8 Generalizations and Extensions of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

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8.1 Fuzzy sets of higher order

Fuzzy sets: a retrospective view

- So far we distinguished between
  - *implicit*, and
  - *explicit*

  description of phenomena when dealing with fuzzy sets

- Typically explicit fuzzy sets we discussed so far were defined in some universe of discourse:
  - each elopement of the universe is associated with a membership degree

Fuzzy sets of order 2

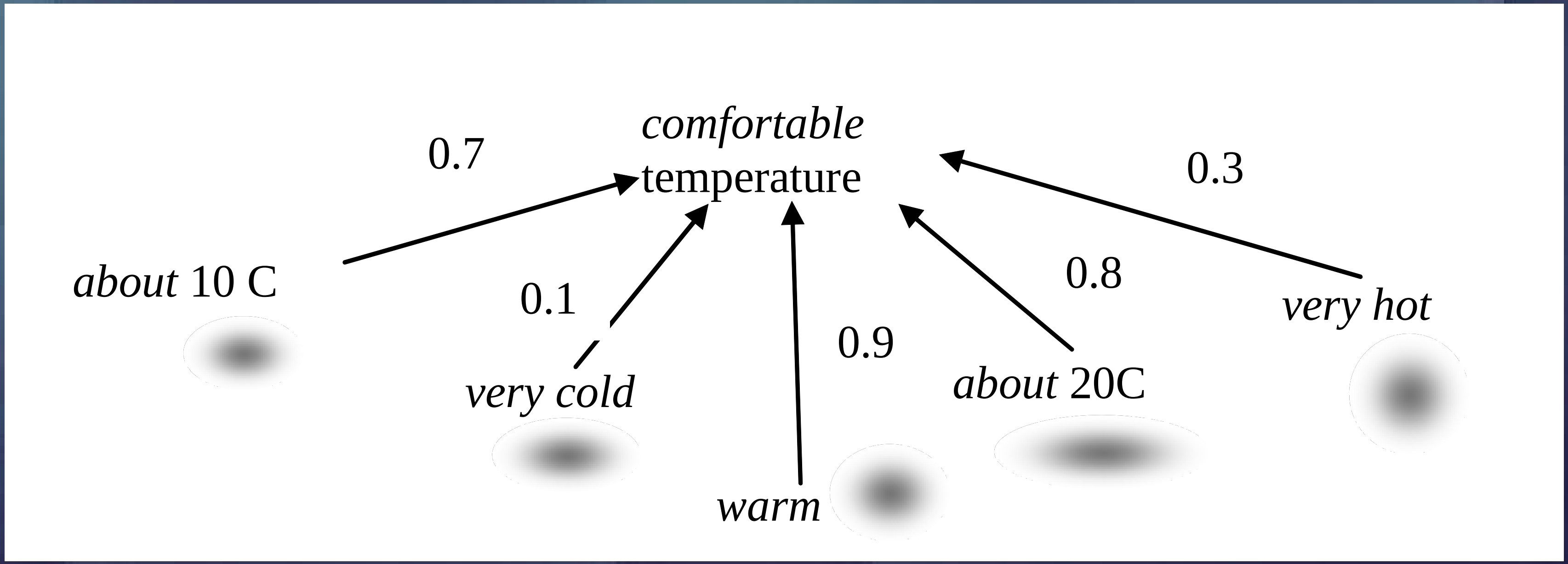
- Defining fuzzy set over a finite family of fuzzy sets
- Example

Describe comfortable temperature given a collection of generic terms (reference fuzzy sets) such as

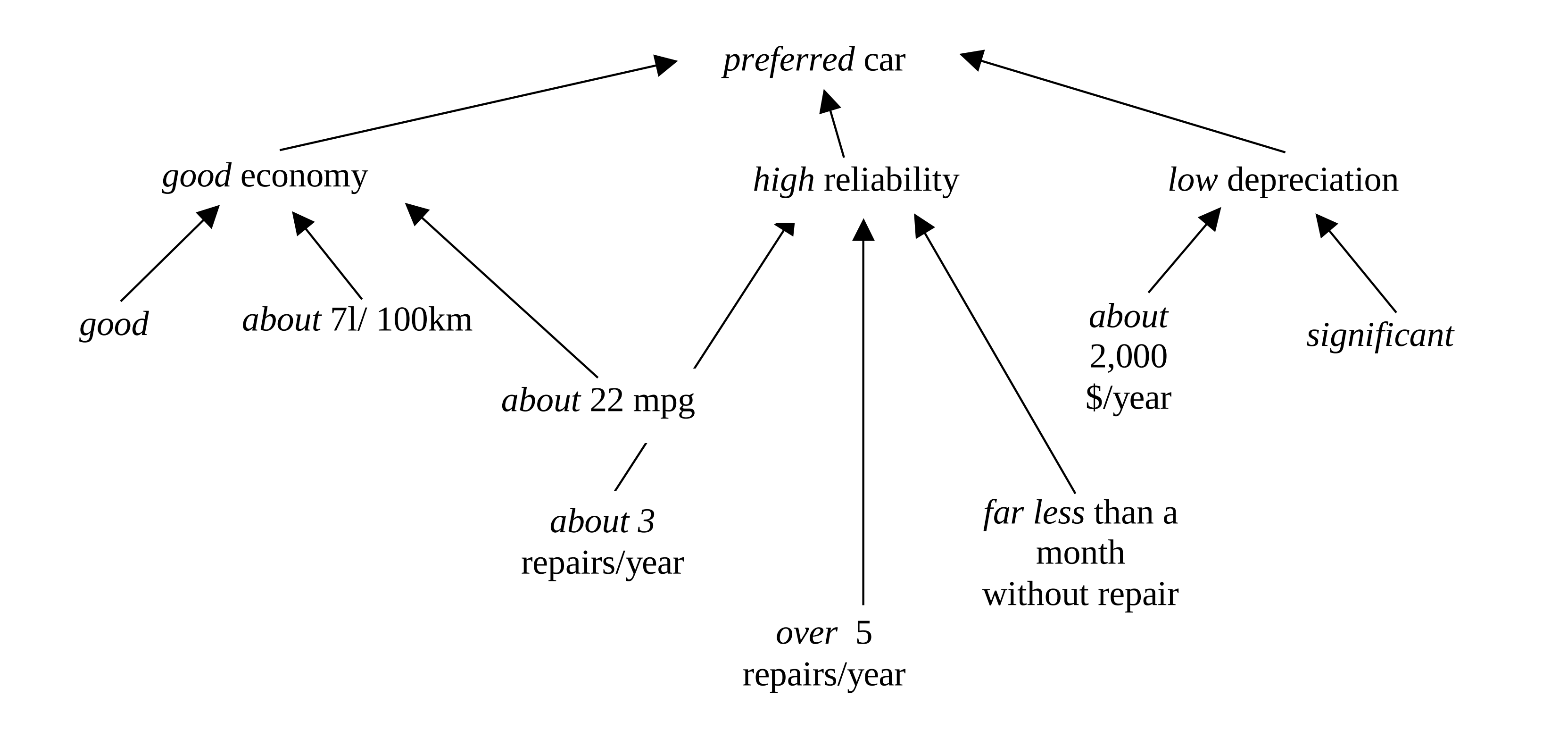
warm,
hot,
cold,
around 15C,

…

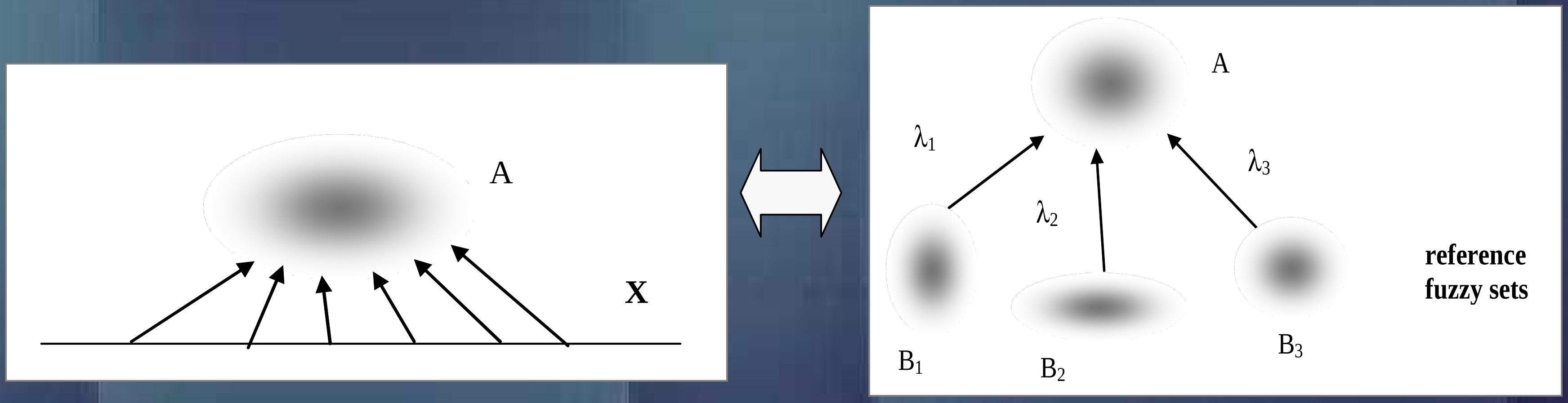
Fuzzy set of order 2



Fuzzy set of order 2



Fuzzy sets of order 2 vs. fuzzy sets: a comparative view



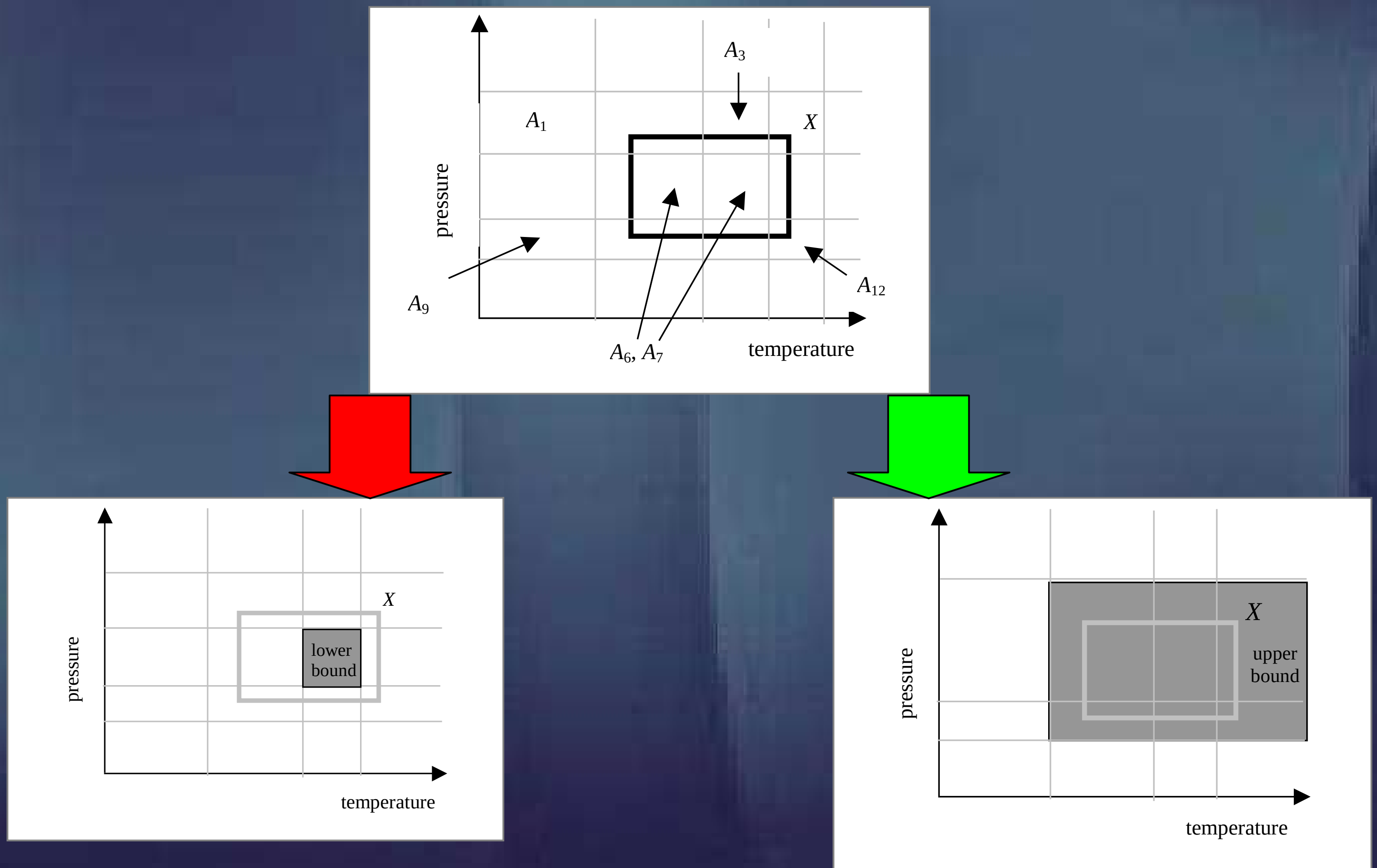
note the fundamental difference in terms of the universes of discourse for fuzzy sets and fuzzy sets of 2^{nd} order

8.2 Rough fuzzy sets and fuzzy rough sets

Fuzzy sets and rough sets

Recall that in rough sets we start with a finite collection of information granules using which we express any given granule in terms of so-called lower and upper bound

Rough sets – an example

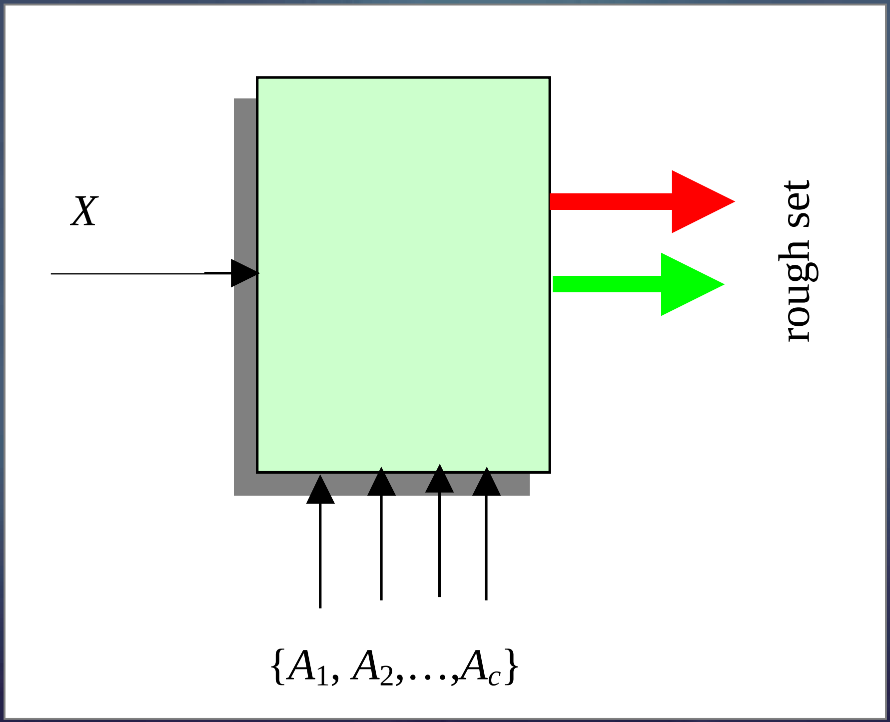


Upper bound $X_{+}=\{A_i \mid A_i \cap X \neq \varnothing\}$

Lower bound $X_{-}=\{A_i \mid A_i \subset X\}$

Pedrycz and Gomide, FSE 2007

Rough sets – schematic representation



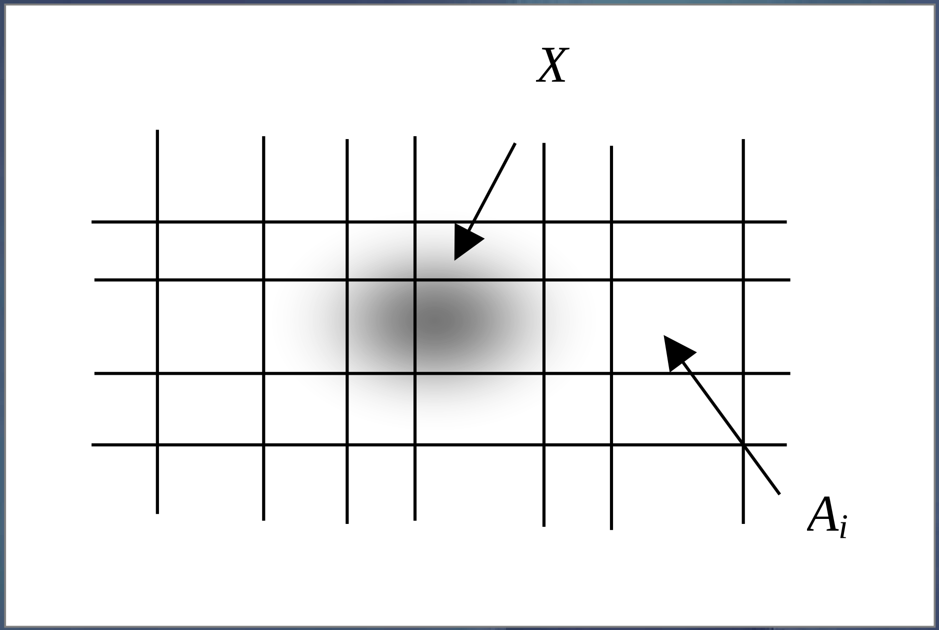
Fuzzy rough sets and rough fuzzy sets

- In rough sets the vocabulary and incoming object were information granules represented as sets.
- Two useful alternatives could be considered:

  1-Reference information granules== sets
  Object to be described == fuzzy set — **Fuzzy rough sets**

  2-Reference information granules== fuzzy sets
  Object to be described == set — **Rough fuzzy sets**

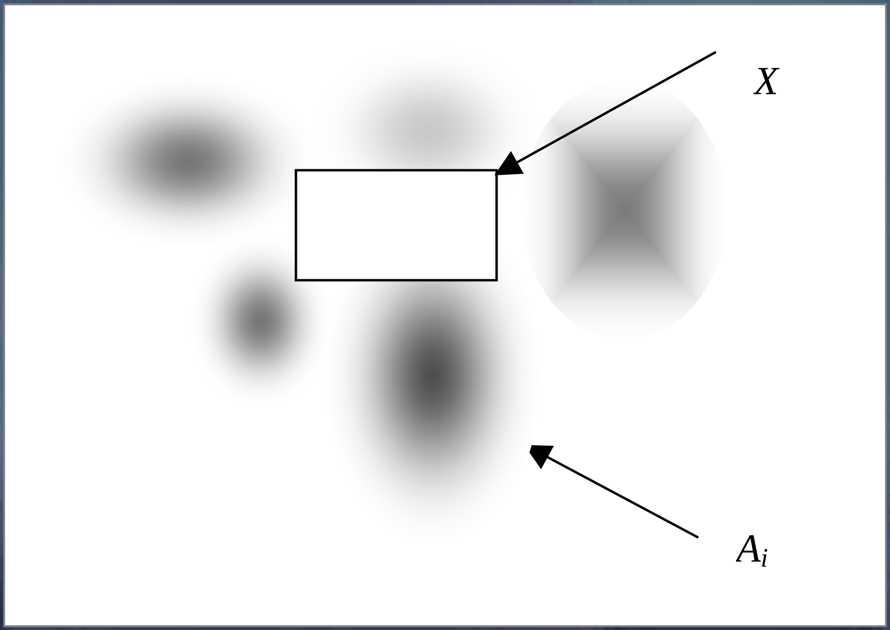
Fuzzy rough sets



$$X_+(A_i) = \sup_x [\min(A_i(x), X(x))] = \sup_{x \in \mathrm{Supp}(A_i)} X(x)$$

$$X_-(A_i) = \inf_x [\max(1 - X(x), A_i(x))]$$

Rough fuzzy sets

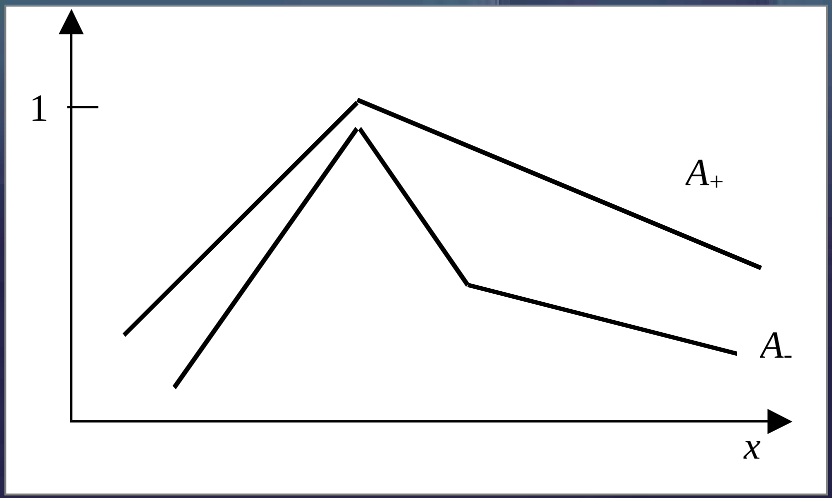


8.3 Interval-valued fuzzy sets

Interval-valued fuzzy sets

- We consider that instead of single membership grades, there are intervals of feasible membership values

- This brings a concept of interval-valued fuzzy sets where the concept of membership is represented in the form of interval



Interval-valued fuzzy sets: operations

- Given $A = (A_-, A^+)$ and $B = (B_-, B_+)$

$$(A \cup B)(x) = (\min(A_+(x), B_+(x)), \max(A_-(x), B_-(x)))$$

Union

$$(A \cap B)(x) = (\max(A_+(x), B_+(x)), \min(A_-(x), B_-(x)))$$

Intersection

$$\overline{A}(x) = (\overline{A}_+(x), A_-(x))$$

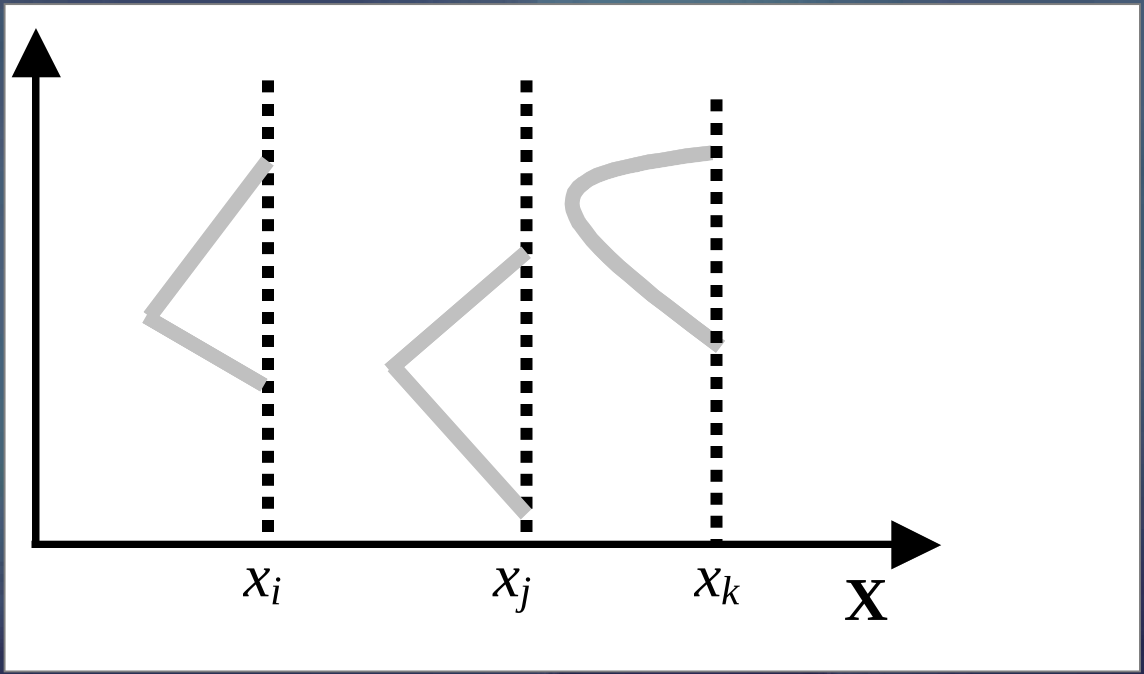
Complement

8.4 Type-2 fuzzy sets

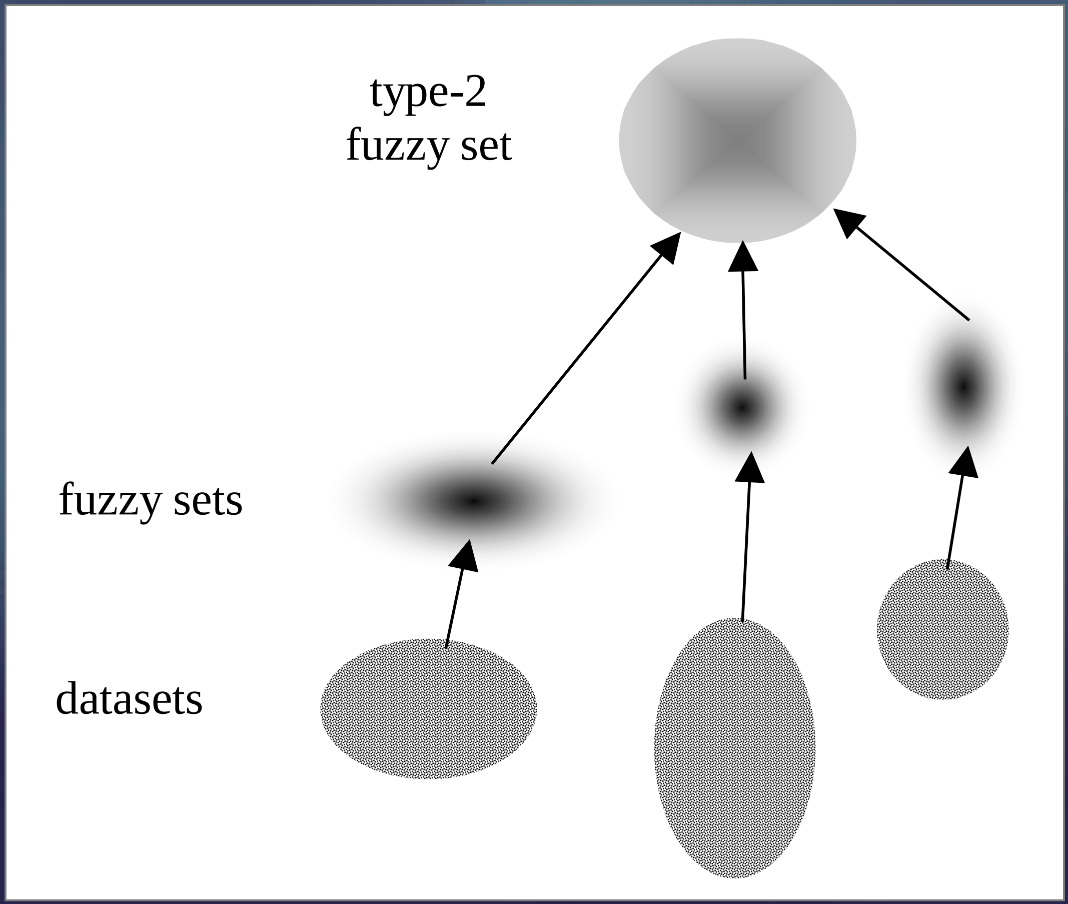
Type-2 fuzzy sets

- Membership degree treated as a single number in [0,1]
- Could the membership itself be a fuzzy set?
- Type-2 fuzzy set: admit membership modeled as fuzzy sets defined in [0,1]

Type-2 fuzzy set: Example



Type-2 fuzzy sets as results of aggregation



Intuitionistic fuzzy set

- Information granule A in which we consider:
  - – degree of membership A^+
  - – degree of non-membership A^-

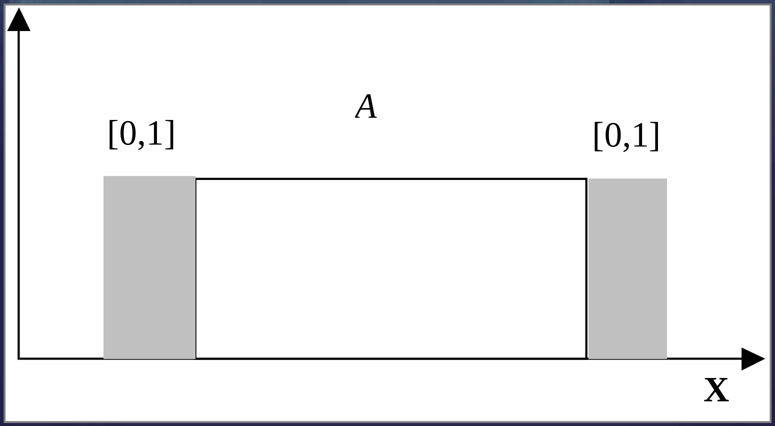
where

$$A^+(x) + A^-(x) \leq 1$$

8.5 Shadowed sets as a three-valued logic characterization of fuzzy sets

Shadowed sets

- Information granule A in which we admit:
- Full membership
- Full exclusion, and
- Shadow – range of [0,1]



Shadowed sets: operations

A: $\mathbf{X} \rightarrow \{0, 1, [0,1]\}$ $S = [0,1]$

$A \backslash B$	0	S	1
0	0	0	0
S	0	S	S
1	0	S	1

intersection

union

$A \backslash B$	0	S	1
0	0	S	1
S	S	S	1
1	0	1	1

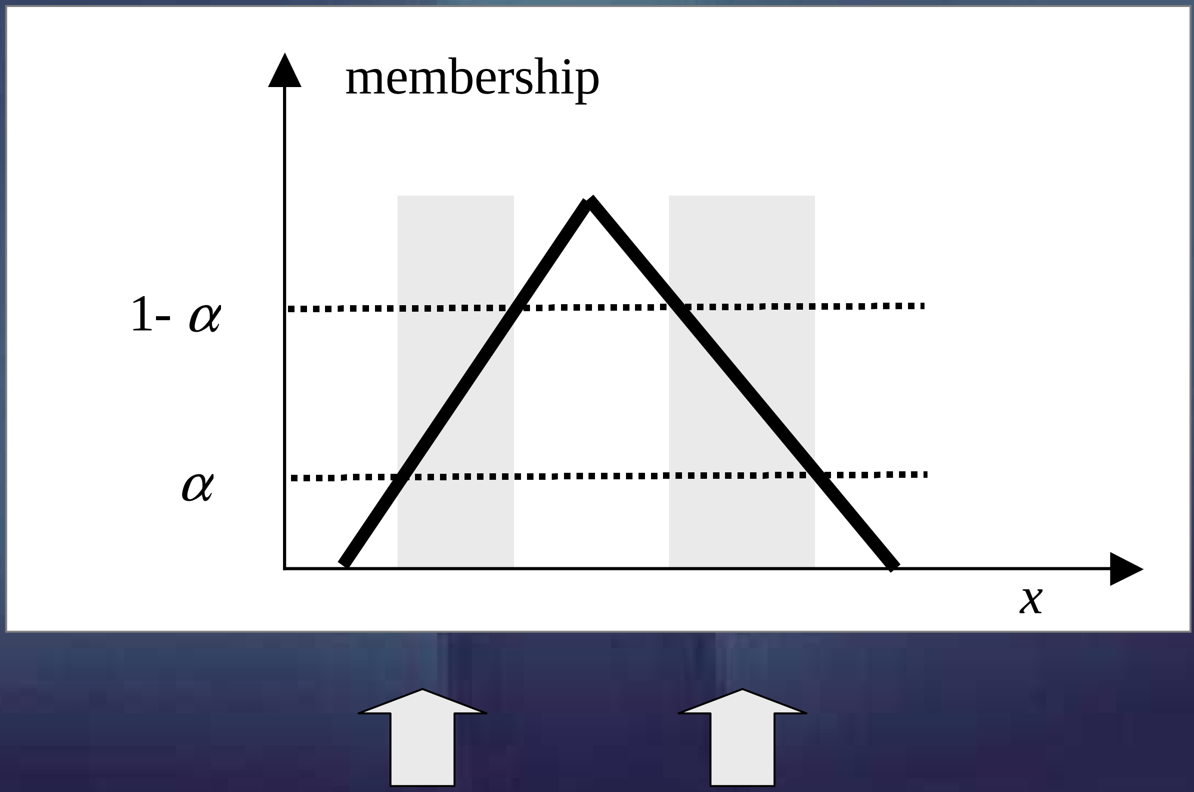
A	$\underline{A}$
0	1
S	S
1	0

complement

Development of shadowed sets

- Shadowed sets are viewed as constructs implied by fuzzy sets:
  - "localization" of membership values by forming shadows and using only 0 - 1 degrees of membership
  - shadowed sets support simpler computing by operating on three logic values

From fuzzy set to shadowed set



reduction of membership + elevation of membership = shadow

From fuzzy set to shadowed set

$$\int_{x:A(x)\leq\alpha} A(x)dx$$

membership reduction

$$\int_{x:A(x)\geq 1-\alpha} (1-A(x))dx$$

membership elevation

$$\int_{x:\alpha<A(x)<1-\alpha} dx$$

shadow

$$V(\alpha) = \left| \int_{x:A(x)\leq\alpha} A(x)dx + \int_{x:A(x)\geq 1-\alpha} (1-A(x))dx + \int_{x:\alpha<A(x)<1-\alpha} dx \right|$$

performance index

$$\alpha_{opt} = \arg\min_{\alpha} V(\alpha)$$

From fuzzy set to shadowed set

- Triangular membership function: $\alpha = \sqrt{2} - 1$
- Parabolic membership function: $\alpha = 0.405$

From fuzzy set to shadowed set: discrete case

$$V(\alpha) = \left| \sum_{k \in \Omega} u_k + \sum_{k \in \Phi} (1 - u_k) - \mathrm{Card}(\Omega) \right|$$

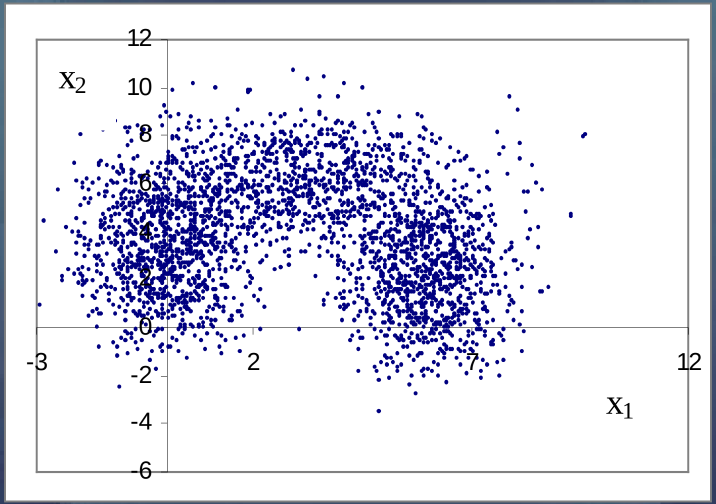
Minimize $V(\alpha)$ w.r.t. α

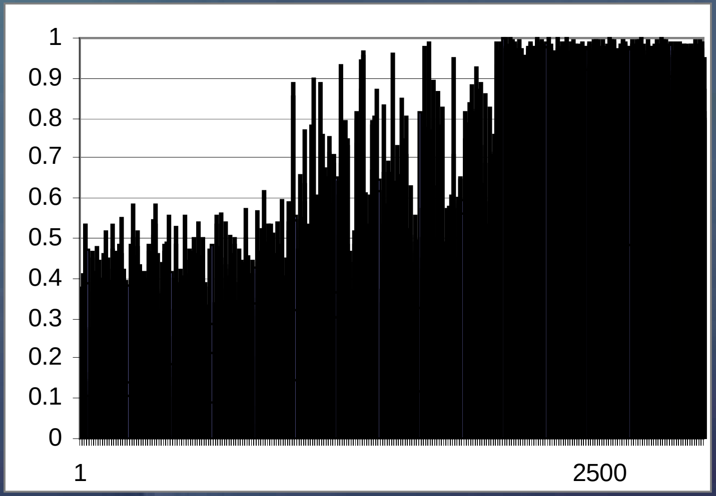
s.t. $u_{\min} \leq \alpha \leq (u_{\min} + u_{\max})/2$

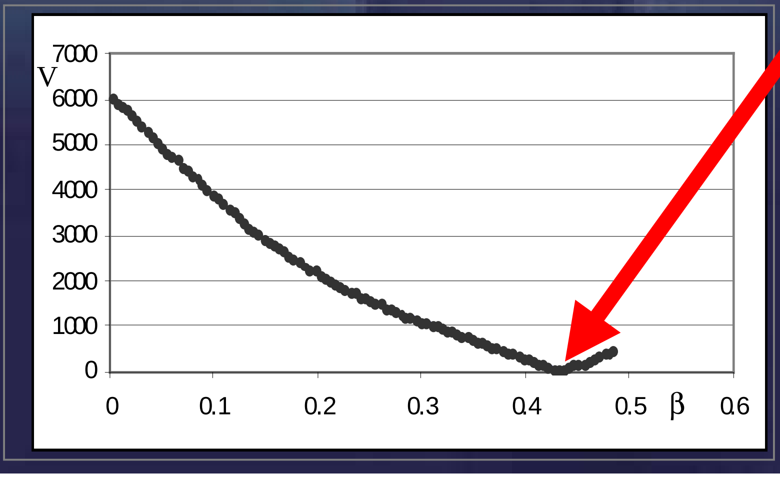
Shadowed sets in fuzzy clustering

- Fuzzy clustering could be conveniently interpreted using shadowed sets
  - elements completely belonging to the cluster
  - elements completely excluded from the cluster
  - elements with uncertainty (*shadow* of the cluster) that are “flagged” in this way and may require further attention

Shadowed sets in fuzzy clustering: Example







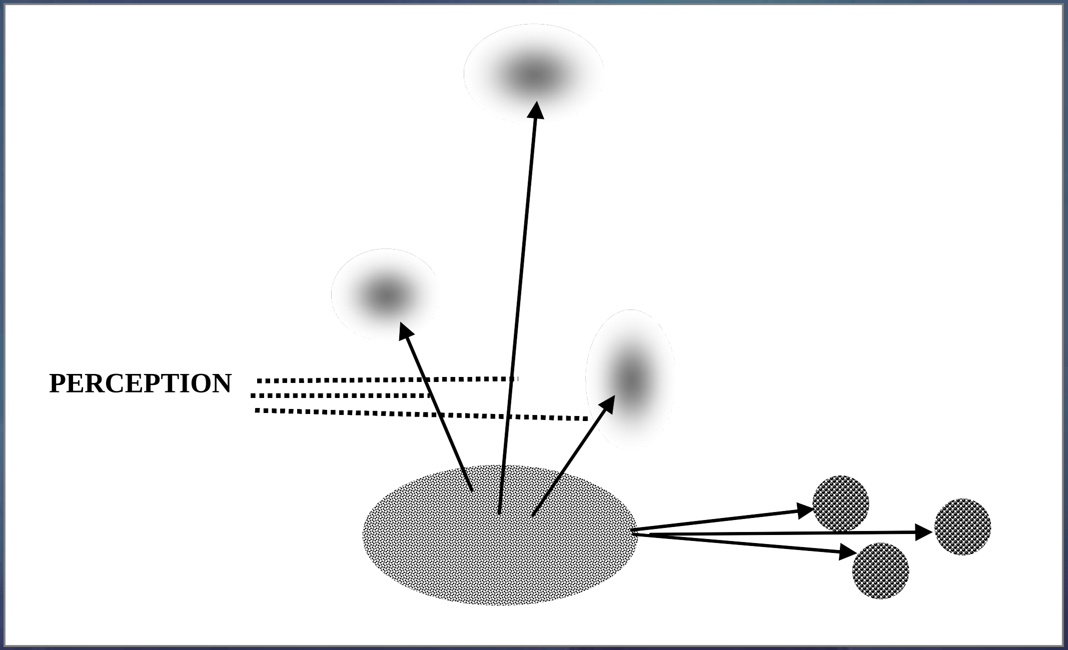
Pedrycz and Gomide, FSE 2007

8.6 Probability and fuzzy sets

Probability and fuzzy sets

- Fuzzy sets and probability are orthogonal concepts:
  - **probability** is concerned with occurrence of Boolean phenomena
  - **fuzzy sets** are concerned with perception of concepts

Probability and fuzzy sets



8.7 Probability of fuzzy events

Probability of fuzzy events

- What is the probability of *low* temperature tomorrow
- What is the probability of *high* inflation in a *short* term
- What is the probability of *small* steady state error of boiler pressure

Probability of fuzzy events

- Underlying probability density function in $\mathbf{X}$: $p(x)$
- Fuzzy event (fuzzy set): A
- Probability of fuzzy event

$$\int_{\mathbf{X}} A(x)dP(x) = \int_{\mathbf{X}} A(x)p(x)dx$$

(assume that the integral does exist)

⇩

This is the expected value $E(A)$ of the membership function of A

Probability of fuzzy events

Variance

$$E^2(A) = \int_{\mathbf{X}} [A(x) - E(A)]^2 p(x)dx$$

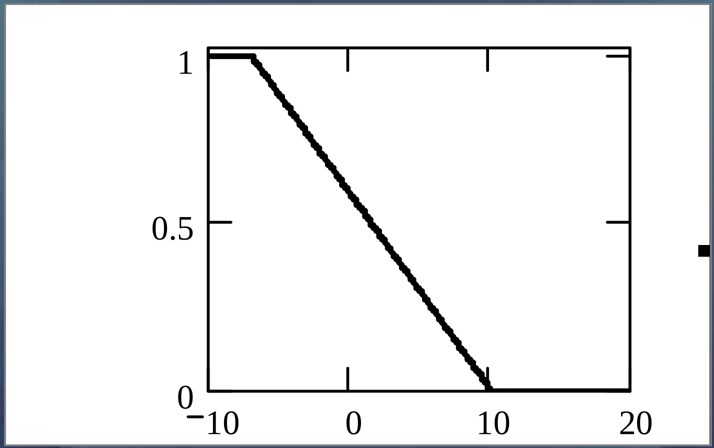
Standard deviation

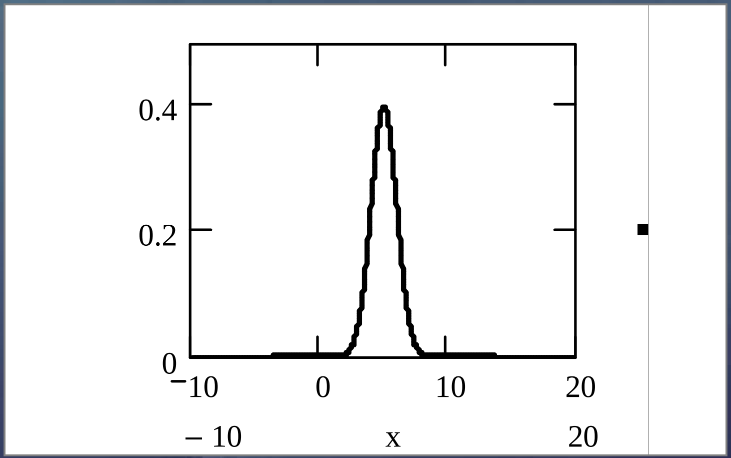
$$\sigma(A) = \sqrt{E^2(A)}$$

High order moments

$$\int_{\mathbf{X}} [A(x) - E(A)]^r p(x)dx \qquad r > 2$$

Probability of fuzzy events: Example





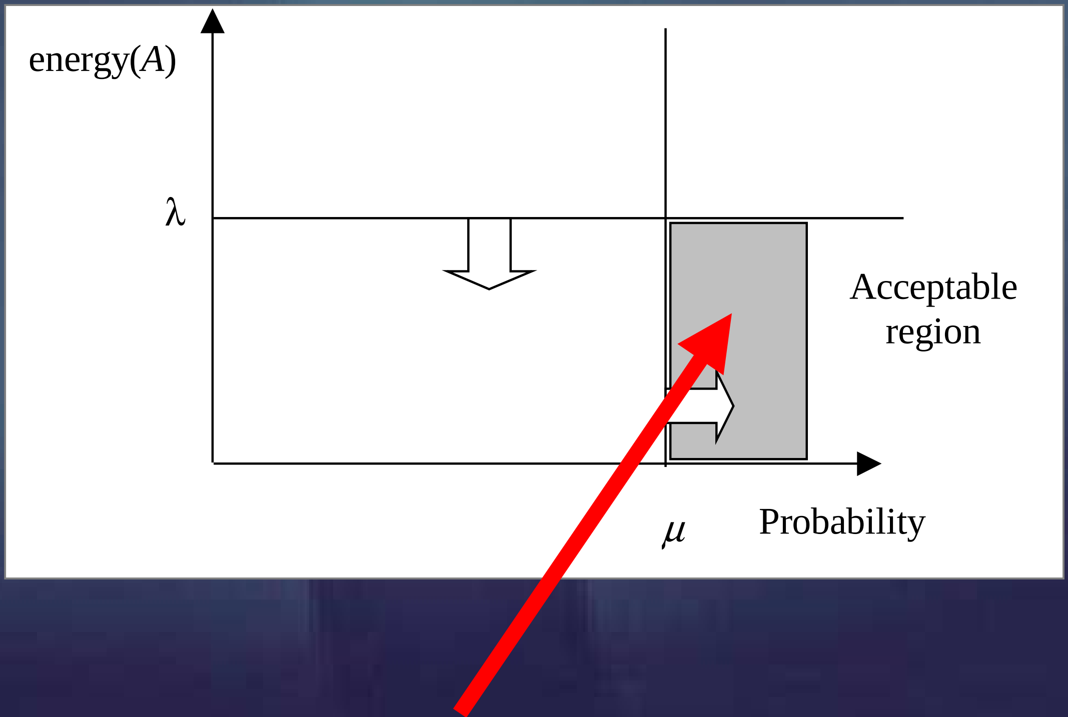
A = *low* temperature

pdf of temperature

$P(A) = 0.294 \quad \sigma(A)\ 3.46\times10^{-3}$

Probability of fuzzy events: orthogonality

Semantic validity

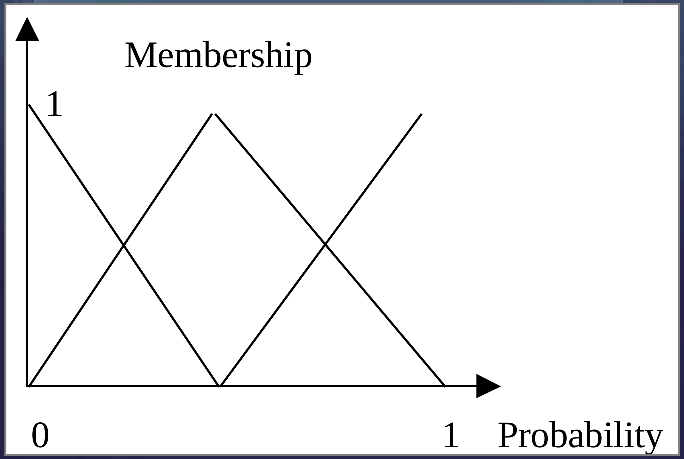


experimental evidence

Linguistically quantified statements

Linguistic probabilities:

low probability, *high* probability, probability *around* 60%...



Linguistically quantified statements: computing

- Random variable a_i with linguistic probabilities P_i

$$Z = \sum_{i=1}^{n} a_i P_i$$

- Extension principle:

$$Z(z) = \sup[\min(P_1(p_1), P_2(p_2), \ldots, P_n(p_n))]$$

$$s.t. \quad z = \sum_{i=1}^{n} a_i p_i$$

$$\sum_{i=1}^{n} p_i = 1$$

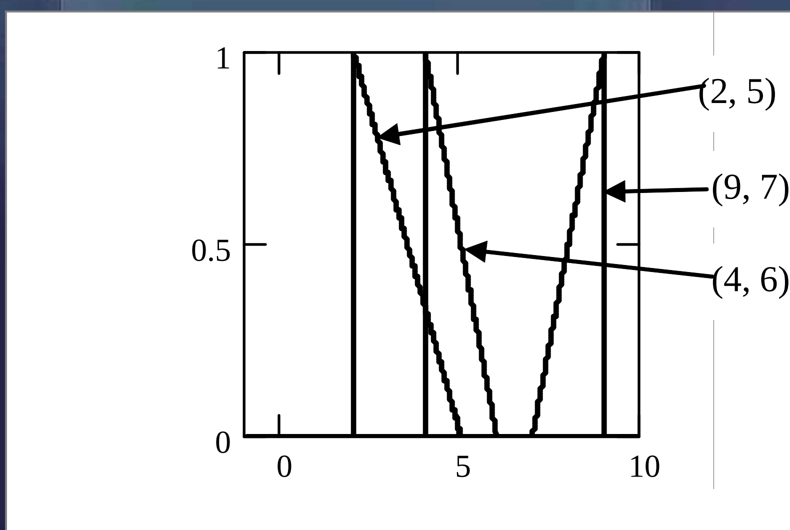
Linguistically quantified statements: Example

$$Z = a_1\ likely + a_2\ unlikely$$

$$Z(z) = likely\left(\frac{z - a_2}{a_1 - a_2}\right)$$

$likely\ (u) = unlikely\ (1 - u)$

$likely\ (u) = u$



$likely\ (u) = u^2$