6 Fuzzy Relations

Fuzzy Systems Engineering
Toward Human-Centric Computing

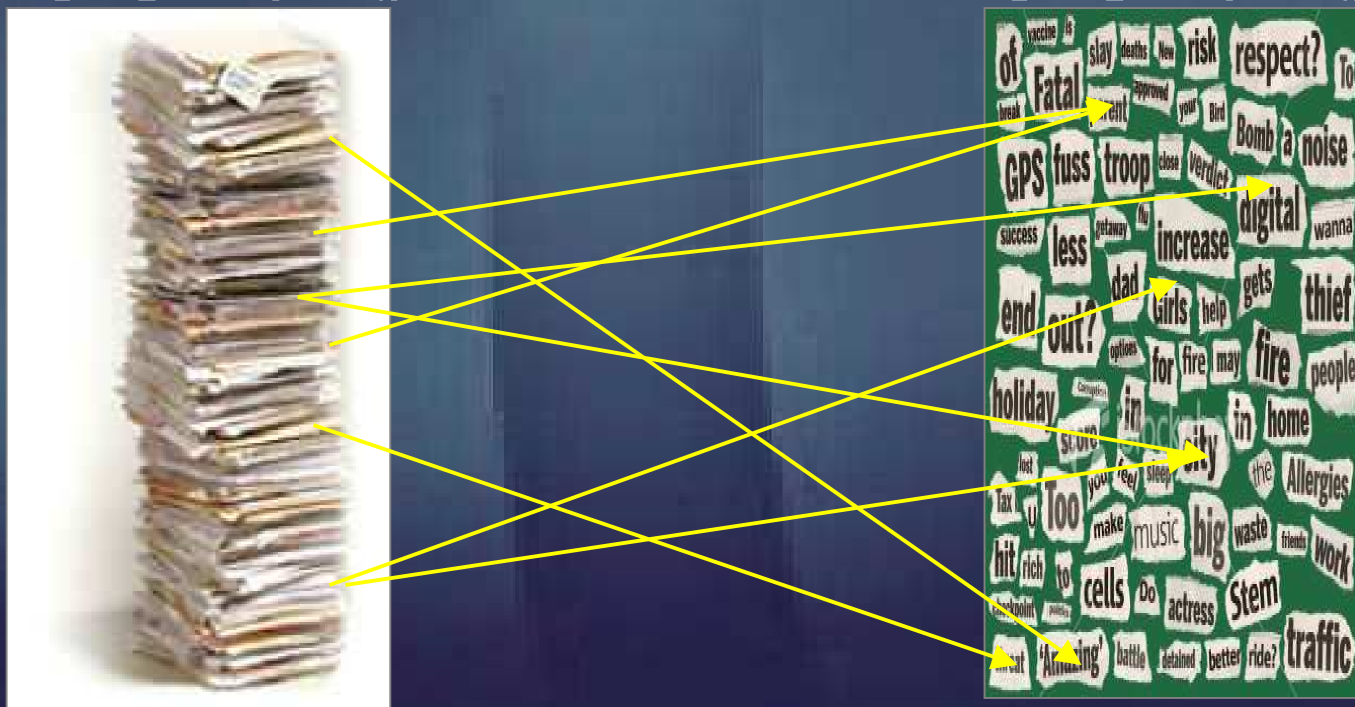
Contents

6.1 The concept of relations

Relation

Docs	Keywords
$\mathbf{X}$	$\mathbf{Y}$
$\{d_1, d_2,\ldots,d_i,\ldots d_n\}$	$\{w_1, w_2,\ldots,j_i,\ldots w_m\}$

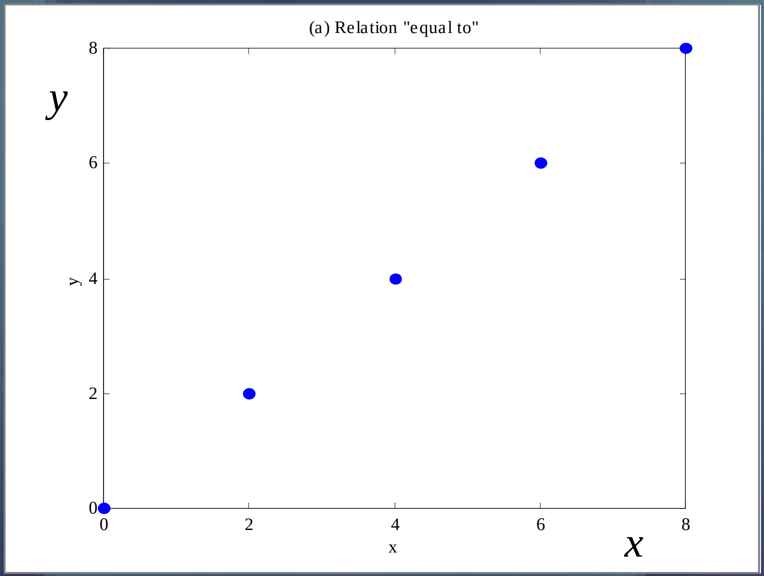
d_i

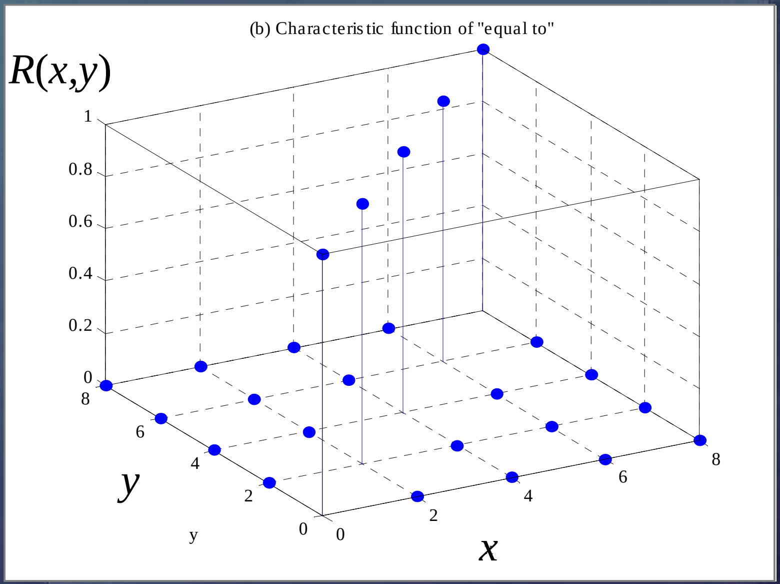


w_j

$$R = \{(d_i, w_j) \mid d_i \in \mathbf{X}, w_j \in \mathbf{Y}\}$$

Relation $R : \mathbf{X} \times \mathbf{Y} \rightarrow \{0,1\}$





$\mathbf{X} = \mathbf{Y} = \{2, 4, 6, 8\}$

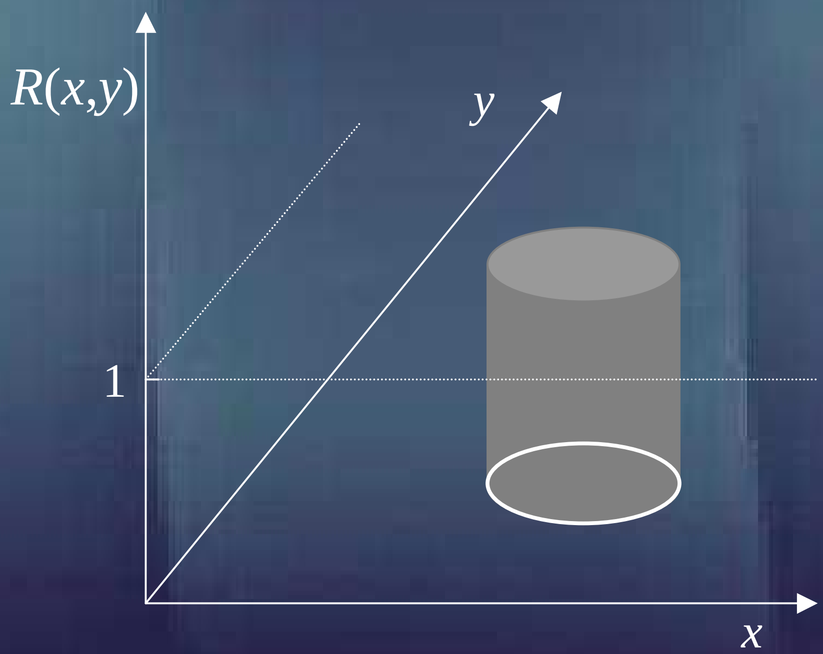
equal to

$R = \{(2,2), (4,4), (6,6), (8,8)\}$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

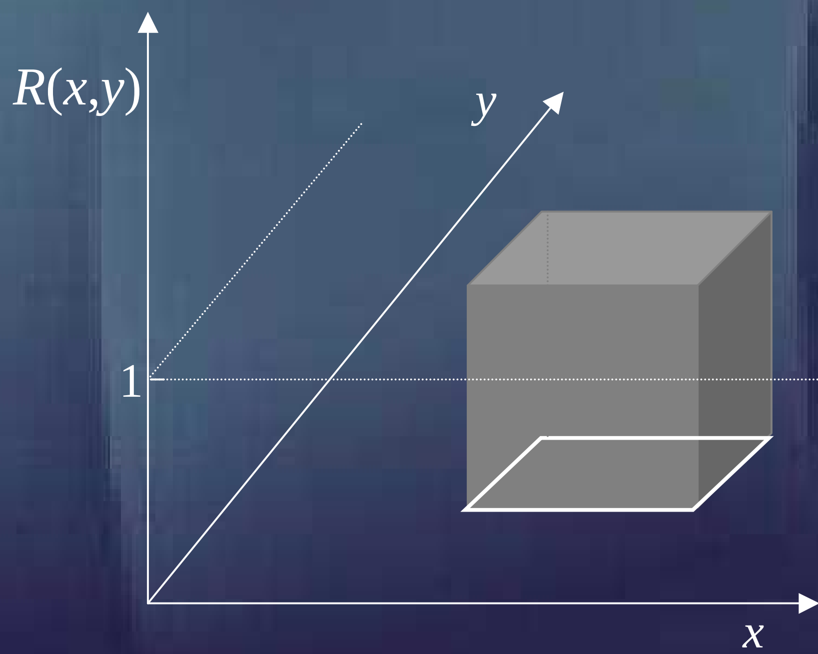
Examples

Circle



$$R(x,y) = \begin{cases} 1 & \text{if } x^2 + y^2 = r^2 \\ 0 & \text{otherwise} \end{cases}$$

Square



$$R(x,y) = \begin{cases} 1 & \text{if } |x| \le 1 \text{ and } |y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

6.2 Fuzzy relations

Fuzzy relation $R : \mathbf{X} \times \mathbf{Y} \rightarrow [0,1]$

Example

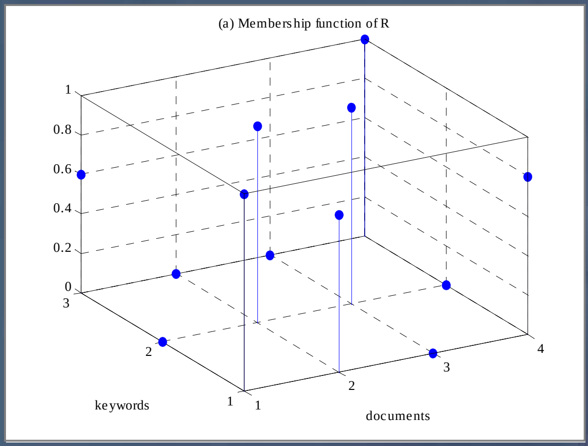
Docs

$\mathbf{D} = \{d_{fs}, d_{nf}, d_{ns}, d_{gf}\}$

Keywords

$\mathbf{W} = \{w_f, w_n, w_g\}$

$R : \mathbf{D} \times \mathbf{W} \rightarrow [0,1]$

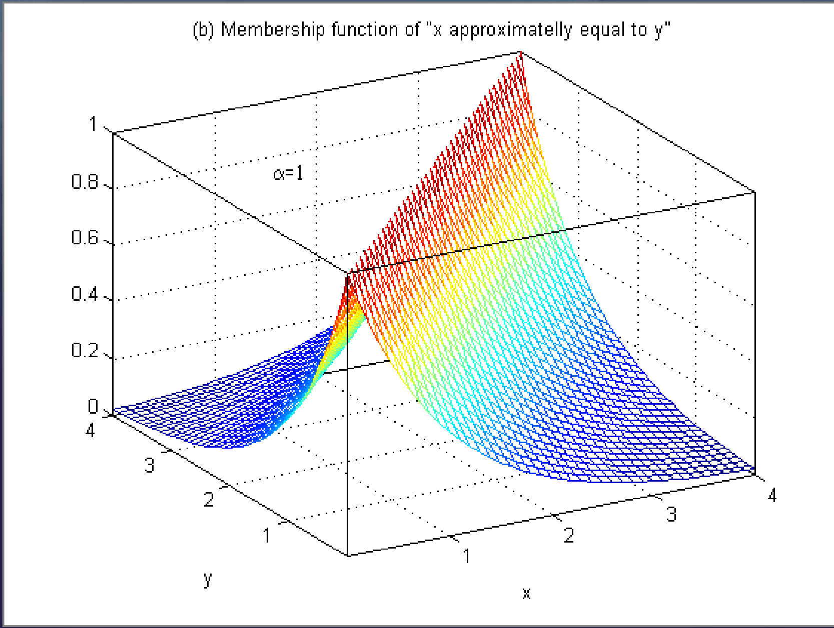


$$\begin{array}{c} \quad w_f \quad w_n \quad w_g \\ R = \begin{bmatrix} 1 & 0 & 0.6 \\ 0.8 & 1 & 0 \\ 0 & 1 & 0 \\ 0.8 & 0 & 1 \end{bmatrix} \begin{array}{l} d_{fs} \\ d_{nf}, \\ d_{ns} \\ d_{gf} \end{array} \end{array}$$

Example

$$R_e(x,y) = exp\left\{\frac{-|x-y|}{\alpha}\right\},\ \alpha > 0$$

X = **Y** = [0,4]



x approximately equal to *y*

$\alpha = 1$

6.3 Properties of fuzzy relations

Fuzzy relation $R : \mathbf{X} \times \mathbf{Y} \rightarrow [0,1]$

Domain

$$\mathrm{dom}R(x) = \sup_{y \in \mathbf{Y}} R(x,y)$$

Codomain

$$\mathrm{cod}R(y) = \sup_{x \in \mathbf{X}} R(x,y)$$

Representation of fuzzy relations

$$R = \bigcup_{\alpha \in [0,1]} \alpha R_{\alpha}$$

$$R(x,y) = \sup_{\alpha \in [0,1]} \{\min[\alpha, R(x,y)]\}$$

Representation theorem

Fuzzy relations $P,Q : \mathbf{X}\times\mathbf{Y} \rightarrow [0,1]$

Equality

$$P(x,y) = Q(x,y) \quad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

Inclusion

$$P(x,y) \leq Q(x,y) \quad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

6.4 Operations on fuzzy relations

Fuzzy relations $P,Q : \mathbf{X}\times\mathbf{Y} \rightarrow [0,1]$

Union: $R = P \cup Q$

$R(x,y) = P(x,y)\ s\ Q(x,y) \quad \forall(x,y) \in \mathbf{X}\times\mathbf{Y}$ (s is a t-conorm)

Intersection: $R = P \cup Q$

$R(x,y) = P(x,y)\ t\ Q(x,y) \quad \forall(x,y) \in \mathbf{X}\times\mathbf{Y}$ (t is a t-norm)

Fuzzy relation $R : \mathbf{X} \times \mathbf{Y} \rightarrow [0,1]$

Standard complement: $\overline{\mathrm{R}}$

$$\overline{R}(x,y) = 1 - R(x,y) \qquad \forall (x,y) \in \mathbf{X} \times \mathbf{Y}$$

Transpose: R^{T}

$$R^{\mathrm{T}}(y,x) = R(x,y) \qquad \forall (x,y) \in \mathbf{X} \times \mathbf{Y}$$

6.5 Cartesian product, projections,and cylindrical extension of fuzzy sets

Cartesian product

$A_1, A_2, ..., A_n$ fuzzy sets on $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$

$$R = A_1 \times A_2 \times ... \times A_n$$

$$R(x_1, x_2,...,x_n) = \min \{A_1(x_1), A_2(x_2), ..., A_n(x_n)\} \quad \forall (x_i,y_i) \in \mathbf{X}_i \times \mathbf{Y}_i$$

Generalization

$$R(x_1, x_2,...,x_n) = A_1(x_1)\ t\ A_2(x_2)\ t\ ...t\ A_n(x_n) \qquad \forall (x_i,y_i) \in \mathbf{X}_i \times \mathbf{Y}_i$$

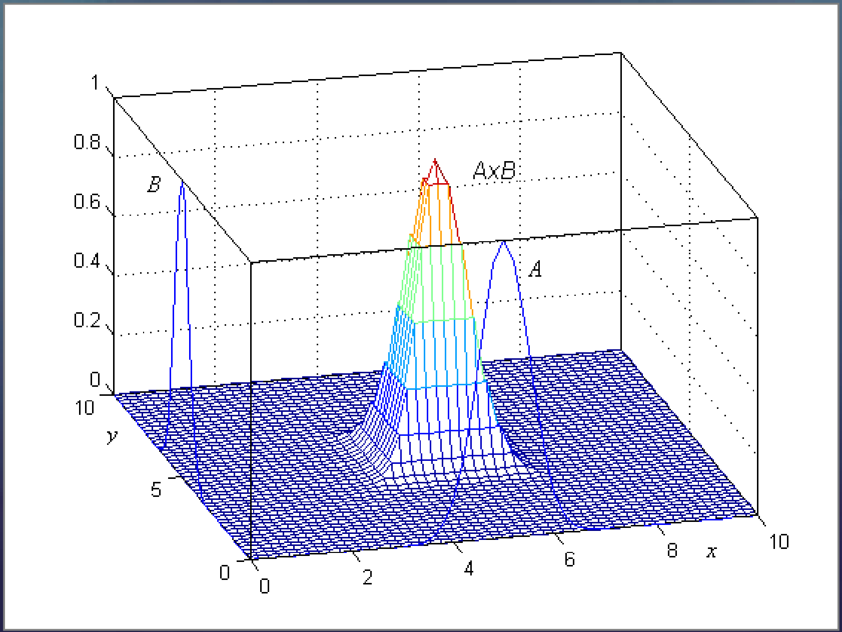
t = t-norm

Examples

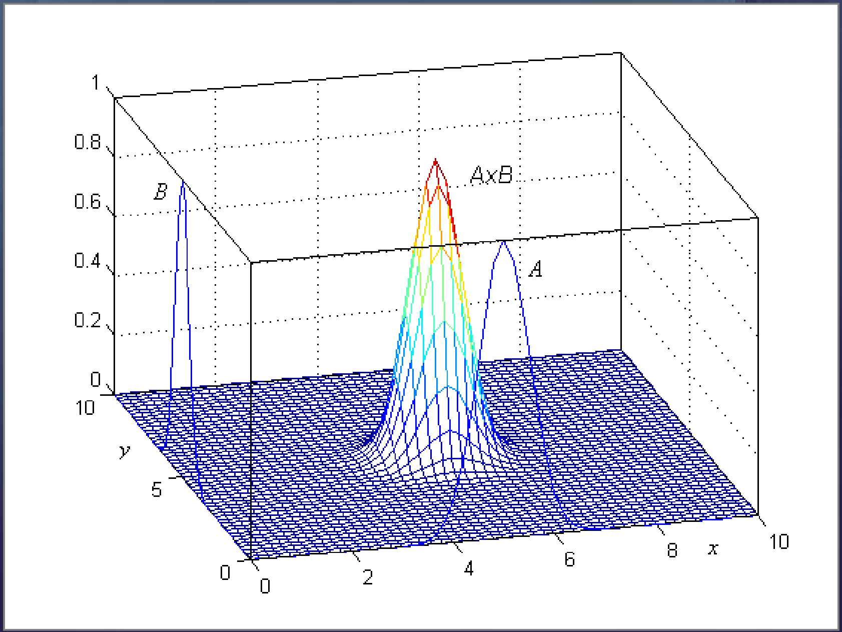
$A(x) = \exp[-2(x-5)^2]$

$B(y) = \exp[-2(y-5)^2]$

$R = A \times B$



$R(x,y) = \min\{A(x), B(y)\}$



$R(x,y) = A(x)B(y)$

Projection of fuzzy relations

$R: \mathbf{X}_1 \times \boldsymbol{X}_2 \times \ldots \times \mathbf{X}_n \rightarrow [0, 1]$

$\mathbf{X} = \mathbf{X}_i \times \boldsymbol{X}_j \times \ldots \times \mathbf{X}_k$

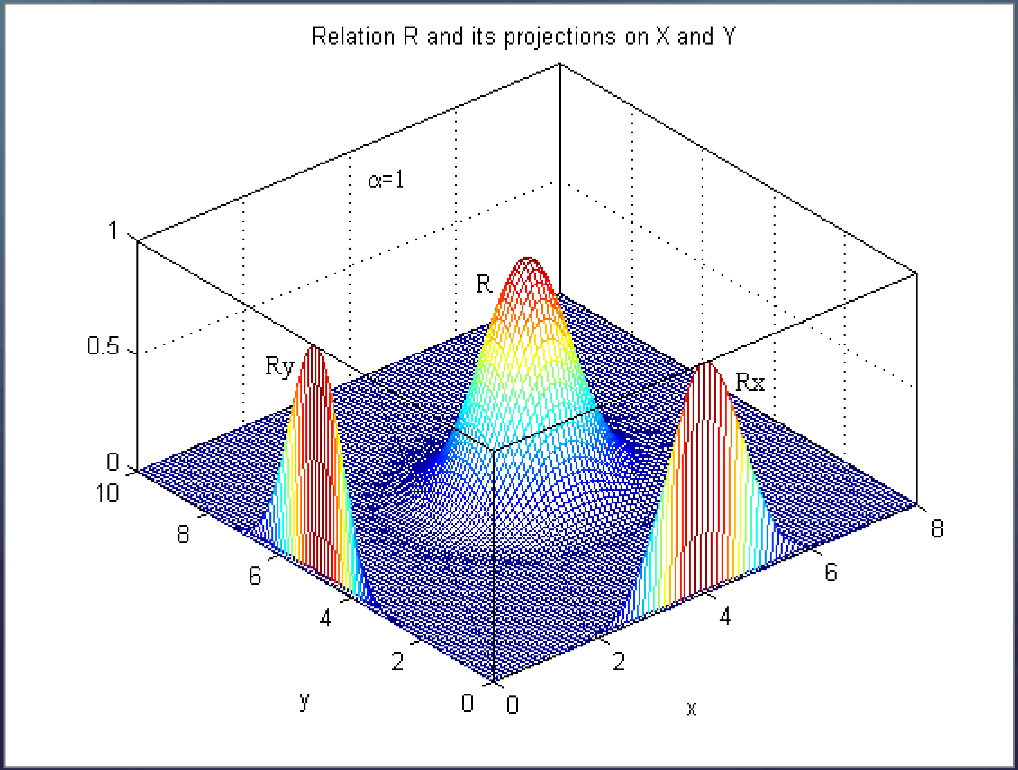
$$R_{\mathbf{X}}(x_i, x_j, \ldots, x_k) = Proj_{\mathbf{X}} R(x_1, x_2, \ldots, x_n) = \sup_{x_t, x_u, \ldots, x_v} R(x_1, x_2, \ldots, x_n)$$

$I = \{i, j, \ldots, k\}, \quad J = \{t, u, \ldots, v\}, \quad I \cup J = N, \quad I \cap J = \varnothing$

$N = \{1, 2, \ldots n\}$

Example

$$R(x, y) = \exp\{-\alpha[(x-4)^2 + (y-5)^2]\}, \quad \alpha = 1$$



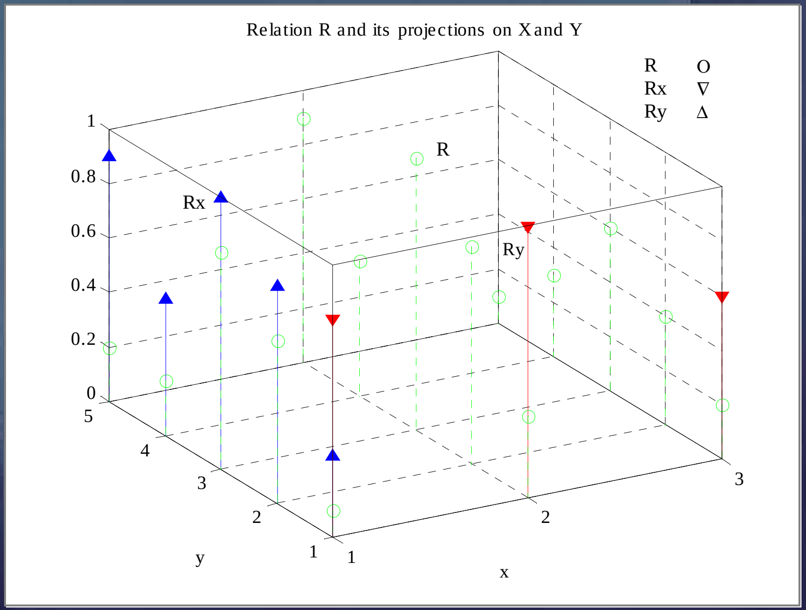
$$R_{\mathbf{X}}(x) = \mathrm{Proj}_{\mathbf{X}} R(x,y) = \sup_{y} R(x,y)$$

$$R_{\mathbf{Y}}(y) = \mathrm{Proj}_{\mathbf{Y}} R(x,y) = \sup_{x} R(x,y)$$

Example

R: $\mathbf{X} \times \mathbf{Y} \rightarrow [0, 1]$, $\mathbf{X} = \{1, 2, 3\}$, $\mathbf{Y} = \{1, 2, 3, 4, 5\}$

$$R(x,y) = \begin{bmatrix} 1.0 & 0.6 & 0.8 & 0.5 & 0.2 \\ 0.6 & 0.8 & 1.0 & 0.2 & 0.9 \\ 0.8 & 0.6 & 0.8 & 0.3 & 0.9 \end{bmatrix}$$

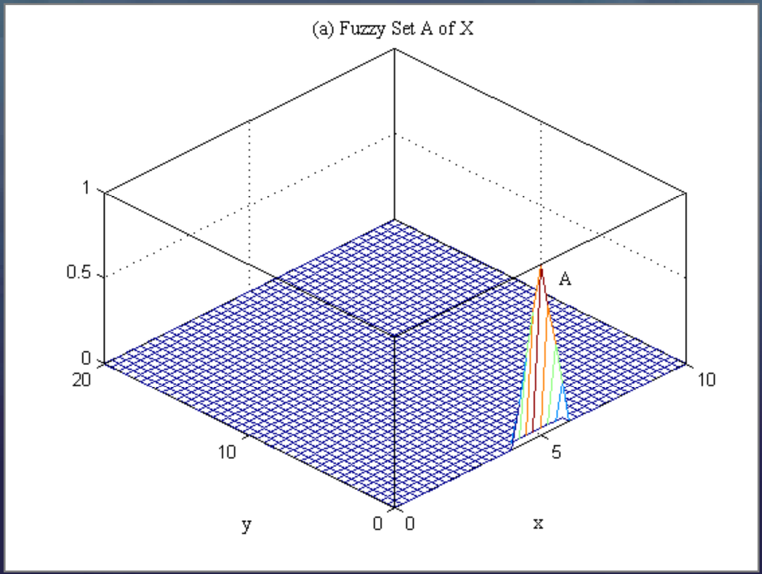


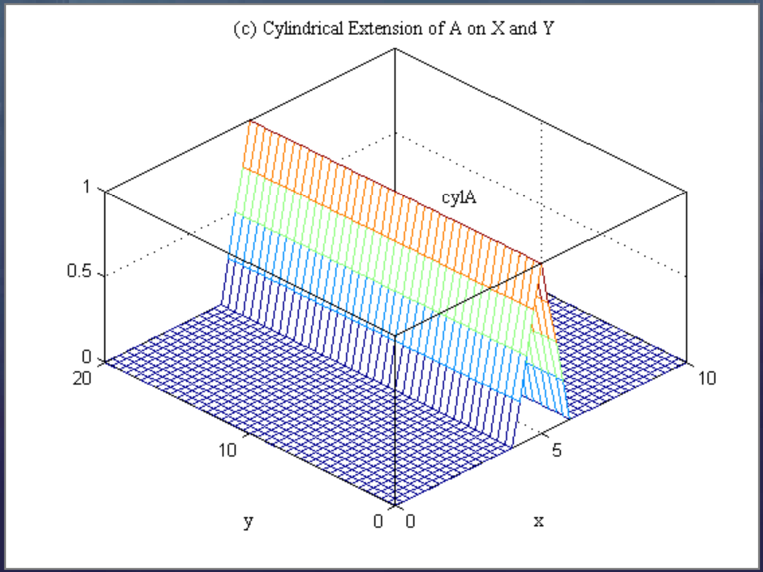
$R_{\mathbf{X}}$ = [1.0, 1.0, 0.9]

$R_{\mathbf{Y}}$ = [1.0, 0.8, 1.0, 0.5, 0.9]

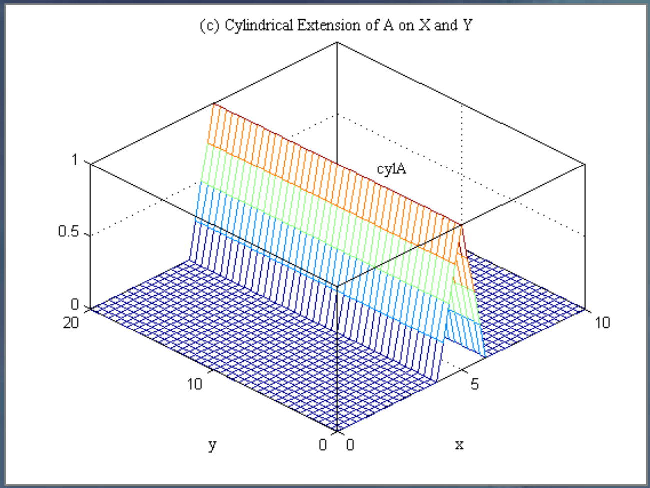
Cylindrical extension

$\text{cyl}A(x,y) = A(x)\ ,\quad \forall x \in \mathbf{X}$

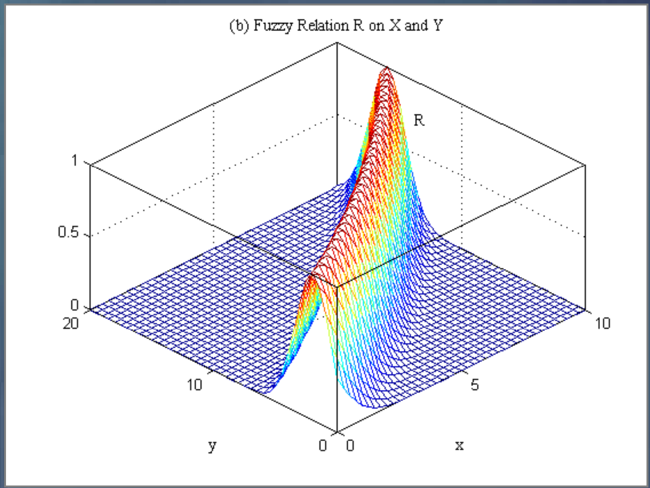




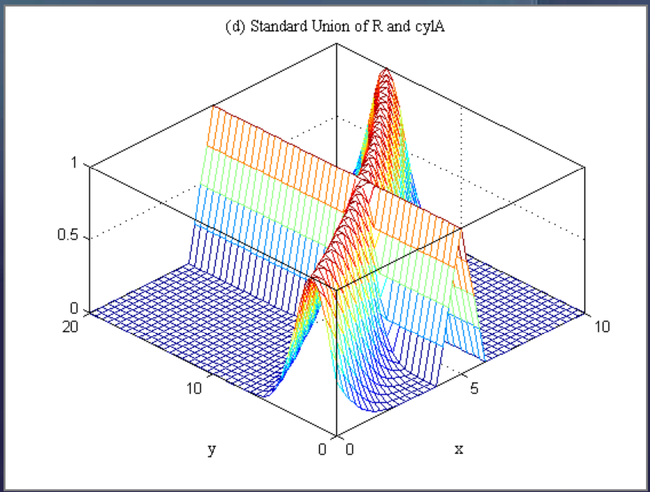
cyl*A*



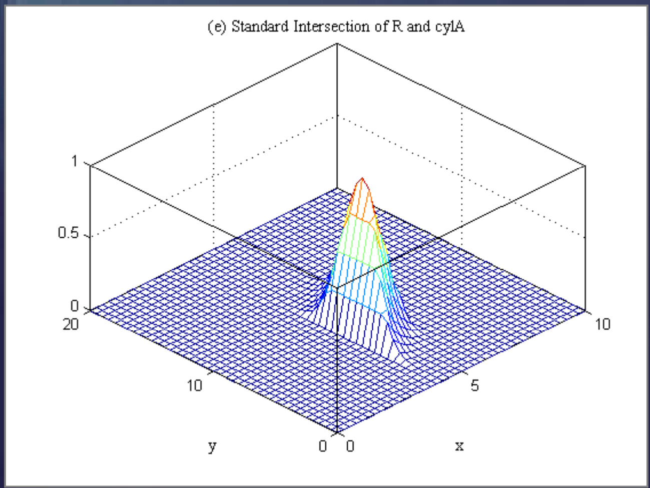
R



cyl*A*∪*R*



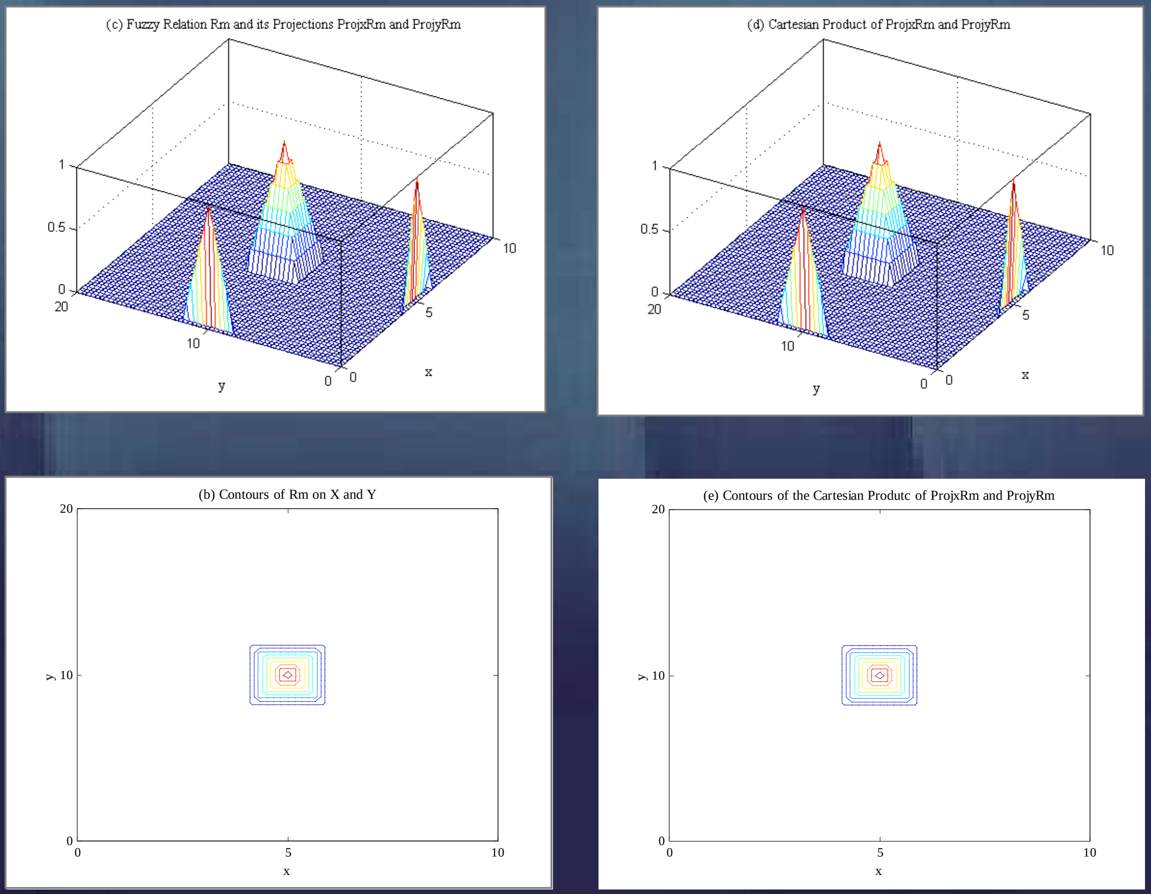
cyl*A*∩*R*



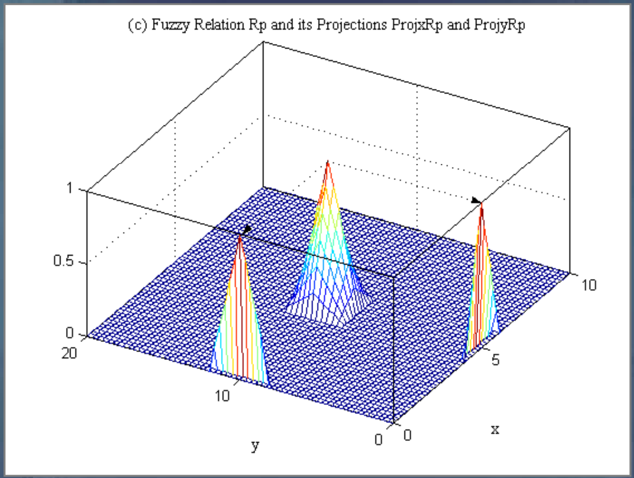
6.6 Reconstruction of fuzzy relations

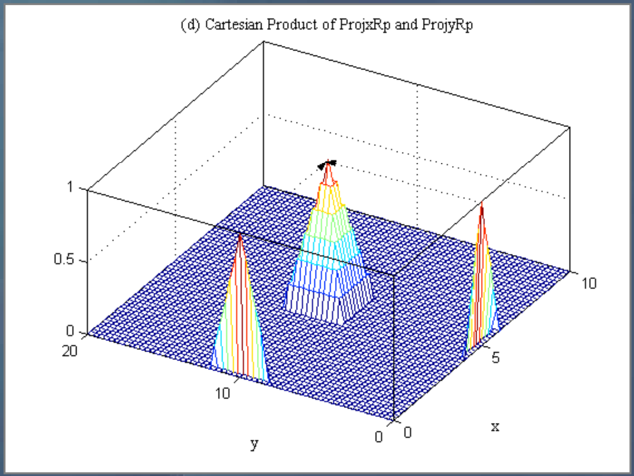
Reconstruction using Cartesian product

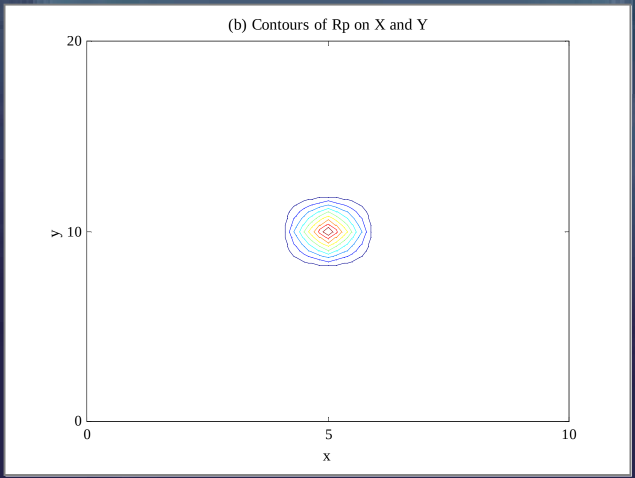
$$\mathrm{Proj}_{\mathbf{X}} R \times \mathrm{Proj}_{\mathbf{Y}} R \supseteq R$$

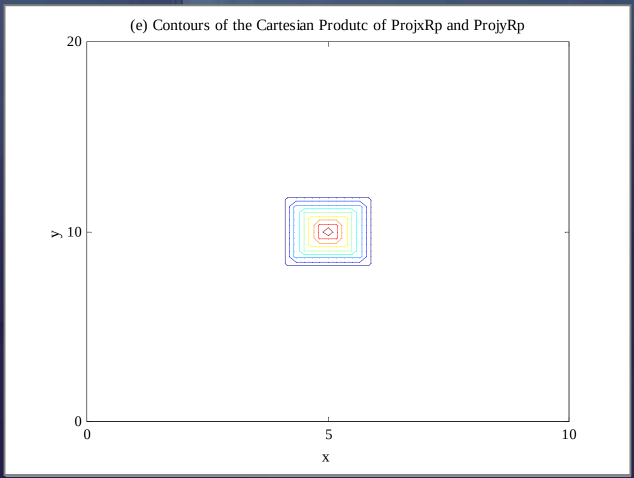


R
noninteractive









R
interactive

6.7 Binary fuzzy relations

Binary fuzzy relation $R : \mathbf{X} \times \mathbf{X} \rightarrow [0,1]$

Features

(a) Reflexivity

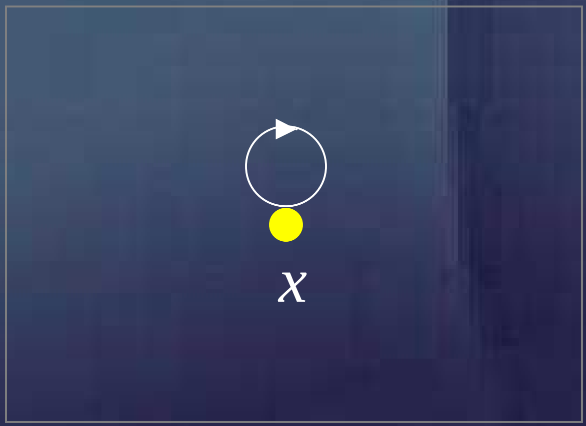
$R(x,x) = 1$

$R(x,x) \supseteq \mathrm{I}$

I = Identity

$R(x,x) \geq \varepsilon$ ε-reflexive

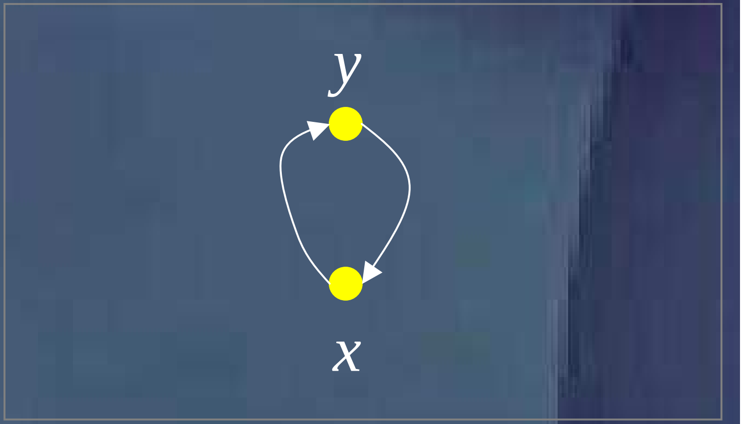
$\max \{R(x,y), R(y,x)\} \leq R(x,x)$ locally reflexive



(b) Symmetry

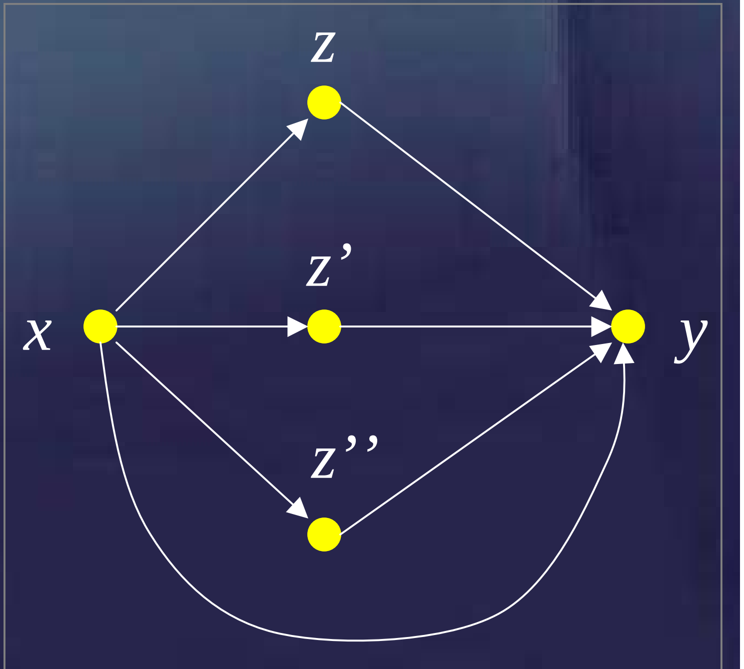
$R(x,y) = R(y,x) \quad \forall \in \times$

$R^{T} = R$



(c) Transitivity

$\sup_{z \in \mathbf{X}} \{R(x, z)\ t\ R(z, y)\} \leq R(x, y)\ \forall x, y, z \in \mathbf{X}$



Pedrycz and Gomide, FSE 2007

Transitive closure

$$\text{trans}(R) = \overleftrightarrow{R} = R \cup R^2 \cup \ldots\ldots \cup R^n$$

$$R^2 = R \circ R \;\ldots\ldots\ldots\; R^p = R \circ R^{p-1}$$

$$R \circ R(x,y) = \max_z \{ R(x,z)\; t\; R(z,y) \}$$

If R is reflexive, then $I \subseteq R \subseteq R^2 \subseteq \ldots \subseteq R^{n-1} = R^n$

I = identity

Floyd-Warshall procedure to compute trans(R)

procedure TRANSITIVE-CLOSUR-W (R) **returns** transitive fuzzy relation

static: fuzzy relation $R = [r_{ij}]$

for i = 1:n **do**
    for j = 1:n **do**
        for k = 1:n **do**
            $r_{jk} \leftarrow \max(r_{jk}, r_{ji} \; t \; r_{ik})$
return R

Equivalence relations

$R : \mathbf{X} \times \mathbf{X} \rightarrow \{0,1\}$

R is an equivalence relation if it is

- reflexive
- symmetric
- transitive

equivalence relations generalize the idea of equality

Equivalence class

$$A_x = \{y \in \mathbf{X} \mid R(x,y) = 1\}$$

$\mathbf{X}/R$ = family of all equivalence classes of R (partition of $\mathbf{X}$)

Similarity relations

$R : \mathbf{X} \times \mathbf{X} \rightarrow [0,1]$

R is a similarity relation if it is

– reflexive

– symmetric

– transitive

Equivalence class

$$P(R) = \{\mathbf{X}/R_\alpha \mid \alpha \in [0, 1]\}$$

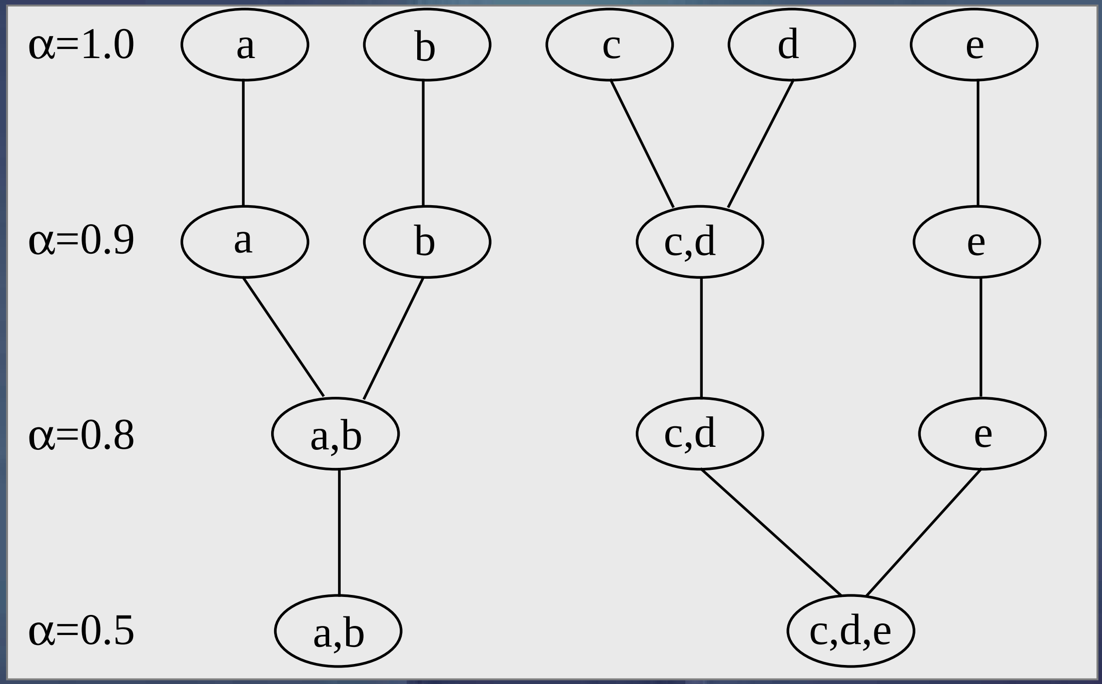
Nested partitions: if $\alpha > \beta$ then $\mathbf{X}/R_\alpha$ finer than $\mathbf{X}/R_\beta$

Example

$$R = \begin{bmatrix} 1.0 & 0.8 & 0 & 0 & 0 \\ 0.8 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0.9 & 0.5 \\ 0 & 0 & 0.9 & 1.0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 1.0 \end{bmatrix}$$

$$R_{0.5} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad R_{0.8} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 1 & 0 \\ 0 & 0 & 1 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Partition tree induced by similarity relation *R*



$$R_{0.5}=\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&1\\0&0&1&1&1\\0&0&1&1&1\end{bmatrix},\quad R_{0.8}=\begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&0&0&1\end{bmatrix},\quad R_{0.9}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1.0&1&0\\0&0&1&1.0&0\\0&0&0&0&1\end{bmatrix}$$

Compatibility relations

$R : \mathbf{X} \times \mathbf{X} \rightarrow \{0,1\}$

R is a compatibility relation if it is

- reflexive
- symmetric

α -Compatibility class: $A \subset \mathbf{X}$ such that

$$R(x,y) = 1 \quad \forall\, x,y \in A$$

Do not necessarily induce partitions

Proximity relations

$R : \mathbf{X} \times \mathbf{X} \to [0,1]$

R is a proximity relation if it is

- reflexive
- symmetric

Compatibility class: $A \subset \mathbf{X}$ such that

$R(x,y) = 1 \quad \forall\, x,y \in A$

Do not necessarily induce partitions

$$R = \begin{bmatrix} 1.0 & 0.7 & 0 & 0 & .6 \\ 0.7 & 1.0 & 0.6 & 0 & 0 \\ 0 & 0.6 & 1.0 & 0.7 & 0.4 \\ 0 & 0 & 0.7 & 1.0 & 0.5 \\ 0.6 & 0 & 0.4 & 0.5 & 1.0 \end{bmatrix}$$