

7 Transformations of Fuzzy Sets

*Fuzzy Systems Engineering
Toward Human-Centric Computing*

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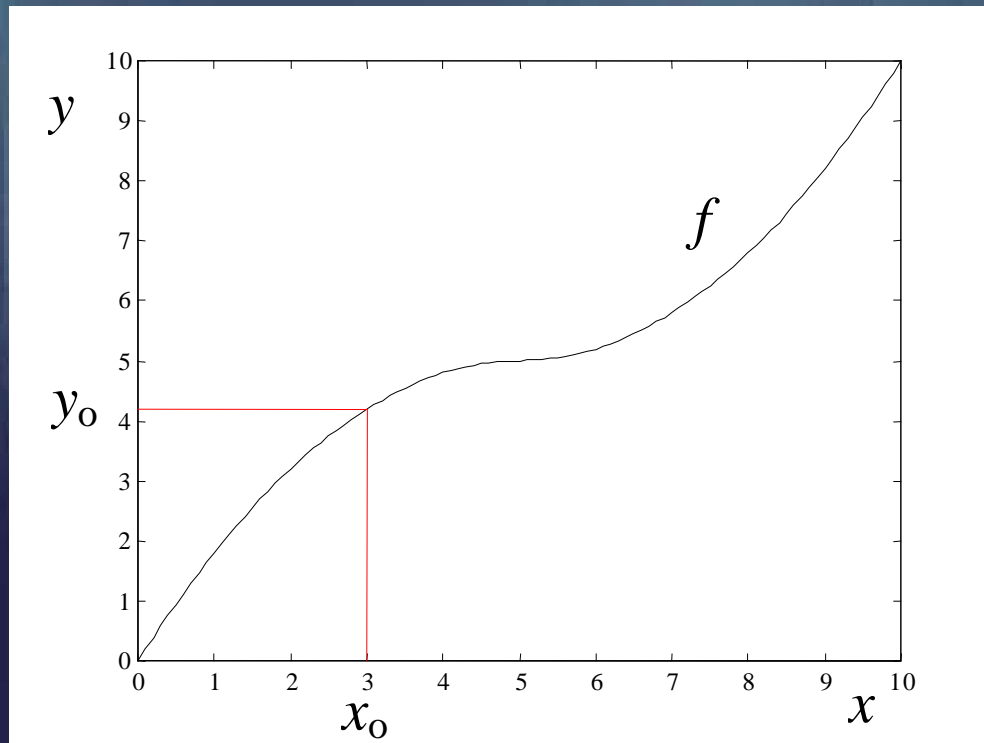
7.1 The extension principle

Extension principle

- Extends point transformations to operations involving
 - sets
 - fuzzy sets
- Given a function $f: \mathbf{X} \rightarrow \mathbf{Y}$ and a set (or fuzzy set) A on \mathbf{X} the extension principle allows to map A into a set (or fuzzy set) on \mathbf{Y} through f

Pointwise transformation

f is a function



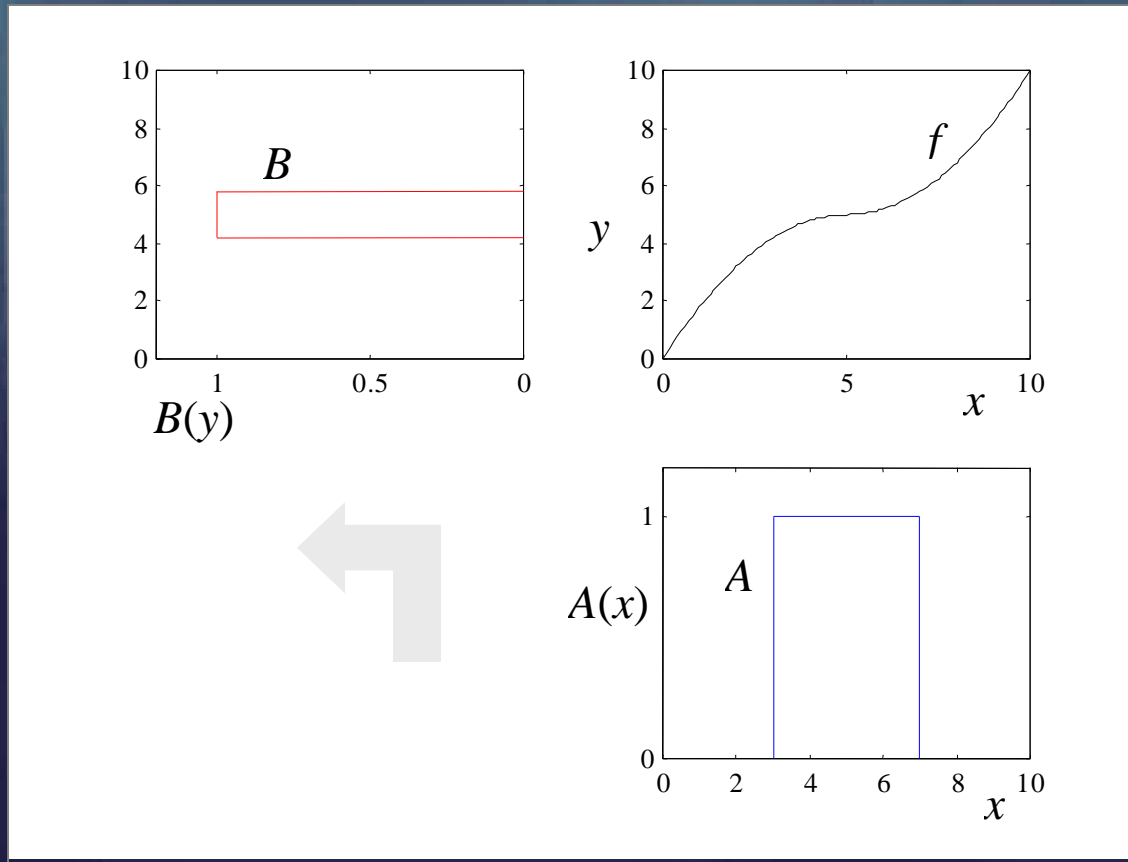
$$f: \mathbf{X} \rightarrow \mathbf{Y}$$

$$y_0 = f(x_0)$$

Set transformation

$$f: \mathbf{X} \rightarrow \mathbf{Y}, \quad A \in P(\mathbf{X})$$

$$B = f(A) = \{ y \in \mathbf{Y} \mid y = f(x), \quad \forall x \in \mathbf{X} \}$$



$$B \in P(\mathbf{Y})$$

$$B(y) = \sup_{x/y=f(x)} A(x)$$

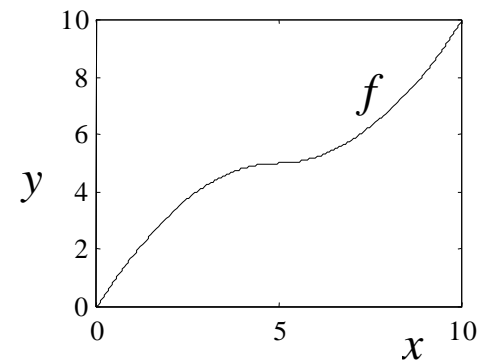
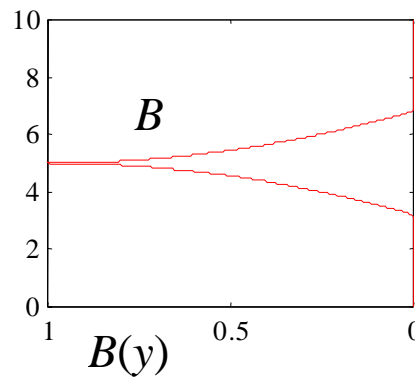
Fuzzy set transformation

$$f: \mathbf{X} \rightarrow \mathbf{Y}, \quad A \in F(\mathbf{X})$$

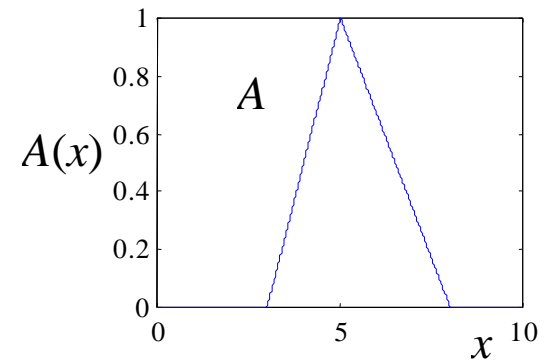
$$B = f(A), \quad B \in F(\mathbf{Y})$$

$$B(y) = \sup_{x/y=f(x)} A(x)$$

$$f(x) = \begin{cases} -0.2(x-5)^2 + 5 & \text{if } 0 \leq x \leq 5 \\ 0.2(x-5)^2 + 5 & \text{if } 5 < x \leq 10 \end{cases}$$



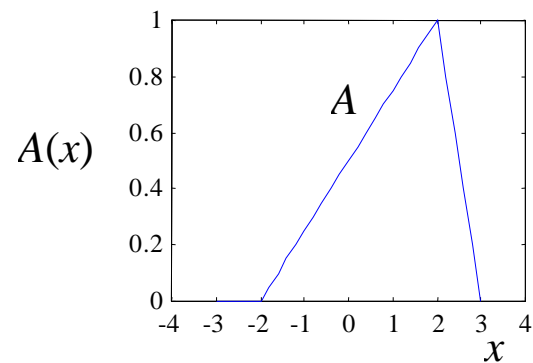
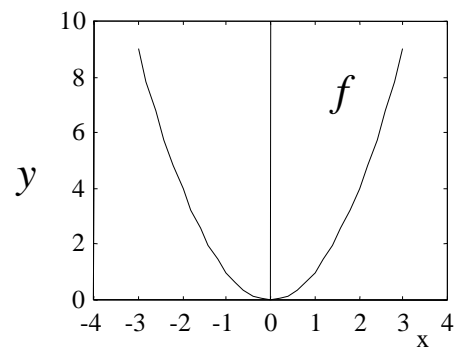
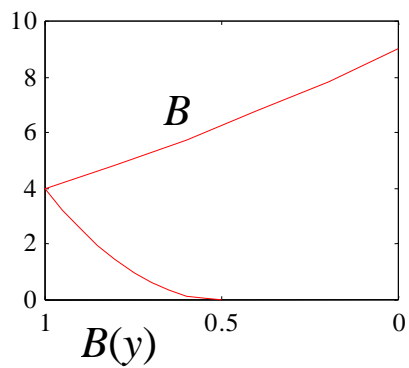
$$A = A(x, 3, 5, 8)$$



Example

$$y = f(x) = x^2$$

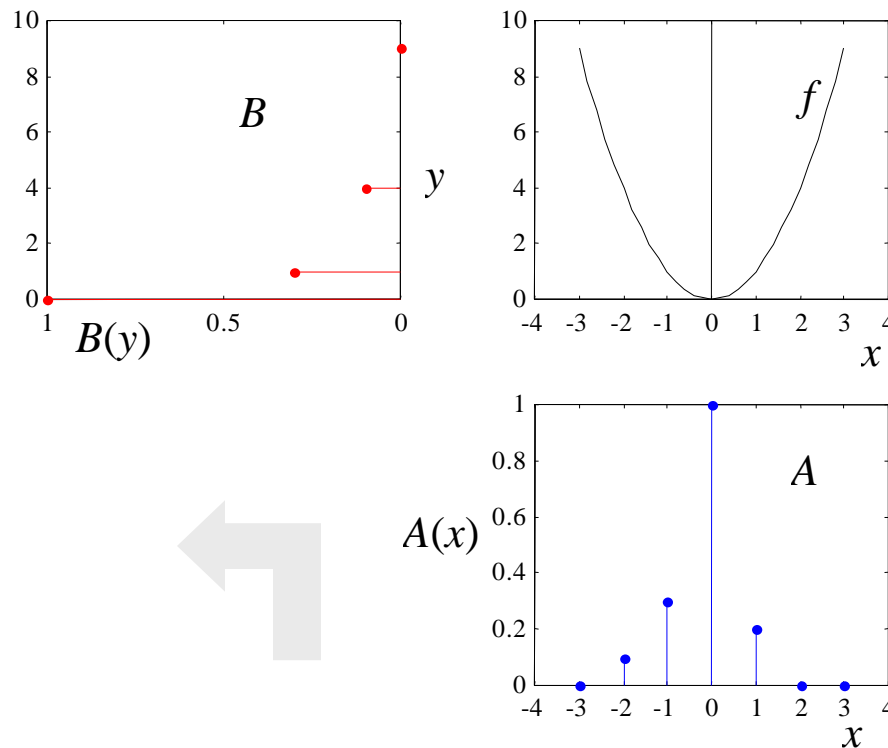
$$A = A(x, -2, 2, 3)$$



Example

$$y = f(x) = x^2$$

$$\mathbf{X} = \{-3, -2, -1, 0, 1, 2, 3\} \quad \mathbf{Y} = \{0, 1, 4, 9\}$$



$$B = \{1/0, \max(0.2, 0.3)/1, \max(0, 0.1)/4, 0/9\} = \{1/0, 0.3/1, 0.1/4, 0/9\}$$

Generalization

$$\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2 \times \dots \times \mathbf{X}_n$$

$$A_i \in F(\mathbf{X}_i), \quad i = 1, \dots, n$$

$$y = f(\mathbf{x}), \quad \mathbf{x} = [x_1, x_2, \dots, x_n]$$

$$B(y) = \sup_{\mathbf{x} | y=f(\mathbf{x})} \{ \min[A_1(x_1), A_2(x_2), \dots, A_n(x_n)] \}$$

$$B \in F(\mathbf{Y})$$

Properties

$$1. B_i = \emptyset \text{ iff } A_i = \emptyset$$

$$2. A_1 \subseteq A_2 \Rightarrow B_1 \subseteq B_2$$

$$3. f(\bigcup_{i=1}^n A_i) = \bigcup_{i=1}^n f(A_i) = \bigcup_{i=1}^n B_i$$

$$4. f(\bigcap_{i=1}^n A_i) \subseteq \bigcap_{i=1}^n f(A_i) = \bigcap_{i=1}^n B_i$$

$$5. B_\alpha \supseteq f(A_\alpha)$$

$$6. B_\alpha^+ = f(A_\alpha^+)$$

$$B_\alpha^+ = \{y \in \mathbf{Y} \mid B(y) > \alpha\}$$

strong α -cut

7.2 Compositions of fuzzy relations

Sup-t composition

Given the fuzzy relations

$$G : \mathbf{X} \times \mathbf{Z} \rightarrow [0,1]$$

$$W : \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1]$$

$$R = G \circ W \quad \text{sup-t composition}$$

$$R(x, y) = \sup_{z \in \mathbf{Z}} \{ \min [G(x, z) \text{ } t \text{ } W(z, y)] \} \quad \forall (x, y) \in \mathbf{X} \times \mathbf{Y}$$

$$R : \mathbf{X} \times \mathbf{Y} \rightarrow [0,1]$$

Example

sup-product composition

$t = \text{product}$

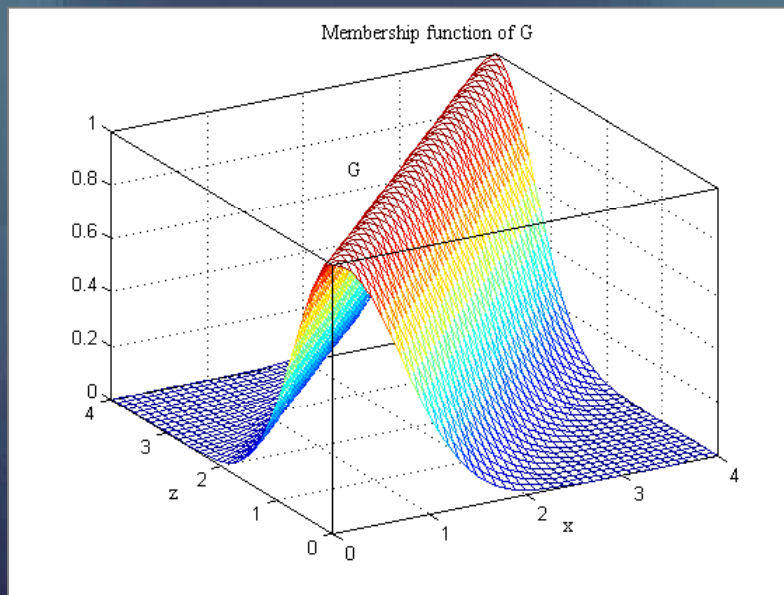
$$G(x, z) = \exp[-(x - z)^2]$$

$$W(z, y) = \exp[-(z - y)^2]$$

$$R(x, y) = \sup_{z \in \mathbf{Z}} \{ e^{-(x-z)^2} e^{-(z-y)^2} \} = \max_{z \in \mathbf{Z}} \{ e^{-(x-z)^2} e^{-(z-y)^2} \}$$

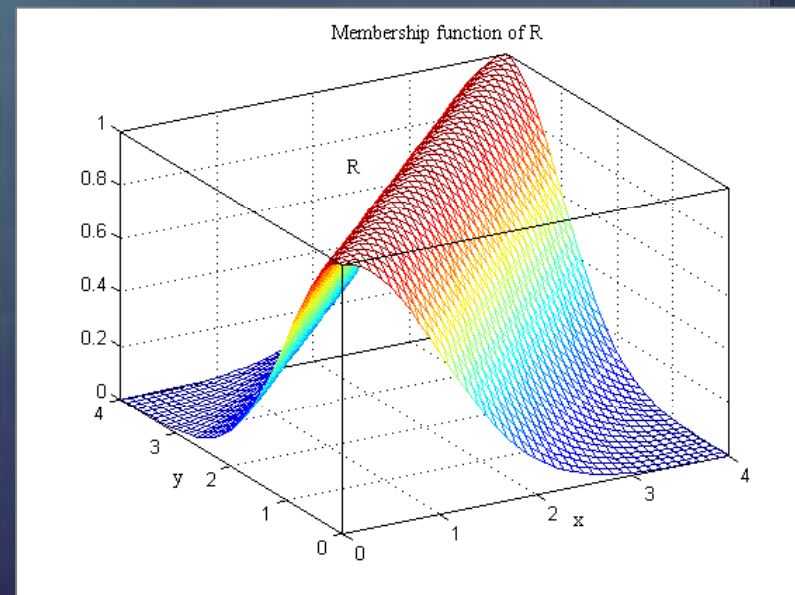
$$R(x, y) = \exp[-(x - y)^2 / 2]$$

$$G : X \times Z$$



$$G(x, z) = \exp[-(x - z)^2]$$

$$R = G \circ W$$



$$R(x, y) = \exp[-(x - y)^2 / 2]$$

Sup-t composition for matrix relations

procedure SUP-T-COMPOSITION (G, W) **returns** composition of fuzzy relations

static: fuzzy relations: $G = [g_{ik}]$, $W = [w_{kj}]$

0_{nm} : $n \times m$ matrix with all entries equal to zero

t : a t-norm

$R = 0_{nm}$

for $i = 1:n$ **do**

for $j = 1:m$ **do**

for $k = 1:p$ **do**

$\text{tope} \leftarrow g_{ik} \, t \, w_{kj}$

$r_{ij} \leftarrow \max(r_{ij}, \text{tope})$

return R

Example

$$G = \begin{bmatrix} 1.0 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.8 & 1.0 & 0.2 \\ 0.8 & 0.3 & 0.4 & 0.3 \end{bmatrix} \quad W = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

$$t = \min = \wedge$$

$$r_{11} = \max(1.0 \wedge 0.6, 0.6 \wedge 0.5, 0.5 \wedge 0.7, 0.5 \wedge 0.3) = \max(0.6, 0.5, 0.5, 0.3) = 0.6$$

.....

$$r_{32} = \max(0.8 \wedge 0.1, 0.3 \wedge 0.7, 0.4 \wedge 0.8, 0.3 \wedge 0.6) = \max(0.1, 0.3, 0.4, 0.3) = 0.4$$

$$R = G \circ W = \begin{bmatrix} 0.6 & 0.6 \\ 0.7 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}$$

Properties

$$1. P \circ (Q \circ R) = (P \circ Q) \circ R$$

associativity

$$2. P \circ (Q \cup R) = (P \circ Q) \cup (P \circ R)$$

distributivity over union

$$3. P \circ (Q \cap R) \subseteq (P \circ Q) \cap (P \circ R)$$

weak distributivity over intersection

$$4. \text{ If } Q \subseteq S \text{ then } P \circ Q \subseteq P \circ S$$

monotonicity

\cup , \cap are standard operations

Interpretations

$$1. B(y) = \sup_{x \in \mathbf{X}} [A(x) \text{ } t \text{ } R_y(x)]$$

possibility

$$2. B(y) = \text{truth}[\exists x \mid A(x) \text{ and } R_y(x)]$$

existential
quantifier

$$3. B(y) = \sup_{x \in \mathbf{X}} [\mathbf{X}(x) \text{ } t \text{ } R(x, y)] = \sup_{x \in \mathbf{X}} [1 \text{ } t \text{ } R(x, y)] = \sup_{x \in \mathbf{X}} R(x, y)$$

projection

Inf-s composition

Given the fuzzy relations

$$G : \mathbf{X} \times \mathbf{Z} \rightarrow [0,1]$$

$$W : \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1]$$

$$R = G \bullet W \quad \text{inf-s composition}$$

$$R(x, y) = \inf_{z \in \mathbf{Z}} \{ \min [G(x, z) \text{ s } W(z, y)] \} \quad \forall (x, y) \in \mathbf{X} \times \mathbf{Y}$$

$$R : \mathbf{X} \times \mathbf{Y} \rightarrow [0,1]$$

procedure INF-S-COMPOSITION(G, W) **returns** composition of fuzzy relations

static: fuzzy relations: $G = [g_{ik}]$, $W = [w_{kj}]$

1_{nm} : $n \times m$ matrix with all entries equal to unity

s : a s-norm

$R = 1_{nm}$

for $i = 1:n$ **do**

for $j = 1:m$ **do**

for $k = 1:p$ **do**

$\text{sope} \leftarrow g_{ik} \text{ } s \text{ } w_{kj}$

$r_{ij} \leftarrow \min(r_{ij}, \text{sope})$

return R

Example

$$G = \begin{bmatrix} 1.0 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.8 & 1.0 & 0.2 \\ 0.8 & 0.3 & 0.4 & 0.3 \end{bmatrix} \quad W = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

s = probabilistic sum

$$\begin{aligned} r_{11} &= \min (1.0+0.6-0.6, 0.6+0.5-0.3, 0.5+0.7-0.35, 0.5+0.3-0.15) \\ &= \min (1.0, 0.8, 0.85, 0.65) = 0.65 \end{aligned}$$

.....

$$\begin{aligned} r_{32} &= \min (0.8+0.1-0.08, 0.3+0.7-0.21, 0.4+0.8-0.32, 0.3+0.6-0.18) \\ &= \min (0.82, 0.79, 0.88, 0.72) = 0.72 \end{aligned}$$

$$R = G \bullet W = \begin{bmatrix} 0.65 & 0.80 \\ 0.44 & 0.64 \\ 0.51 & 0.72 \end{bmatrix}$$

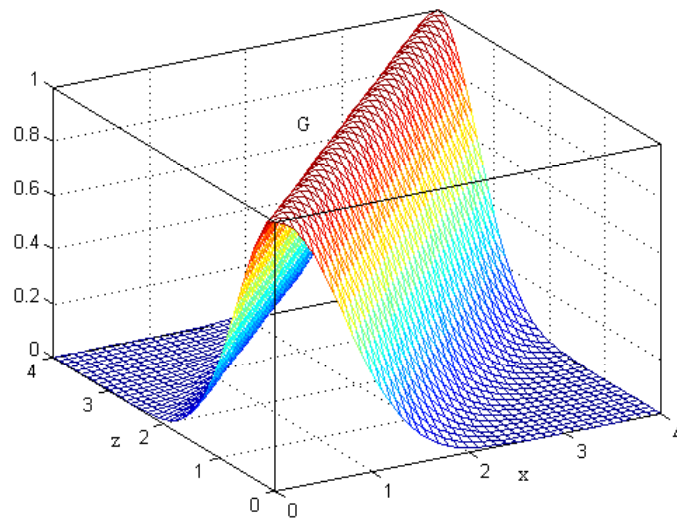
Example

$$G(x, z) = \exp[-(x - z)^2], \quad W(z, y) = \exp[-(z - y)^2]$$

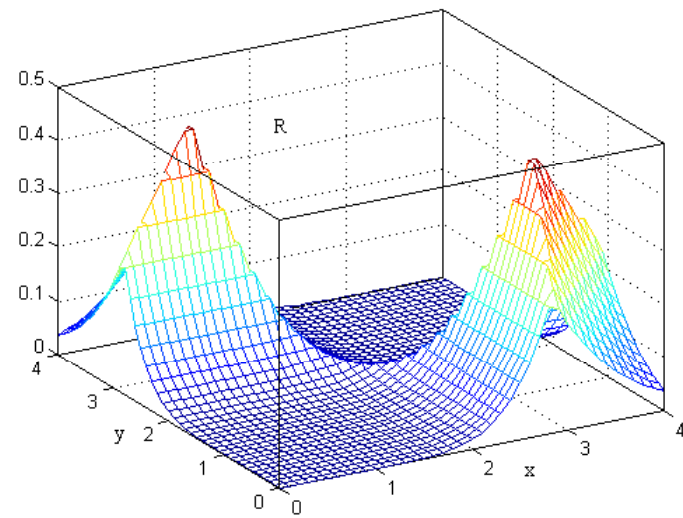
$$G : X \times Z$$

$$R = G \bullet W$$

(a) Membership function of G



(b) Membership function of R



Properties

1. $P \bullet (Q \bullet R) = (P \bullet Q) \bullet R$

associativity

2. $P \bullet (Q \cup R) \supseteq (P \bullet Q) \cup (P \bullet R)$

weak distributivity over union

3. $P \bullet (Q \cap R) = (P \bullet Q) \cap (P \bullet R)$

distributivity over intersection

4. If $Q \subseteq S$ then $P \bullet Q \supseteq P \bullet S$

monotonicity

\cup , \cap are standard operations

Interpretations

$$1. B(y) = \inf_{x \in \mathbf{X}} [A(x) s R_y(x)] = \inf_{x \in \mathbf{X}} [R_y(x) s A(x)] = \inf_{x \in \mathbf{X}} [R_y(x) s \overline{A(x)}]$$

necessity

$$2. B(y) = \text{truth}[\forall x \mid A(x) \text{ or } R_y(x)]$$

universal
quantifier

Inf- φ composition

Given the fuzzy relations

$$G : \mathbf{X} \times \mathbf{Z} \rightarrow [0,1]$$

$$W : \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1]$$

$$R = G \varphi W \quad \text{inf-}\varphi \text{ composition}$$

$$a \varphi b = \{c \in [0,1] \mid a \varphi c \leq b\}, \quad \forall a, b \in [0,1]$$

$$\varphi : [0,1] \rightarrow [0,1]$$

$$R(x, y) = \inf_{z \in \mathbf{Z}} \{G(x, z) \varphi W(z, y)\}$$

$$\forall (x, y) \in \mathbf{X} \times \mathbf{Y}$$

Example

$$G = \begin{bmatrix} 1.0 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.8 & 1.0 & 0.2 \\ 0.8 & 0.3 & 0.4 & 0.3 \end{bmatrix} \quad W = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

If t is the bounded difference: $a \, t \, b = \max(0, a + b - 1)$

then $a \, \varphi \, b = \min(1, 1 - a + b)$ Lukasiewicz implication

$$R = G \varphi W = \begin{bmatrix} 0.6 & 0.1 \\ 0.7 & 0.6 \\ 0.8 & 0.3 \end{bmatrix}$$

Properties

$$1. P\varphi(Q\varphi R) = (P \circ Q)\varphi R$$

associative

$$2. P\varphi(Q \cup R) \supseteq (P\varphi Q) \cup (P\varphi R)$$

weak distributivity over union

$$3. P\varphi(Q \cap R) = (P\varphi Q) \cap (P\varphi R)$$

distributivity over intersection

$$4. \text{ If } Q \subseteq S \text{ then } P\varphi Q \subseteq P\varphi S$$

monotonicity

\cup , \cap are standard operations

Interpretation

$$B(y) = \inf_{x \in \mathbf{X}} [A(x) \phi R_y(x)] = \inf_{x \in \mathbf{X}} [A(x) \Rightarrow R_y(x)]$$

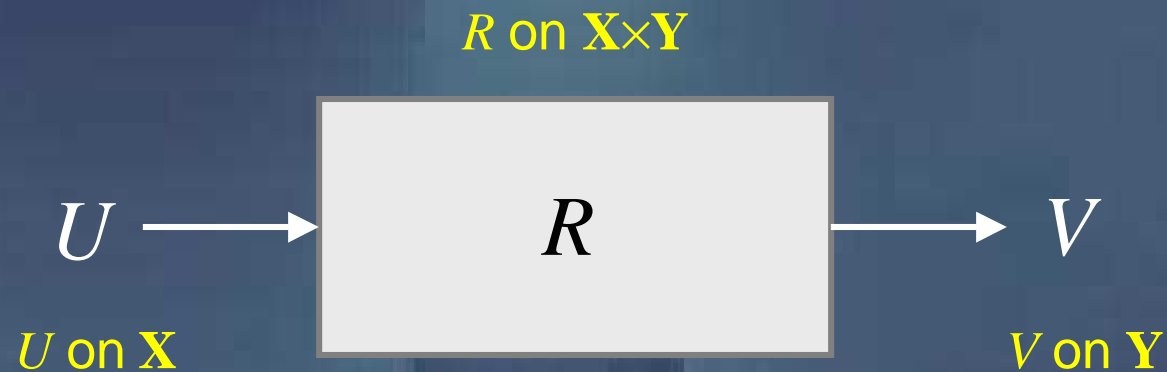
$$= \inf_{x \in \mathbf{X}} [A(x) \Rightarrow R_y(x)] = \inf_{x \in \mathbf{X}} [A(x) \subset R_y(x)]$$

\Downarrow

$$B(y) = \forall x [A(x) \Rightarrow R_y(x)]$$

7.3 Fuzzy relational equations

Single–input, single–output fuzzy system



Fundamental problems

- given U and V , determine R
- given V and R , determine U

estimation

inverse

Solution to the estimation problem

Sup-t composition

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}$$

$$\mathbf{Y} = \{y_1, y_2, \dots, y_m\}$$

$$U: \mathbf{X} \rightarrow [0,1] \qquad U = [u_1, u_2, \dots, u_i, \dots, u_n] = [u_i] \ (1 \times n)$$

$$V: \mathbf{Y} \rightarrow [0,1] \qquad V = [v_1, v_2, \dots, v_j, \dots, v_m] = [v_j] \ (1 \times m)$$

$$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1] \qquad R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}] \ (n \times m)$$

$$S_e = \{R \in F(\mathbf{X}) \times F(\mathbf{Y}) \mid V = U \circ R\}$$

solution set

$$a \varphi b = \sup \{c \in [0,1] \mid a \text{ t } c \leq b\}$$

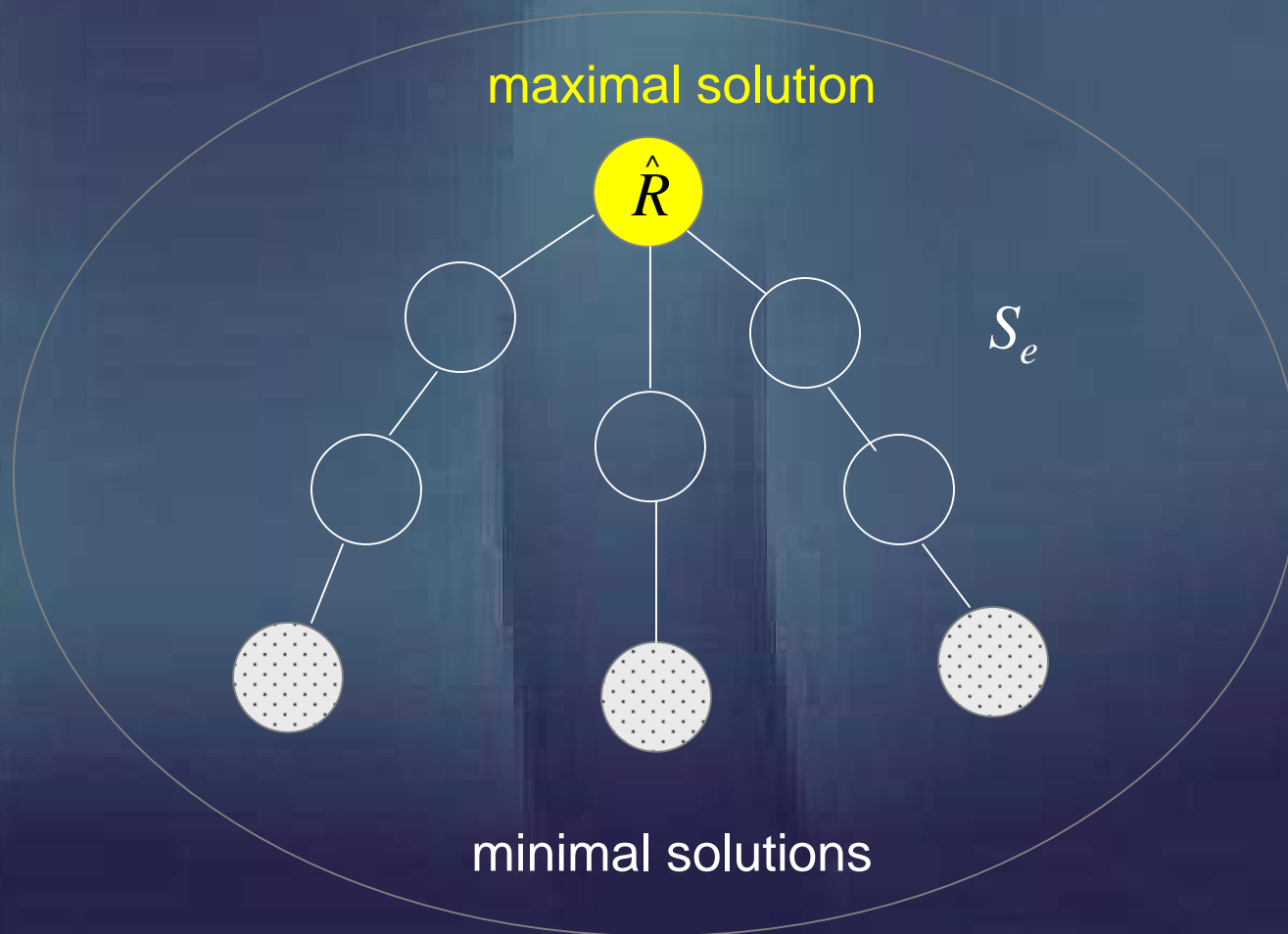
φ operator

Proposition

if $S_e \neq \emptyset$, then the unique maximal solution \hat{R} of the sup-t relational equation $V = U \circ R$ is

$$\hat{R} = U^T \varphi V$$

\hat{R} is maximal (in the sense that, if $R \in S_e$, then $R \subseteq \hat{R}$)



```
procedure ESTIMATION-SOLUTION ( $U, V$ ) returns fuzzy relation
static: fuzzy unary relations  $U = [u_i]$ ,  $V = [v_j]$ 
          $t$ : a t-norm
define  $\varphi$  operator

for  $i = 1:n$  do
  for  $j = 1:m$  do
     $\hat{r}_{ij} \leftarrow u_i \varphi v_j$ 

return  $\hat{R}$ 
```

Example

$$U = [0.8, 0.5, 0.3] \quad V = [0.4, 0.2, 0.0, 0.7]$$

$$t = \min \Rightarrow a \varphi b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

$$\hat{R} = \begin{bmatrix} 0.8 \\ 0.5 \\ 0.3 \end{bmatrix} \varphi [0.4 \quad 0.2 \quad 0.0 \quad 0.7]$$

$$\hat{R} = \begin{bmatrix} 0.8\varphi 0.4 & 0.8\varphi 0.2 & 0.8\varphi 0.0 & 0.8\varphi 0.7 \\ 0.5\varphi 0.4 & 0.5\varphi 0.2 & 0.5\varphi 0.0 & 0.5\varphi 0.7 \\ 0.3\varphi 0.4 & 0.3\varphi 0.2 & 0.3\varphi 0.0 & 0.3\varphi 0.7 \end{bmatrix}$$



$$\hat{R} = \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \\ 0.4 & 0.2 & 0.0 & 1.0 \\ 1.0 & 0.2 & 0.0 & 1.0 \end{bmatrix}$$

maximal solution

$$R_1 = \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \\ 0.0 & 0.0 & 0.0 & 0.5 \\ 0.3 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0.0 & 0.2 & 0.0 & 0.7 \\ 0.4 & 0.0 & 0.0 & 0.2 \\ 0.6 & 0.2 & 0.0 & 1.0 \end{bmatrix}$$

$$R_1 \in S_e \text{ and } R_2 \in S_e$$

$$R_1 \subset \hat{R} \text{ and } R_2 \subset \hat{R}$$

Fuzzy relational system

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}$$

$$U_k: \mathbf{X} \rightarrow [0,1] \quad U_k = [u_{1k}, u_{2k}, \dots, u_{ik}, \dots, u_{nk}] = [u_{ik}] (1 \times n)$$

$$V_k: \mathbf{Y} \rightarrow [0,1] \quad V_k = [v_{1k}, v_{2k}, \dots, v_{jk}, \dots, v_{mk}] = [v_{jk}] (1 \times m)$$

$$k = 1, \dots, N$$

$$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1] \quad R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}] (n \times m)$$

$$V_k = U_k \circ R, \quad k = 1, \dots, N$$

$$S_e^k = \{R \in F(\mathbf{X}) \times F(\mathbf{Y}) \mid V_k = U_k \circ R\} \neq \emptyset$$

$$S_e^N = \bigcap_{k=1}^N S_e^k \neq \emptyset$$



$$\hat{R} = \bigcap_{k=1}^N \hat{R}_k$$

maximal solution

$$\hat{R}_k = U_k^T \varphi V_k$$

Relation-relation fuzzy equations

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}, \quad \mathbf{Z} = \{z_1, z_2, \dots, z_p\}$$

$$U: \mathbf{Z} \times \mathbf{X} \rightarrow [0,1] \quad U = [u_{ki}] \quad (p \times n)$$

$$V: \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1] \quad V = [v_{kj}] \quad (p \times m)$$

$$V = U \circ R$$

$$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1] \quad R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}] \quad (n \times m)$$

Let

$$U^k = [u_{k1}, u_{k2}, \dots, u_{ki}, \dots, u_{kn}] \quad (1 \times n) \quad k\text{-th row of } U$$

$$V^k = [v_{k1}, v_{k2}, \dots, v_{ki}, \dots, v_{km}] \quad (1 \times n) \quad k\text{-th row of } V$$

$$R^j = [r_{1j}, r_{2j}, \dots, r_{ij}, \dots, r_{nj}]^T \quad (n \times 1) \quad j\text{-th column of } R$$

then

$$V = U \circ R =$$

$$\begin{bmatrix} V^1 \\ V^2 \\ \vdots \\ V^p \end{bmatrix} = \begin{bmatrix} U^1 \\ U^2 \\ \vdots \\ U^p \end{bmatrix} \circ \begin{bmatrix} R^1 & R^2 & \dots & R^m \end{bmatrix} = \begin{bmatrix} U^1 \circ R^1 & U^1 \circ R^2 & \dots & U^1 \circ R^m \\ U^2 \circ R^1 & U^2 \circ R^2 & \dots & U^2 \circ R^m \\ \vdots & \vdots & \dots & \vdots \\ U^p \circ R^1 & U^p \circ R^2 & \dots & U^p \circ R^m \end{bmatrix}$$

$$V^1 = \begin{bmatrix} U^1 \circ R^1 & U^1 \circ R^2 & \dots & U^1 \circ R^m \end{bmatrix} = U^1 \circ R$$

$$V^2 = \begin{bmatrix} U^2 \circ R^1 & U^2 \circ R^2 & \dots & U^2 \circ R^m \end{bmatrix} = U^2 \circ R$$

\vdots

$$V^p = \begin{bmatrix} U^p \circ R^1 & U^p \circ R^2 & \dots & U^p \circ R^m \end{bmatrix} = U^p \circ R$$

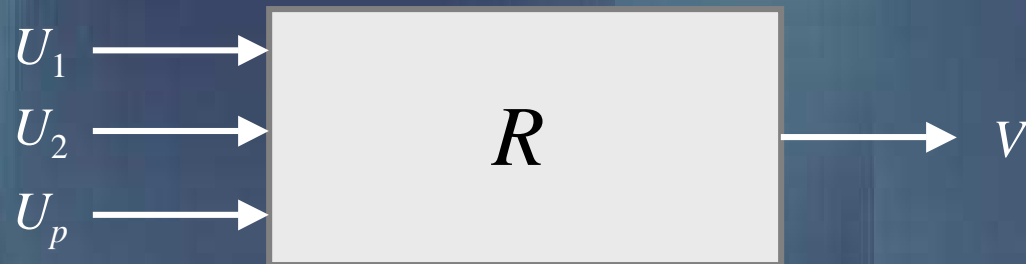
Therefore, using the previous result we get

$$\hat{R} = \bigcap_{k=1}^p \hat{R}_k$$

$$\hat{R}_k = U^{kT} \varphi V^k$$

$$U^{kT} = (U^k)^T$$

Multi-input, single-output fuzzy equations



$$U_i \in F(\mathbf{X}_i), i = 1, \dots, p$$

$$V \in F(\mathbf{Y})$$

$$R \in F(\mathbf{X}_1 \times \mathbf{X}_2 \times \dots \times \mathbf{X}_p \times \mathbf{Y})$$



$$V = U_1 \circ U_2 \circ \dots \circ U_p \circ R$$

If $U = U_1 t U_2 t \dots t U_p$ then

$$V = U \circ R$$



$$\hat{R} = U^T \phi V$$

Solution to the estimation problem

Inf-s composition

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}$$

$$U: \mathbf{X} \rightarrow [0,1] \quad U = [u_i] \ (1 \times n)$$

$$V: \mathbf{Y} \rightarrow [0,1] \quad V = [v_j] \ (1 \times m)$$

$$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1] \quad R = [r_{jj}] \ (n \times m)$$

$$V = U \bullet R$$

$$S_e^s = \{R \in F(\mathbf{X}) \times F(\mathbf{Y}) \mid V = U \bullet R\}$$

solution set

$$a \beta b = \inf \{c \in [0,1] \mid a s c \geq b\}$$

β operator

Proposition

if $S_e^s \neq \emptyset$, then the unique minimal \hat{R} solution R of the sup-t relational equation $V = U \bullet R$ is

$$\hat{R} = U^T \beta V$$

\hat{R} is minimal (in the sense that, if $R \in S_e^s$, then $\hat{R} \subseteq R$)

Solution to the inverse problem

Sup-t composition

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}$$

$$U: \mathbf{X} \rightarrow [0,1] \quad U = [u_i] \ (1 \times n)$$

$$V: \mathbf{Y} \rightarrow [0,1] \quad V = [v_j] \ (1 \times m)$$

$$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1] \quad R = [r_{jj}] \ (n \times m)$$

$$V = U \circ R$$

$$S_i = \{U \in F(\mathbf{X}) \mid V = U \circ R\}$$

solution set

$$v_j \theta s_{ji} = \min (v_j \varphi s_{ji}, j=1, \dots, m), i = 1, \dots, n$$

θ operator

Proposition

if $S_i \neq \emptyset$, then the unique maximal solution \hat{U} of the sup-t relational equation $V = U \circ R$ is

$$\hat{U} = V \theta R^T$$

\hat{U} is maximal (in the sense that, if $U \in S_i$, then $U \subseteq \hat{U}$)

procedure INVERSE-SOLUTION (R, V) **returns** fuzzy unary relation

static: fuzzy relations: $R=[r_{ij}]$, $V=[v_j]$

M : large number

t : a t-norm

define: φ operator

for $i = 1:n$ **do**

$u \leftarrow M$

for $j = 1:m$ **do**

$u \leftarrow \min(u, v_j \varphi r_{ij})$

$\hat{u}_i \leftarrow u$

return \hat{U}

Example

$$V = [0.4, 0.2, 0.0, 0.7]$$

$$\hat{R} = \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \\ 0.4 & 0.2 & 0.0 & 1.0 \\ 1.0 & 0.2 & 0.0 & 1.0 \end{bmatrix}$$

$$t = \min \Rightarrow a \varphi b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

$$\hat{U} = [0.4 \quad 0.2 \quad 0.0 \quad 0.7] \theta \begin{bmatrix} 0.4 & 0.4 & 1.0 \\ 0.2 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.7 & 1.0 & 1.0 \end{bmatrix}$$

$$= \min \left\{ [0.4 \quad 0.2 \quad 0.0 \quad 0.7] \theta \begin{bmatrix} 0.4 & 0.4 & 1.0 \\ 0.2 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.7 & 1.0 & 1.0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \min(0.4 \varphi 0.4, 0.2 \varphi 0.2, 0.0 \varphi 0.0, 0.7 \varphi 0.7) \\ \min(0.4 \varphi 0.4, 0.2 \varphi 0.2, 0.0 \varphi 0.0, 1.0 \varphi 0.7) \\ \min(1.0 \varphi 0.4, 0.2 \varphi 0.2, 0.0 \varphi 0.0, 1.0 \varphi 0.7) \end{bmatrix}^T = [1.0 \quad 0.7 \quad 0.4]$$

Relation-relation fuzzy equations

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}, \quad \mathbf{Z} = \{z_1, z_2, \dots, z_p\}$$

$$U: \mathbf{Z} \times \mathbf{X} \rightarrow [0,1] \quad U = [u_{ki}] \quad (p \times n)$$

$$V: \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1] \quad V = [v_{kj}] \quad (p \times m)$$

$$V = U \circ R$$

$$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1] \quad R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}] \quad (n \times m)$$

As before, let

$$U^k = [u_{k1}, u_{k2}, \dots, u_{ki}, \dots, u_{kn}] \quad (1 \times n) \quad k\text{-th row of } U$$

$$V^k = [v_{k1}, v_{k2}, \dots, v_{ki}, \dots, v_{km}] \quad (1 \times n) \quad k\text{-th row of } V$$

$$R^j = [r_{1j}, r_{2j}, \dots, r_{ij}, \dots, r_{nj}]^T \quad (n \times 1) \quad j\text{-th column of } R$$

thus

$$V = U \circ R =$$

$$\begin{bmatrix} V^1 \\ V^2 \\ \vdots \\ V^p \end{bmatrix} = \begin{bmatrix} U^1 \\ U^2 \\ \vdots \\ U^p \end{bmatrix} \circ \begin{bmatrix} R^1 & R^2 & \dots & R^m \end{bmatrix} = \begin{bmatrix} U^1 \circ R^1 & U^1 \circ R^2 & \dots & U^1 \circ R^m \\ U^2 \circ R^1 & U^2 \circ R^2 & \dots & U^2 \circ R^m \\ \vdots & \vdots & \dots & \vdots \\ U^p \circ R^1 & U^p \circ R^2 & \dots & U^p \circ R^m \end{bmatrix}$$

$$V^1 = \begin{bmatrix} U^1 \circ R^1 & U^1 \circ R^2 & \dots & U^1 \circ R^m \end{bmatrix} = U^1 \circ R$$

$$V^2 = \begin{bmatrix} U^2 \circ R^1 & U^2 \circ R^2 & \dots & U^2 \circ R^m \end{bmatrix} = U^2 \circ R$$

\vdots

$$V^p = \begin{bmatrix} U^p \circ R^1 & U^p \circ R^2 & \dots & U^p \circ R^m \end{bmatrix} = U^p \circ R$$

Using the previous result we get

$$\hat{U}_i = V^i \theta R^T, i = 1, \dots, p$$

Multi-input, single-output fuzzy equations

$$U_i \in F(\mathbf{X}_i), i = 1, \dots, p$$

$$V \in F(\mathbf{Y})$$

$$R \in F(\mathbf{X}_1 \times \mathbf{X}_2 \times \dots \times \mathbf{X}_p \times \mathbf{Y})$$

$$V = U_1 \circ U_2 \circ \dots \circ U_p \circ R$$

$$V(y) = \sup_{x \in \mathbf{X}} [U_1(x_1) \mathbin{\text{t}} U_2(x_2) \mathbin{\text{t}} \dots \mathbin{\text{t}} U_p(x_p) \mathbin{\text{t}} R(x_1, x_2, \dots, x_n, y)]$$

$$\hat{U}_i = V \theta R_i^T$$

$$R_i = U_1 \circ \dots \circ U_{i-1} \circ U_{i+1} \circ \dots \circ U_p \circ R$$

Solvability conditions for maximal solutions

- $hgt(U) \geq hgt(V)$ estimation problem
- $\max_i r_{ij} \geq v_j$ necessary condition for inverse problem
- concise and practically relevant solvability is difficult (in general)
- if system is not solvable, then look for approximate solution

7.4 Associative memories

Sup-t fuzzy associative memories

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}$$

$$U_k: \mathbf{X} \rightarrow [0,1] \quad U_k = [u_{1k}, u_{2k}, \dots, u_{ik}, \dots, u_{nk}] = [u_{ik}] (1 \times n)$$

$$V_k: \mathbf{Y} \rightarrow [0,1] \quad V_k = [v_{1k}, v_{2k}, \dots, v_{jk}, \dots, v_{mk}] = [v_{jk}] (1 \times m)$$

$$k = 1, \dots, N$$

U_k and V_k are patterns to be encoded into **memory R**

Sup-t fuzzy associative memories

- Encoding

$$R = \bigcap_{k=1}^N R_k, \quad R_k = U_k^T \varphi V_k$$

- Decoding

$$V_k = U_k \circ R$$

Semioverlapping fuzzy sets

- U_1, U_2, \dots, U_N form a partition
- adjacent and overlap at $\frac{1}{2}$
- $\text{hgt}(U_k \cap U_{k-1}) = 0.5$ and $\sum_k U_k(\mathbf{x}) = 1 \quad \forall \mathbf{x} \in \mathbf{X}$
- $\mathbf{x} = [x_1, x_2, \dots, x_n]$

Proposition

if fuzzy patterns U_k are semioverlapped, then the pairwise encoding of U_k and V_k , $k = 1, \dots, N$ using

$$R = \bigcap_{k=1}^N R_k, \quad R_k = U_k^T \phi V_k$$

produces perfect recall realized as

$$V_k = U_k \circ R$$

Inf-s fuzzy associative memories

$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}, \quad \mathbf{Y} = \{y_1, y_2, \dots, y_m\}$$

$$U_k: \mathbf{X} \rightarrow [0,1] \quad U_k = [u_{1k}, u_{2k}, \dots, u_{ik}, \dots, u_{nk}] = [u_{ik}] (1 \times n)$$

$$V_k: \mathbf{Y} \rightarrow [0,1] \quad V_k = [v_{1k}, v_{2k}, \dots, v_{jk}, \dots, v_{mk}] = [v_{jk}] (1 \times m)$$

$$k = 1, \dots, N$$

U_k and V_k are patterns to be encoded into memory R

Inf-s fuzzy associative memories

- Encoding

$$R = \bigcup_{k=1}^N R_k, \quad R_k = U_k^T \beta V_k$$

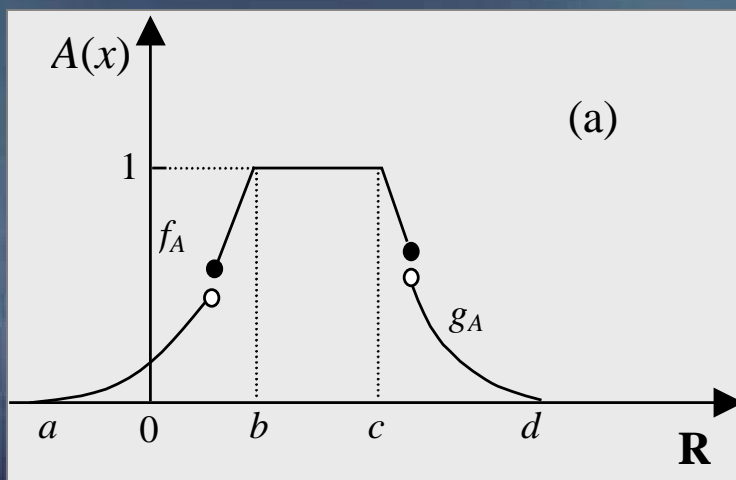
- Decoding

$$V_k = U_k \bullet R$$

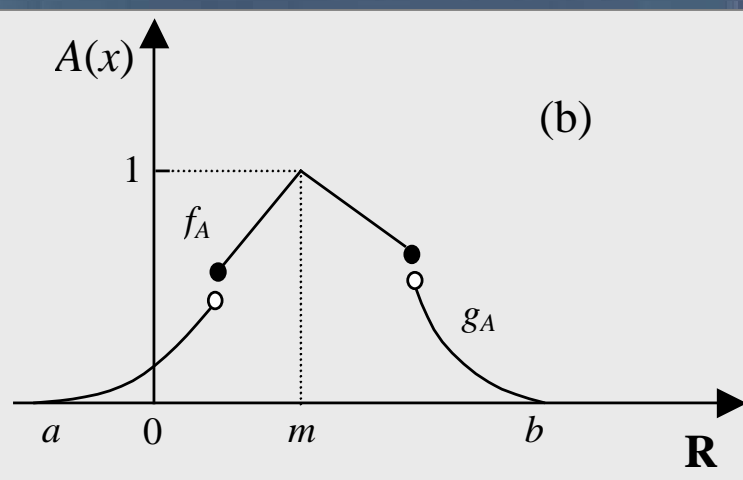
7.5 Fuzzy numbers and fuzzy arithmetic

Algebraic operations on fuzzy numbers

Fuzzy interval



Fuzzy number

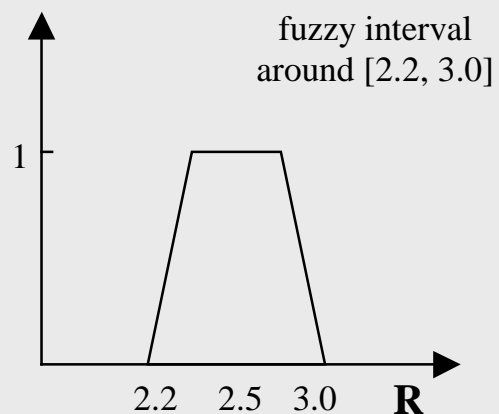
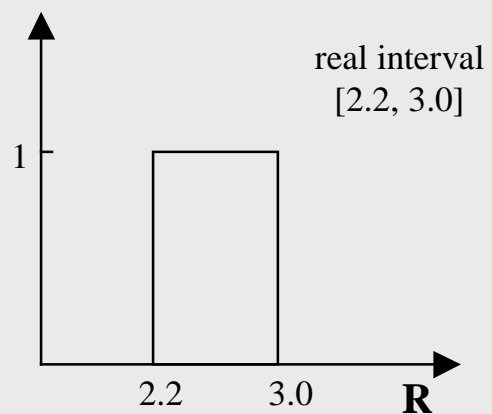
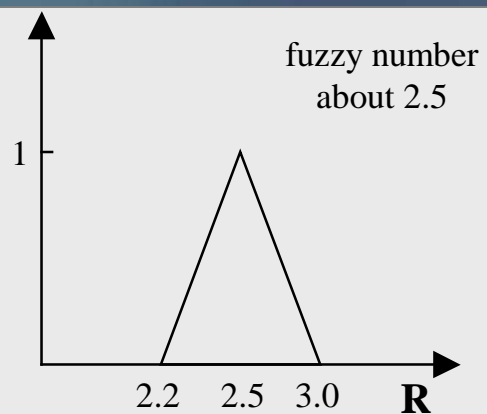
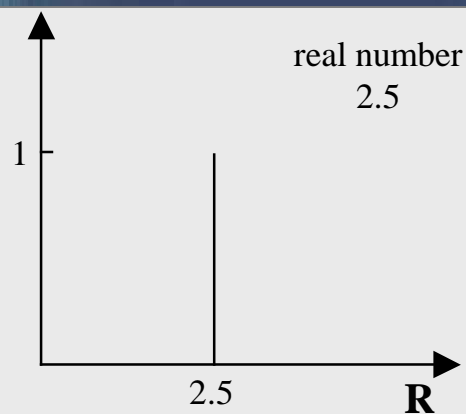


$$A(x) = \begin{cases} f_A(x) & \text{if } x \in [a, b) \\ 1 & \text{if } x \in [b, c] \\ g_A(x) & \text{if } x \in (c, d] \\ 0 & \text{otherwise} \end{cases}$$

f_A right semicontinuous

g_A left semicontinuous

Examples



Computing with fuzzy numbers

- *Consider a 2 h travel at a speed of about 110 km/h. What was the distance you traveled?*
- *In a given manufacturing process, there are five operations completed in series. Each manufacturing task has durations of about T_1, T_2, \dots, T_n time units. What is the completion time of the process?*
- **Two fundamental methods to perform algebraic operations**
 - based on interval arithmetic and α -cuts
 - extension principle

Interval arithmetic and α -cuts

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] / [c, d] = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$$

If $*$ is any of the four basic algebraic operations
and A and B are fuzzy sets on \mathbf{R} and $\alpha \in [0,1]$, then

$$(A * B)_{\alpha} = A_{\alpha} * B_{\alpha}$$

$$A * B = \bigcup_{\alpha \in [0,1]} (A * B)_{\alpha}$$

$$(A * B)(x) = \sup_{\alpha \in [0,1]} [\alpha(A * B)(x)]$$

Example

$$A(x,a,m,b), \quad B(x,c,n,d)$$

triangular fuzzy numbers

$$A_{\alpha} = [(m-a)\alpha + a, (m-b)\alpha + b], \quad B_{\alpha} = [(n-c)\alpha + c, (n-d)\alpha + d]$$

$$A = A(x,1,2,3), \quad B = B(x,2,3,5)$$

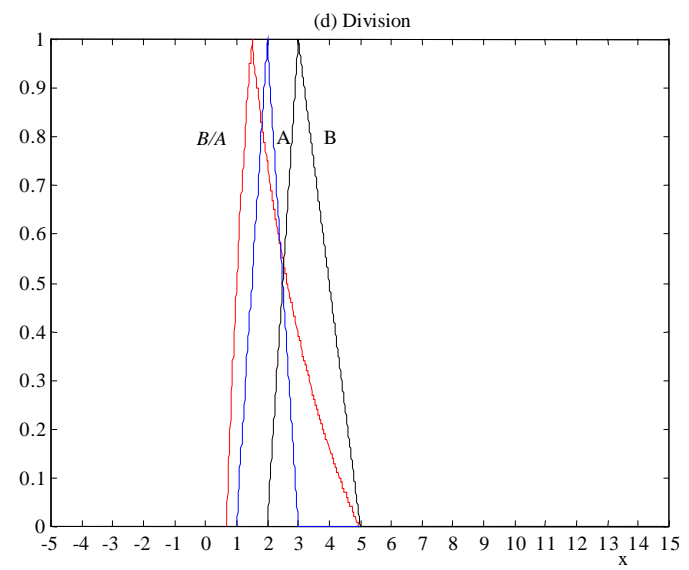
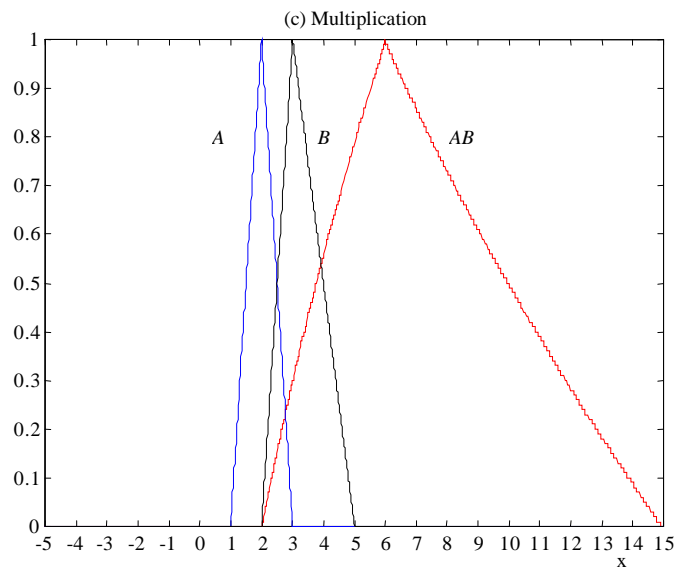
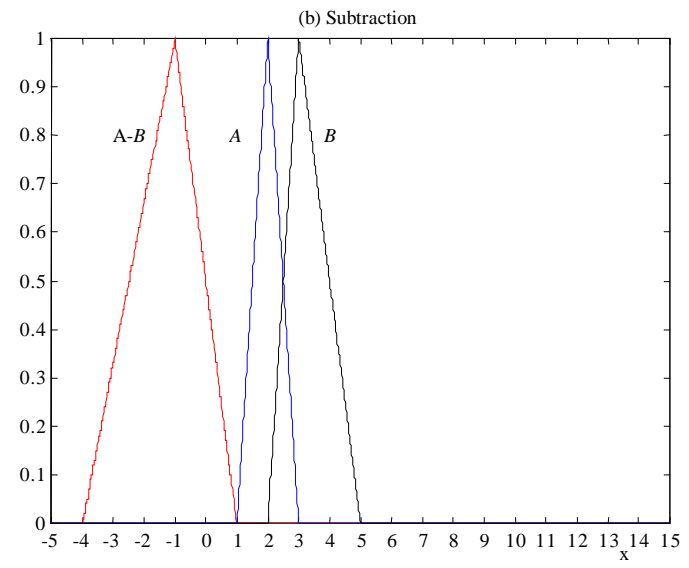
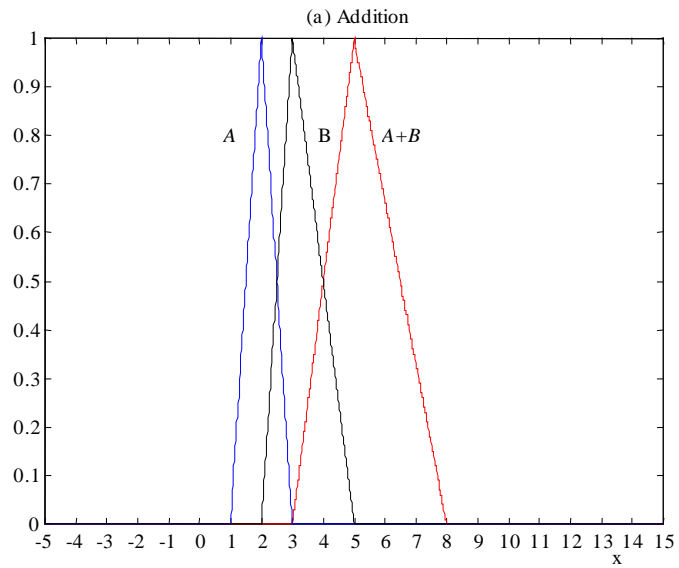
$$A_{\alpha} = [\alpha + 1, -\alpha + 3], \quad B_{\alpha} = [\alpha + 2, -2\alpha + 5]$$

$$(A+B)_{\alpha} = [2\alpha + 3, -3\alpha + 3]$$

$$(A-B)_{\alpha} = [3\alpha - 4, -2\alpha + 1]$$

$$(AB)_{\alpha} = [(\alpha + 1)(\alpha + 2), (-\alpha + 3)(-2\alpha + 5)]$$

$$(A/B)_{\alpha} = [(\alpha + 1)/(-2\alpha + 5), (-\alpha + 3)/(\alpha + 2)]$$



Fuzzy arithmetic and the extension principle

Extension principle and standard operations on real numbers

$$(A * B)(z) = \sup_{z=x*y} \min[A(x), B(y)] , \quad \forall z \in \mathbf{R}$$

$$* \in \{+, -, \cdot, /\}$$

In general, if t is a t-norm and $*: \mathbf{R}^2 \rightarrow \mathbf{R}$ then

$$(A * B)(z) = \sup_{z=x*y} [A(x) t B(y)] , \quad \forall z \in \mathbf{R}$$

$$t_1 \leq t_2 \Rightarrow a t_1 b \leq a t_2 b, \quad \forall a, b \in [0,1]$$

$$\sup_{z=x*y} [A(x) t_d B(y)] \leq \sup_{z=x*y} [A(x) t B(y)] \leq \sup_{z=x*y} [A(x) t_m B(y)], \quad \forall z \in \mathbf{R}$$

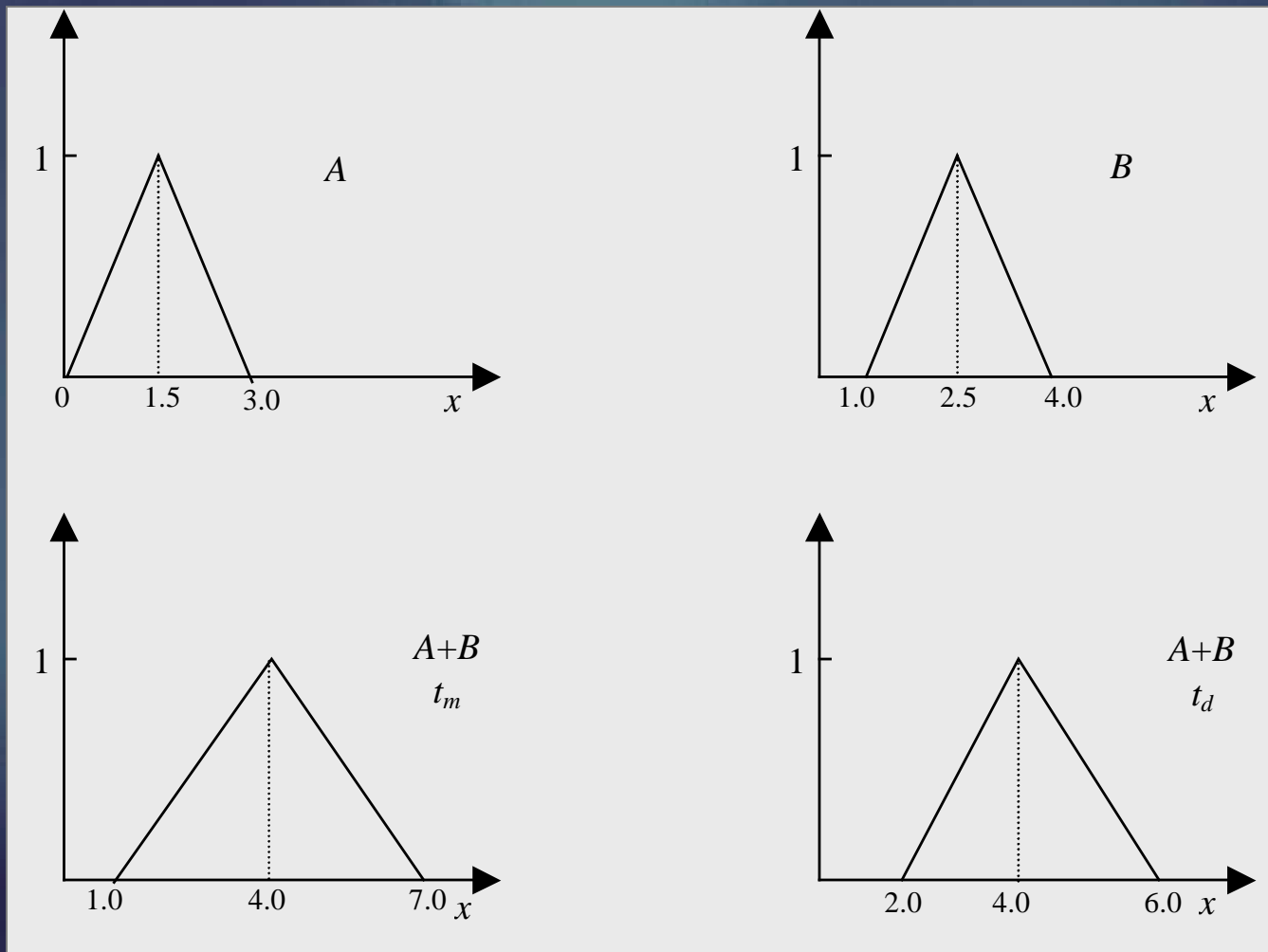
$${}^{t_d}(A * B)(z) \leq {}^t(A * B)(z) \leq {}^{t_d}(A * B)(z), \quad \forall z \in \mathbf{R}$$

Example

$A(x,a,m,b), \quad B(x,c,n,d)$ triangular fuzzy numbers

${}^{tm}(A+B) = (A+B)$ using *minimum* t-norm

${}^{td}(A+B) = (A+B)$ using *drastic product* t-norm



Different choices of t-norms, different results

Proposition

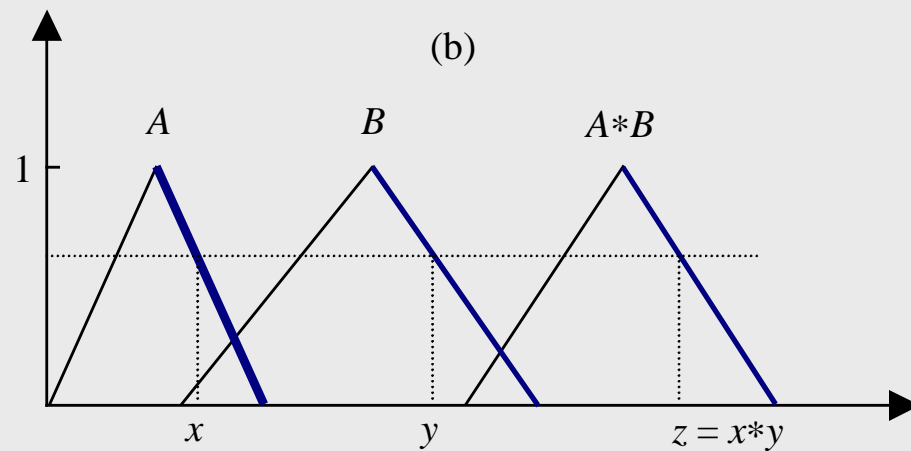
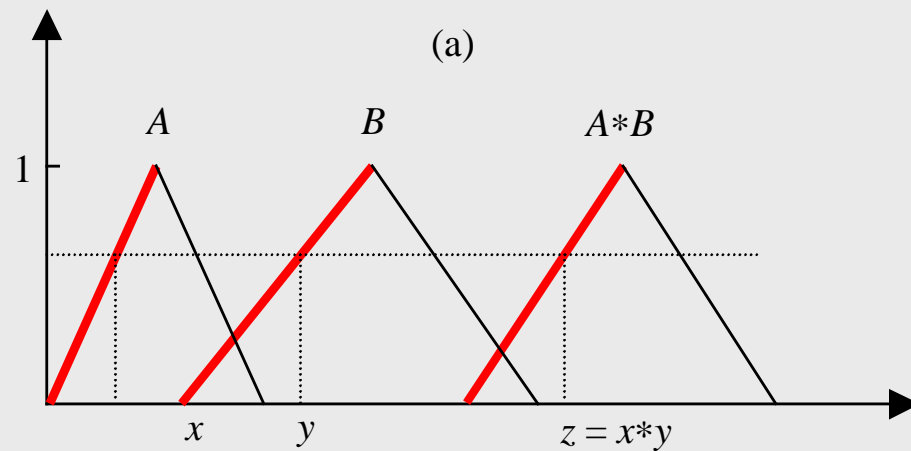
For any fuzzy numbers A and B and a continuous monotone binary operation $*$ on \mathbf{R} , the following equality holds for all α -cuts with $\alpha \in [0,1]$:

$$(A*B)_{\alpha} = A_{\alpha} * B_{\alpha}$$

(Nguyen and Walker, 1999)

Important consequences of the proposition:

1. A_α and B_α closed and bounded $\forall \alpha \Rightarrow (A*B)_\alpha$ closed and bounded
2. A and B normal $\Rightarrow (A*B)$ normal
3. Computation of $(A*B)$ can be done combining the increasing and decreasing parts of the membership functions of A and B .



Computation of $(A*B)$
combining the increasing
and decreasing parts of
the membership functions

Computing with triangular fuzzy numbers

- $A(x,a,m,b)$ and $B(x,c,n,d) \rightarrow$ triangular fuzzy numbers
- membership functions

$$A(x) = \begin{cases} \frac{x-a}{m-a} & \text{if } x \in [a, m) \\ \frac{b-x}{b-m} & \text{if } x \in [m, b] \\ 0 & \text{otherwise} \end{cases} \quad B(x) = \begin{cases} \frac{x-c}{n-c} & \text{if } x \in [c, n) \\ \frac{d-x}{d-n} & \text{if } x \in [n, d] \\ 0 & \text{otherwise} \end{cases}$$

Addition

$$C(z) = \sup_{z=x+y} \min[A(x), B(y)], \quad \forall z \in \mathbf{R}^1.$$

1. $C(z) = 1$ for $z = m + n$

2. $z < m + n$

$$x < m \text{ and } y < n$$

$$A(x) = B(y) = \alpha$$

$$\frac{x-a}{m-a} = \alpha \quad \text{and} \quad \frac{y-c}{n-c} = \alpha \quad x \in [a, m), y \in [c, n)$$

$$x = (m-a)\alpha + a \quad y = (n-c)\alpha + c \quad \text{and from } z = x + y$$

$$\alpha = \frac{z - (a + c)}{(m + n) - (a + c)}$$

$$3. \quad z > m + n$$

$$x > m \text{ and } y > n$$

$$A(x) = B(y) = \alpha$$

$$\frac{b-x}{b-m} = \alpha \quad \text{and} \quad \frac{d-y}{d-n} = \alpha \quad x \in [m, b], \quad y \in [n, d]$$

$$x = (m-b)\alpha + b \quad y = (n-d)\alpha + d \quad \text{and from } z = x + y$$

$$\alpha = \frac{(b+d)}{(b+d)-(m+n)}$$

$$4. \quad C(x) = \begin{cases} \frac{z-(a+c)}{(M+n)-(a+c)} & \text{if } z < m+n \\ 1 & \text{if } z = m+n \\ \frac{(b+d)-z}{(b+d)-(m+n)} & \text{if } z > m+n \end{cases}$$

$$C = A + B$$



$$C(x) = C(x, a+c, m+n, b+d)$$

Multiplication

- Looking at the increasing part of the membership function

$$x = (m - a)\alpha + a$$

$$y = (n - c)\alpha + c$$

$$z = xy = [(m - a)\alpha + a][(n - c)\alpha + c]$$

$$z = (m - a)(n - c)\alpha^2 + (m - a)\alpha c + a(n - c)\alpha + ac = f_1(\alpha)$$

if $ac \leq z \leq mn$ then the membership function of $D = A.B$ is

$$D(z) = f_1^{-1}(z)$$

- Looking at the decreasing part of the membership function

$$x = (m - b)\alpha + b$$

$$y = (n - d)\alpha + d$$

$$z = xy = [(m - b)\alpha + b][(n - d)\alpha + d]$$

$$z = (m - b)(n - d)\alpha^2 + (m - b)\alpha d + b(n - d)\alpha + bd = f_2(\alpha)$$

if $mn \leq z \leq bd$ then the membership function of $D = A.B$ is

$$D(z) = f_2^{-1}(z)$$