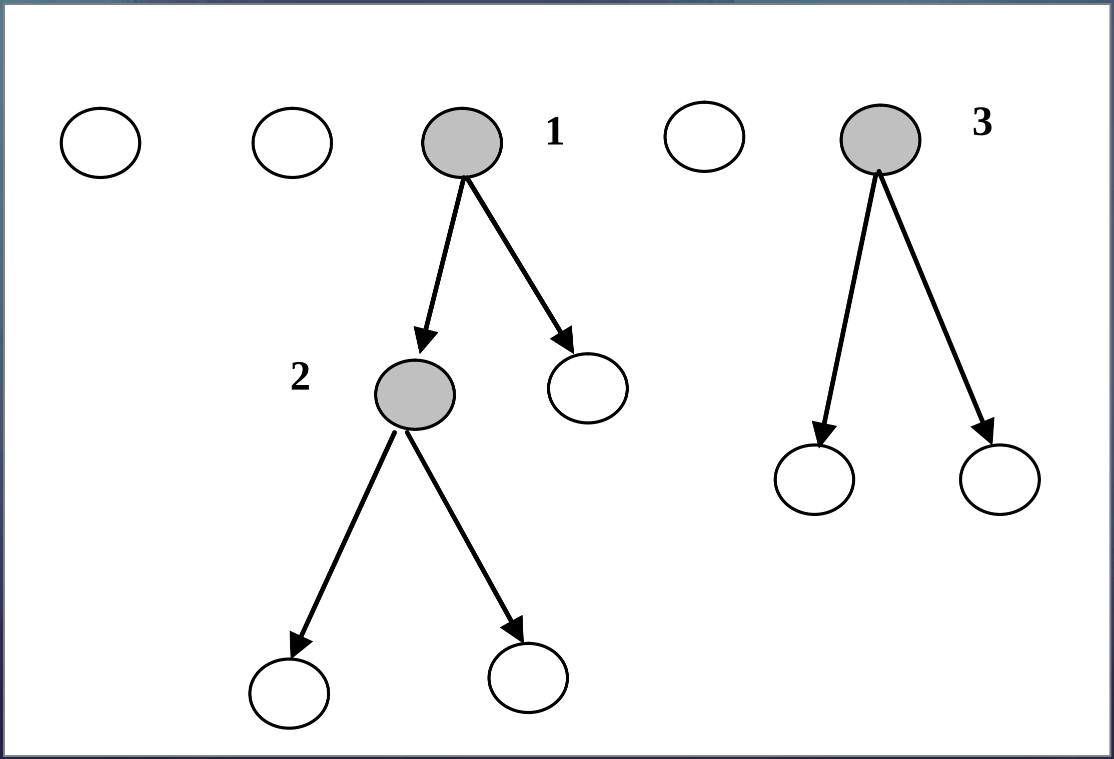
Cluster sharing: a separation measure

$$\varphi(u_1, u_2, \ldots, u_c) = 1 - c^c \prod_{i=1}^{c} u_i$$

- Data fully belongs to a single cluster (1- 0)

- Data belongs to all clusters at the same level (1/c)

Hierarchical format of FCM: Successive refinements of clusters



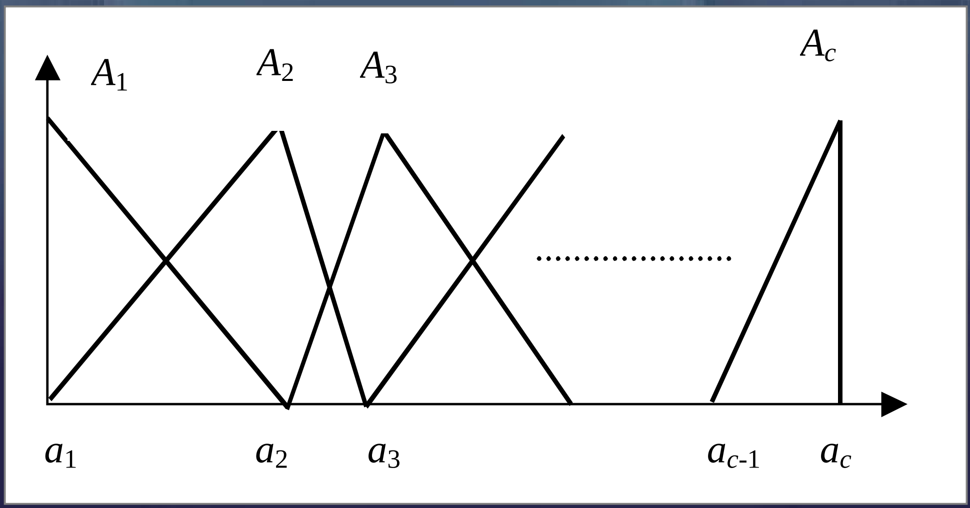
$$V_i = \sum_{k=1}^{N} u_{ik}^m ||\mathbf{x}_k - \mathbf{v}_i||^2$$

$$\mathbf{X}(i_o) = \{\mathbf{x}_k \in \mathbf{X} \mid u_{iok} = \max u_{ik}\}$$

4.10 Fuzzy equalization

Fuzzy equalization

Construct triangular fuzzy sets $A_1, A_2, \ldots, A_c$ defined in $\mathbf{R}$ such that they come with the same level of experimental evidence (support)



$$\sum_{k=1}^{N} A_1(x_k) = \frac{N}{2(c-1)}$$

$$\sum_{k=1}^{N} A_2(x_k) = \frac{N}{(c-1)}$$

$$\sum_{k=1}^{N} A_{c-1}(x_k) = \frac{N}{(c-1)}$$

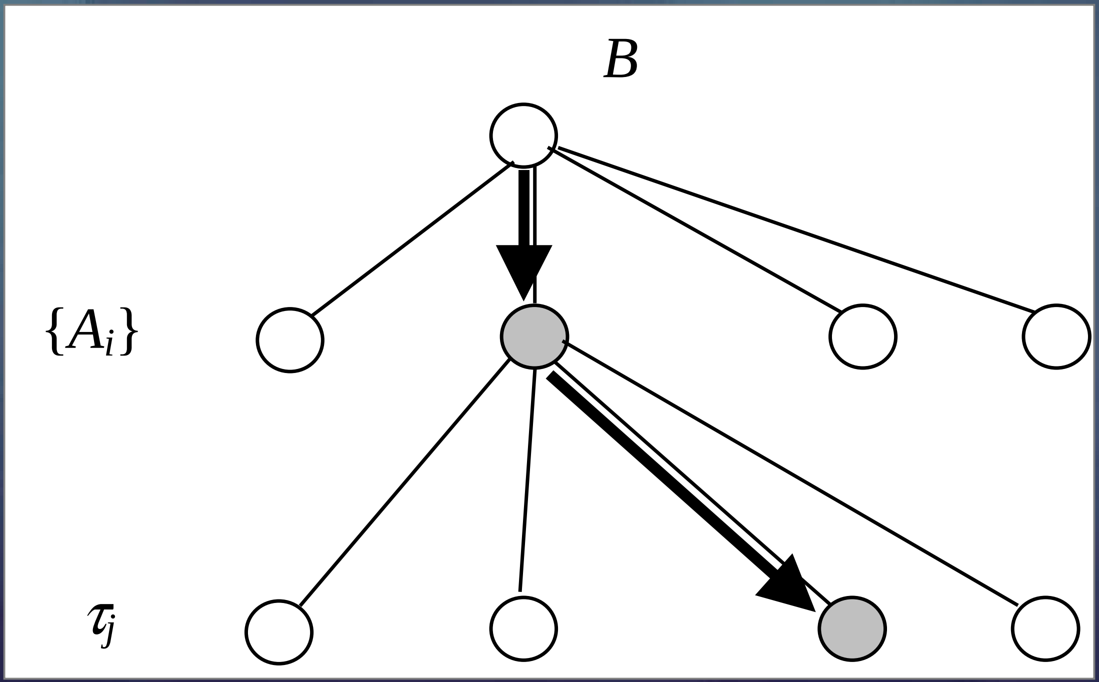
$$\sum_{k=1}^{N} A_c(x_k) = \frac{N}{2(c-1)}$$

4.11 Linguistic approximation

Linguistic approximation

- Given is a family of reference fuzzy sets $\{A_i\}$ defined in some space $\mathbf{X}$
- We have at disposal is a family of linguistic modifiers τ_j, say
  more or less (dilution),
  very (concentration)
- Represent (approximate) B in $\mathbf{X}$ with the use of reference fuzzy sets and linguistic modifiers == linguistic approximation

Linguistic approximation: optimization



$$B \approx \tau_i(A_j)$$

4.12 Design guidelines for the construction of fuzzy sets

Construction of fuzzy sets: Design guidelines (1)

- Strive for highly visible and well-defined semantics of information granules
- Keep number of information granules low (7 ± 2 fuzzy sets)
- There are several views at fuzzy sets and, depending on them, consider the use of various estimation techniques
- Fuzzy sets are context-sensitive constructs and require careful calibration
- Calibration mechanisms are reflective of human-centric fuzzy sets

Construction of fuzzy sets: Design guidelines (2)

- Major categories of approaches to design of membership functions are data-driven and expert(user)-driven
- User-driven membership estimation uses statistics of data implicitly
- Granular term-fuzzy sets exists once there is experimental evidence behind
- Development of fuzzy sets can be carried out in a stepwise manner