

5 Operations and Aggregations of Fuzzy Sets

*Fuzzy Systems Engineering
Toward Human-Centric Computing*

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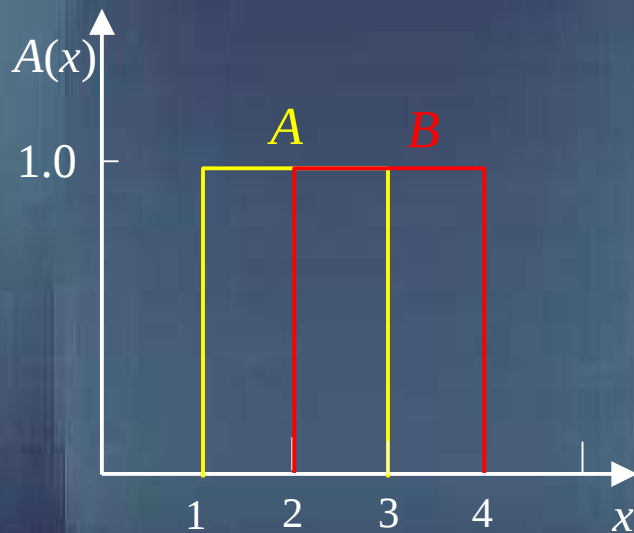
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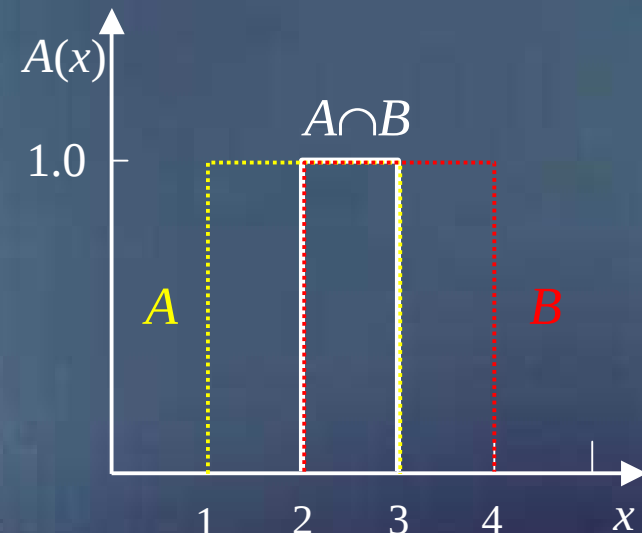
5.1 Standard operations on sets and fuzzy sets

Intersection of sets



$$A = \{x \in \mathbf{R} \mid 1 \leq x \leq 3\}$$

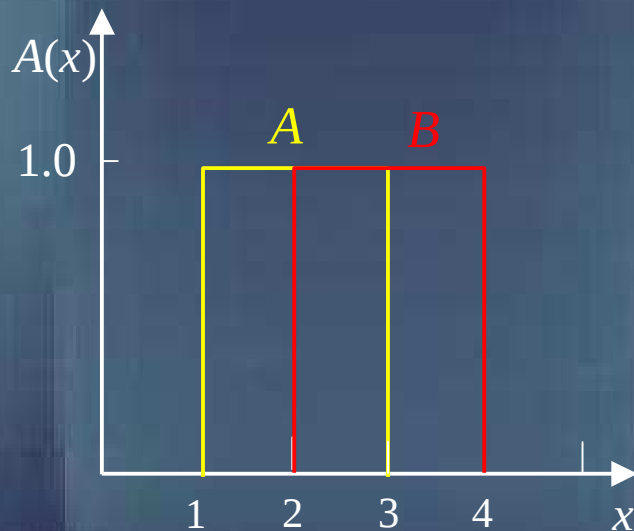
$$B = \{x \in \mathbf{R} \mid 2 \leq x \leq 4\}$$



$$A \cap B: \{x \in \mathbf{R} \mid 2 \leq x \leq 3\}$$

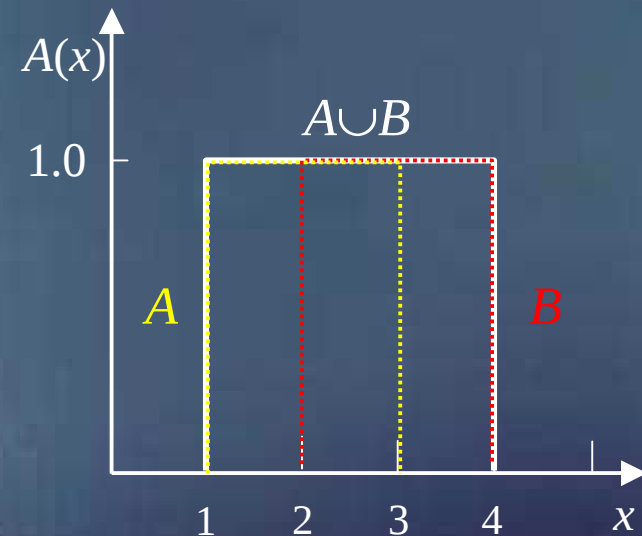
$$(A \cap B)(x) = \min [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

Union of sets



$$A = \{x \in \mathbf{R} \mid 1 \leq x \leq 3\}$$

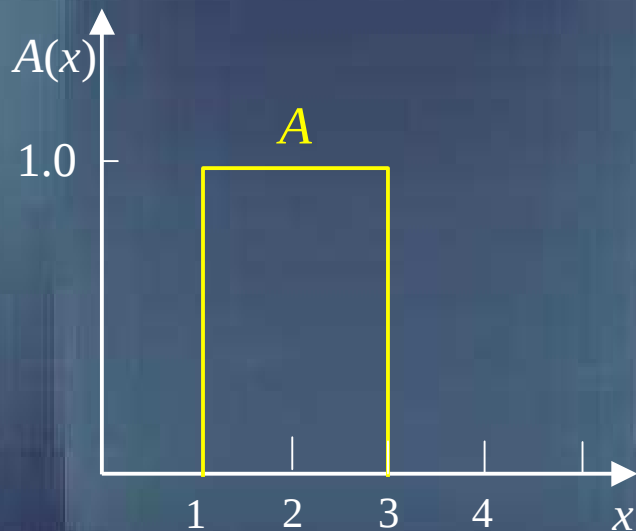
$$B = \{x \in \mathbf{R} \mid 2 \leq x \leq 4\}$$



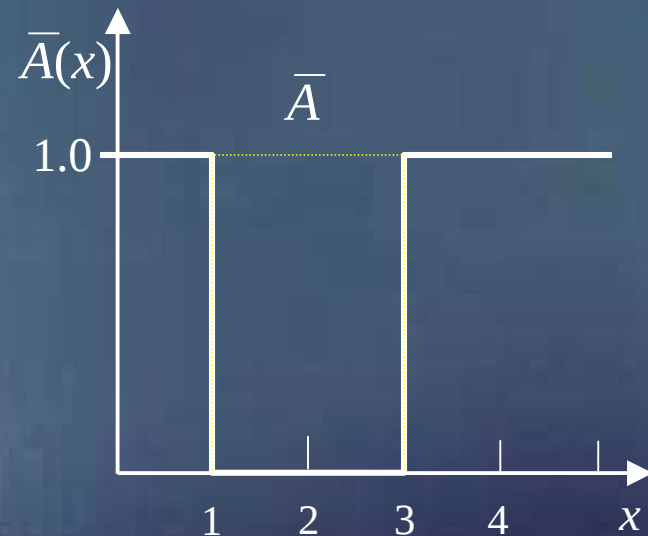
$$A \cup B: \{x \in \mathbf{R} \mid 1 \leq x \leq 4\}$$

$$(A \cup B)(x) = \max [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

Complement of sets



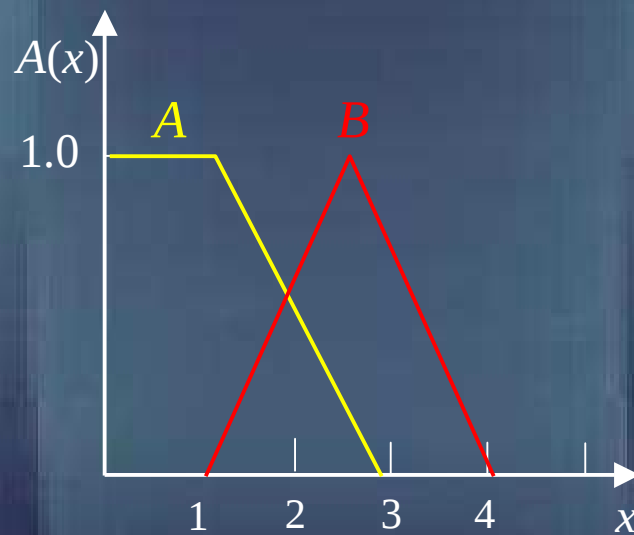
$$A = \{x \in \mathbf{R} \mid 1 \leq x \leq 3\}$$



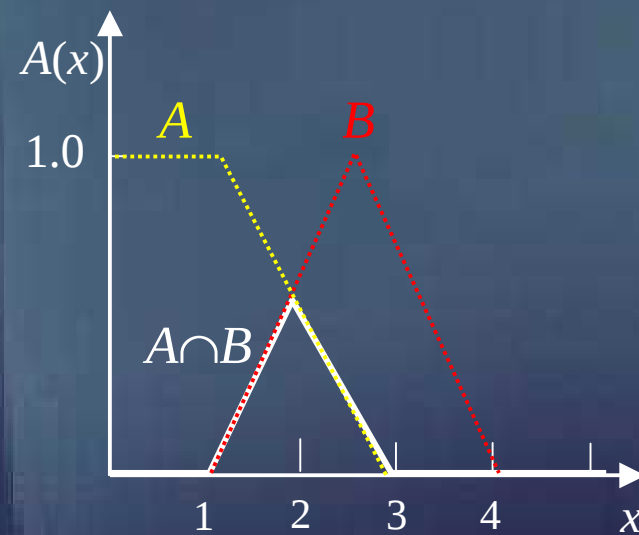
$$\bar{A}(x) = \{x \in \mathbf{R} \mid x < 1, x > 3\}$$

$$\bar{A}(x) = 1 - A(x) \quad \forall x \in \mathbf{X}$$

Intersection of fuzzy sets

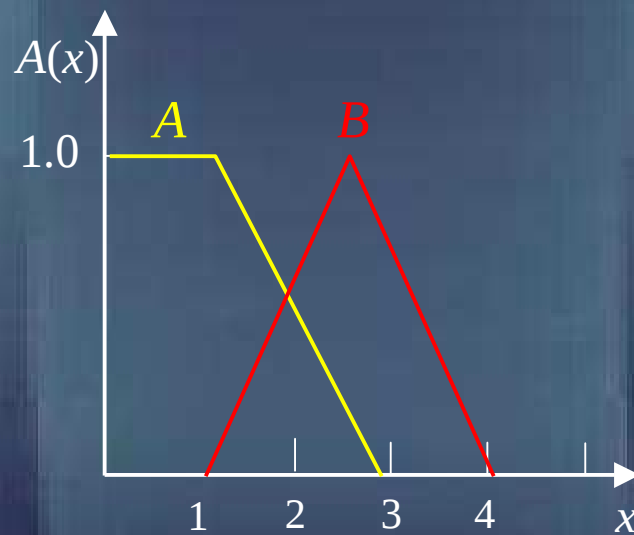


$$(A \cap B)(x) = \min [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

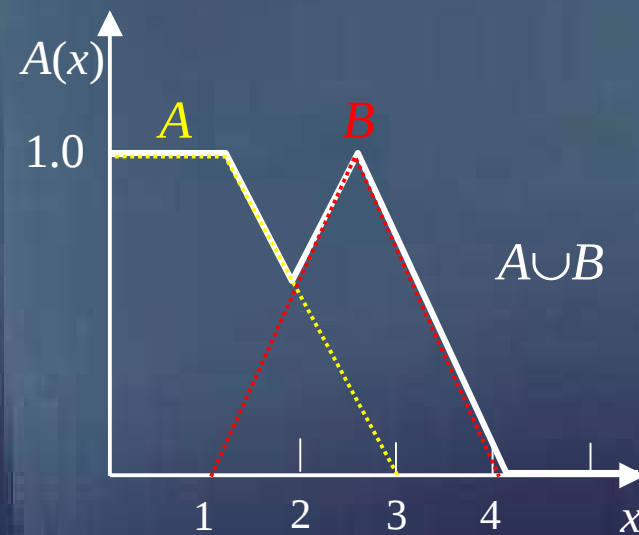


Standard intersection

Union of fuzzy sets

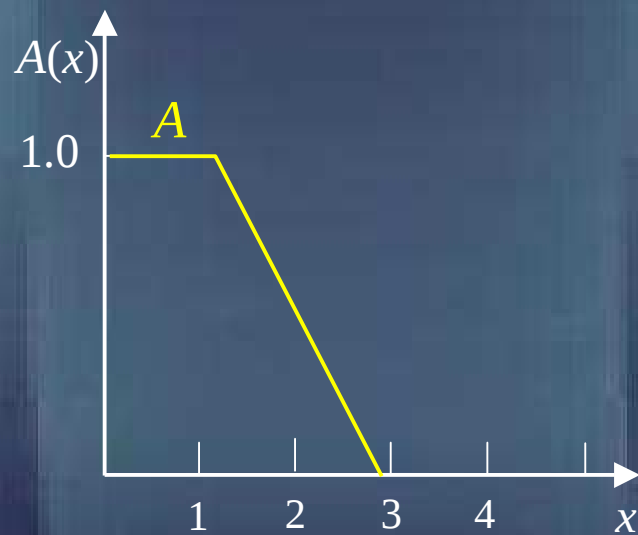


$$(A \cup B)(x) = \max [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

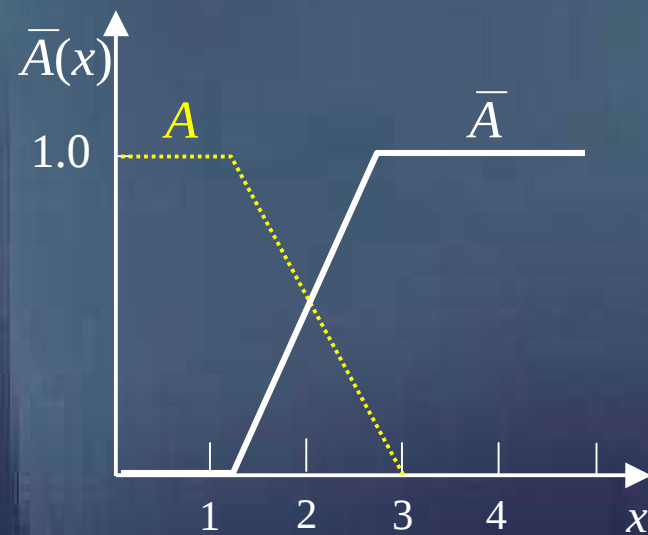


Standard union

Complement of fuzzy sets



$$\bar{A}(x) = 1 - A(x) \quad \forall x \in X$$



Standard complement

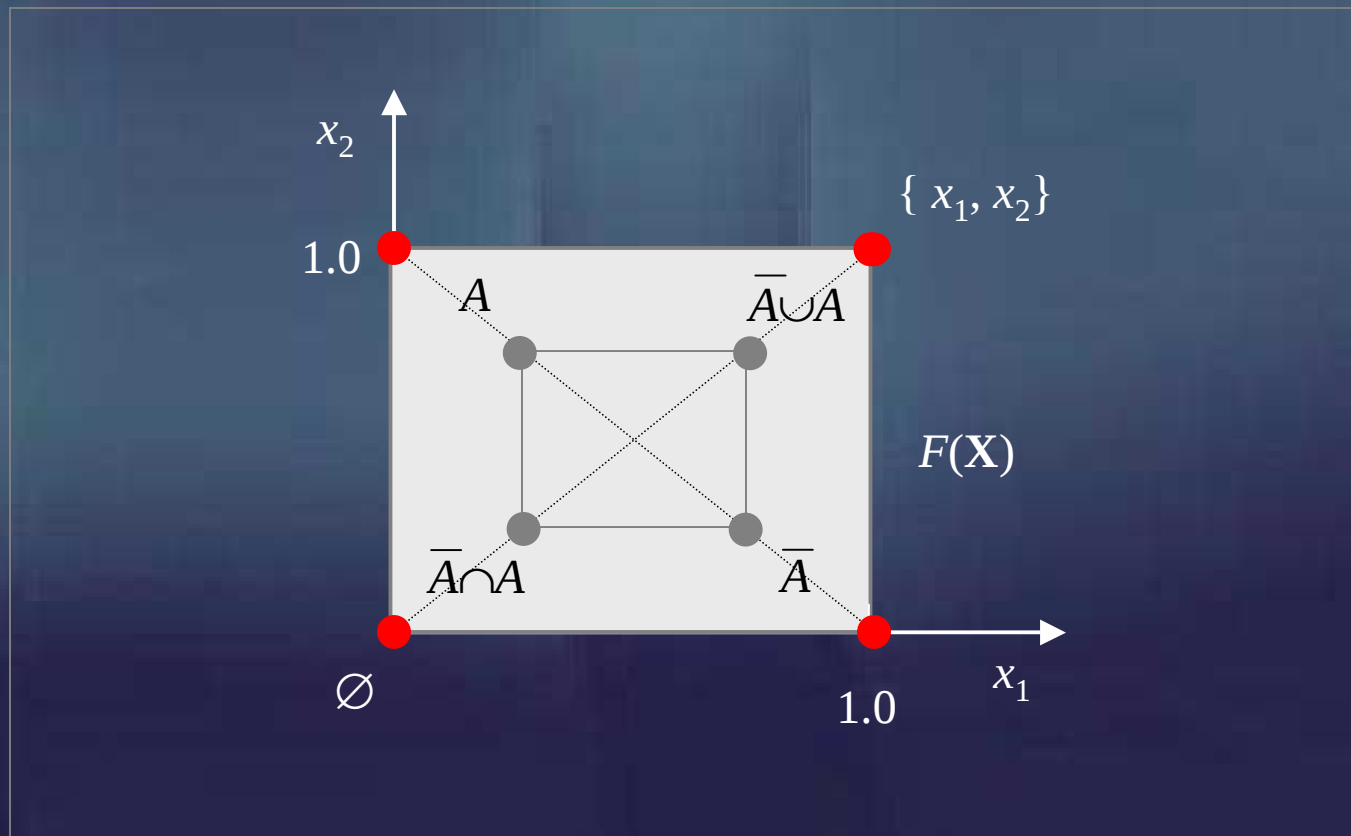
Basic properties of sets and fuzzy sets

1 Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
2 Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$
	$A \cap (B \cap C) = (A \cap B) \cap C$
3 Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4 Idempotency	$A \cup A = A$
	$A \cap A = A$
5 Boundary Conditions	$A \cup \phi = A$ and $A \cup X = X$
	$A \cap \phi = \phi$ and $A \cap X = A$
6 Involution	$\overline{\overline{A}} = A$
7 Transitivity	if $A \subset B$ and $B \subset C$ then $A \subset C$

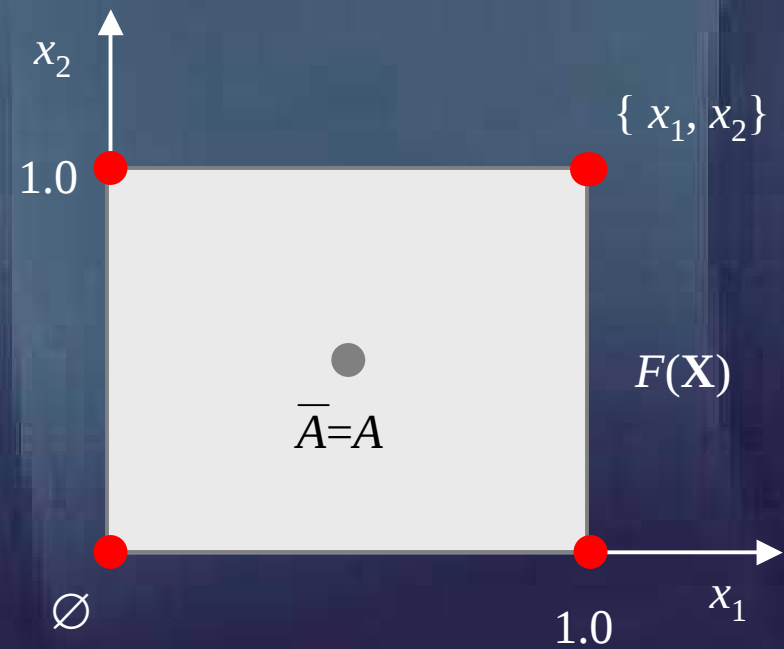
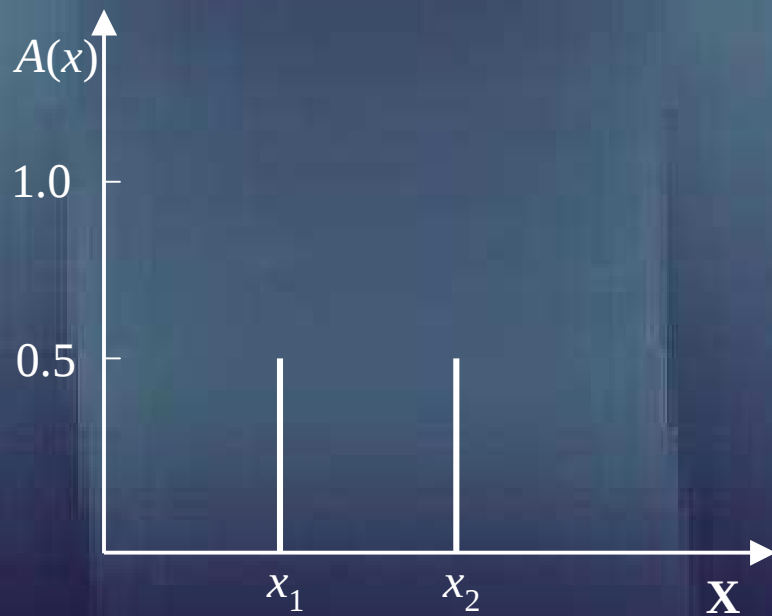
Noncontradiction and excluded middle for standard operations

	Sets	Fuzzy sets
8-Noncontradiction	$\bar{A} \cap A = \emptyset$	$\bar{A} \cap A \neq \emptyset$
9-Excluded middle	$\bar{A} \cup A = \mathbf{X}$	$\bar{A} \cup A \neq \mathbf{X}$

Geometric view of standard operations



Example



5.2 Generic requirements for operations on fuzzy sets

Set operation is a binary operator

$$[0,1] \times [0,1] \rightarrow [0,1]$$

- Requirements:

- commutativity
- associativity
- identity

- Identity:

- its form depends on the operation

5.3 Triangular norms

Definition

$$t : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity: $a \, t \, b = b \, t \, a$
- Associativity: $a \, t \, (b \, t \, c) = (a \, t \, b) \, t \, c$
- Monotonicity: if $b \leq c$ then $a \, t \, b \leq a \, t \, c$
- Boundary conditions: $a \, t \, 1 = a$
 $a \, t \, 0 = 0$

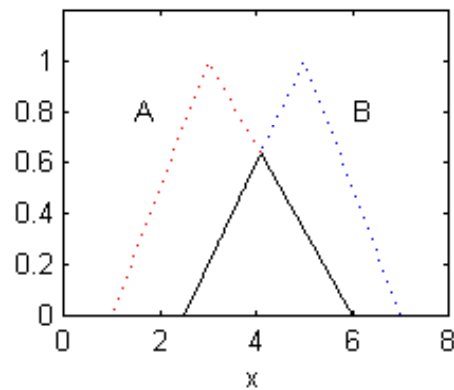
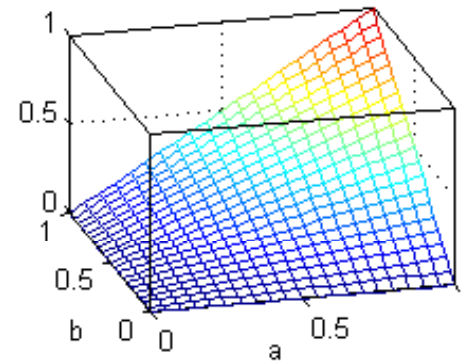
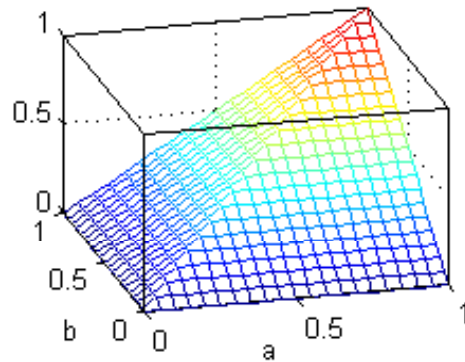
Examples

$$a \, t_m \, b = \min(a, b)$$

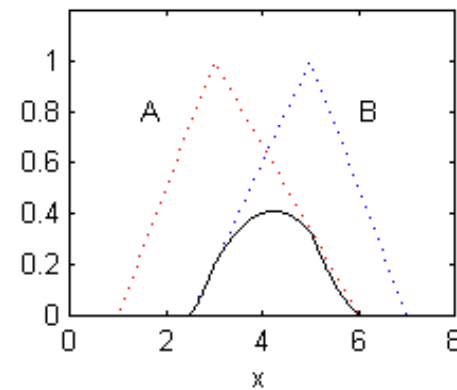
$$a \, t_p \, b = ab$$

minimum

product



(a)

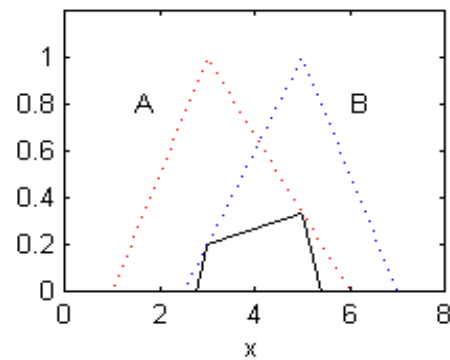
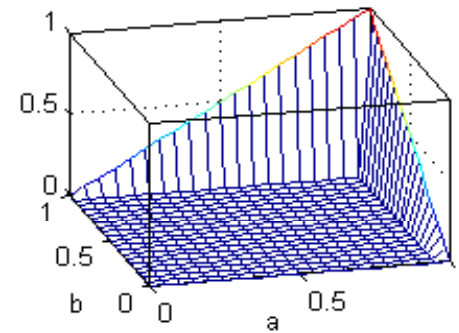
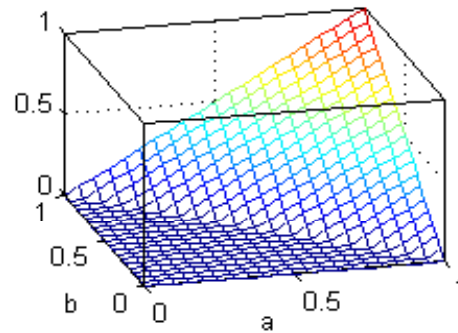


(b)

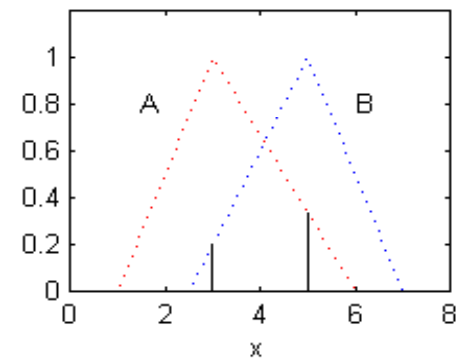
Lukasiewicz

$$a \text{ } t_l \text{ } b = \max(a+b-1, 0)$$

$$a \text{ } t_d \text{ } b = \begin{cases} a & \text{if } b=1 \\ b & \text{if } a=1 \\ 0 & \text{otherwise} \end{cases}$$



(c)



(d)

drastic
product

$$at_d b \leq atb \leq \min(a, b)$$

lower and upper bounds

$$ata \leq a$$

Archimedean (if t is continuous)

$$a^n = 0$$

nilpotent (e.g. Lukasiewicz)

$$a t a = a^2, \dots, a^{n-1} t a = a^n$$

Constructors of t-norms

- Monotonic function transformation
- Additive and multiplicative generators
- Ordinal sums

Monotonic function transformation

- If $h: [0,1] \rightarrow [0,1]$ is a strictly increasing bijection

then $t_h(a,b) = h^{-1}(t(h(a),h(b)))$ is a t-norm

- h is a scaling transformation

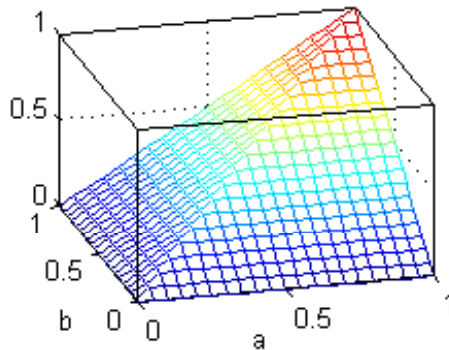
Obs: h bijective means both, h injective (one-to-one)
and h surjective (onto)

Examples

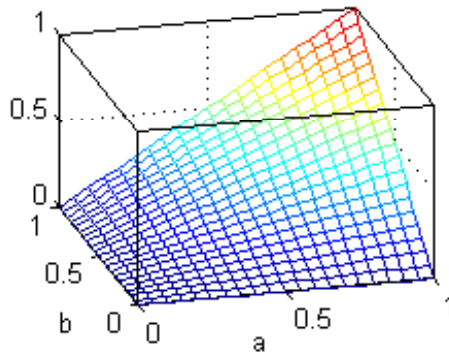
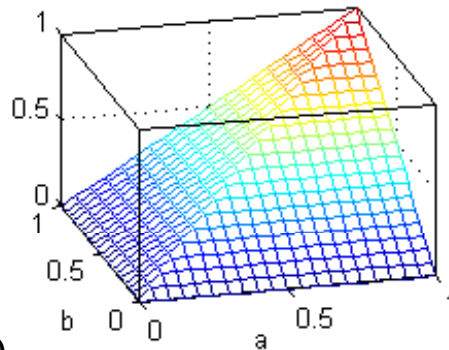
$$h = x^2, x \in [0,1]$$

t

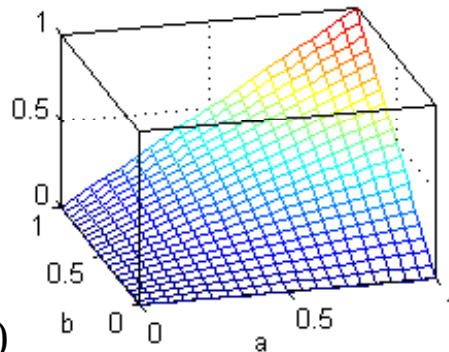
t_h



(a)



(b)



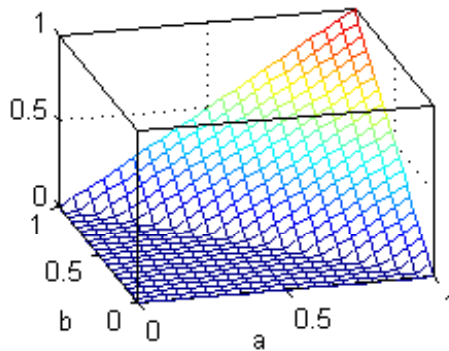
minimum

product

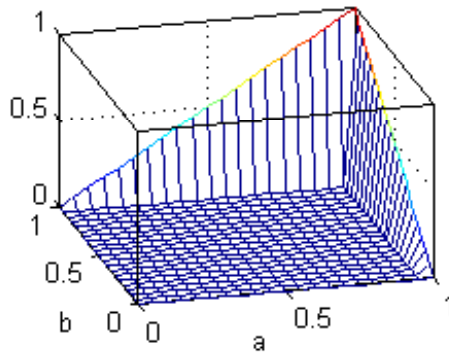
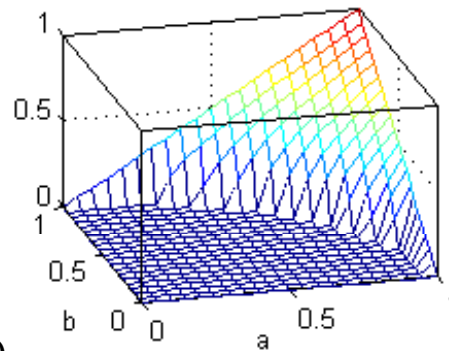
$$h = x^2, x \in [0,1]$$

t

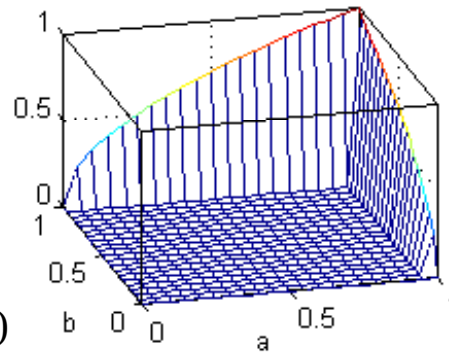
t_h



(c)



(d)



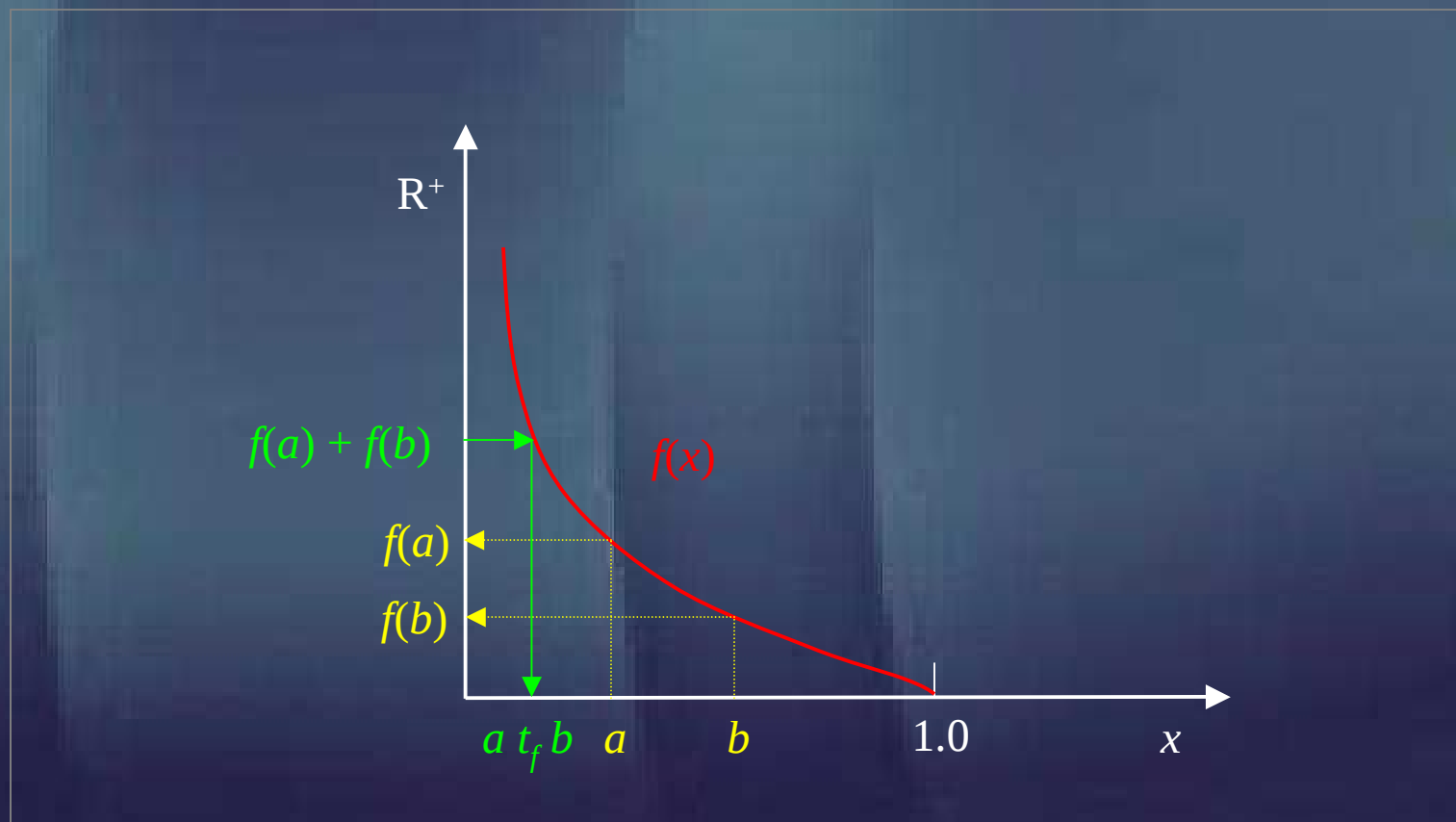
Lukasiewicz

drastic product

Additive generators

- $f: [0,1] \rightarrow [0, \infty), f(1) = 0$
 - continuous
 - strictly decreasing
- $a \, t_f \, b = f^{-1}(f(a) + f(b)) \Leftrightarrow$ is a Archimedean t-norm

Additive generators of t-norms



$$\mathbb{R}^+ \equiv [0, \infty),$$

Example

$$f = -\log(x)$$

$$\begin{aligned} f(a) + f(b) &= -\log(a) - \log(b) \\ &= -(\log(a) + \log(b)) \\ &= -\log(ab) \end{aligned}$$

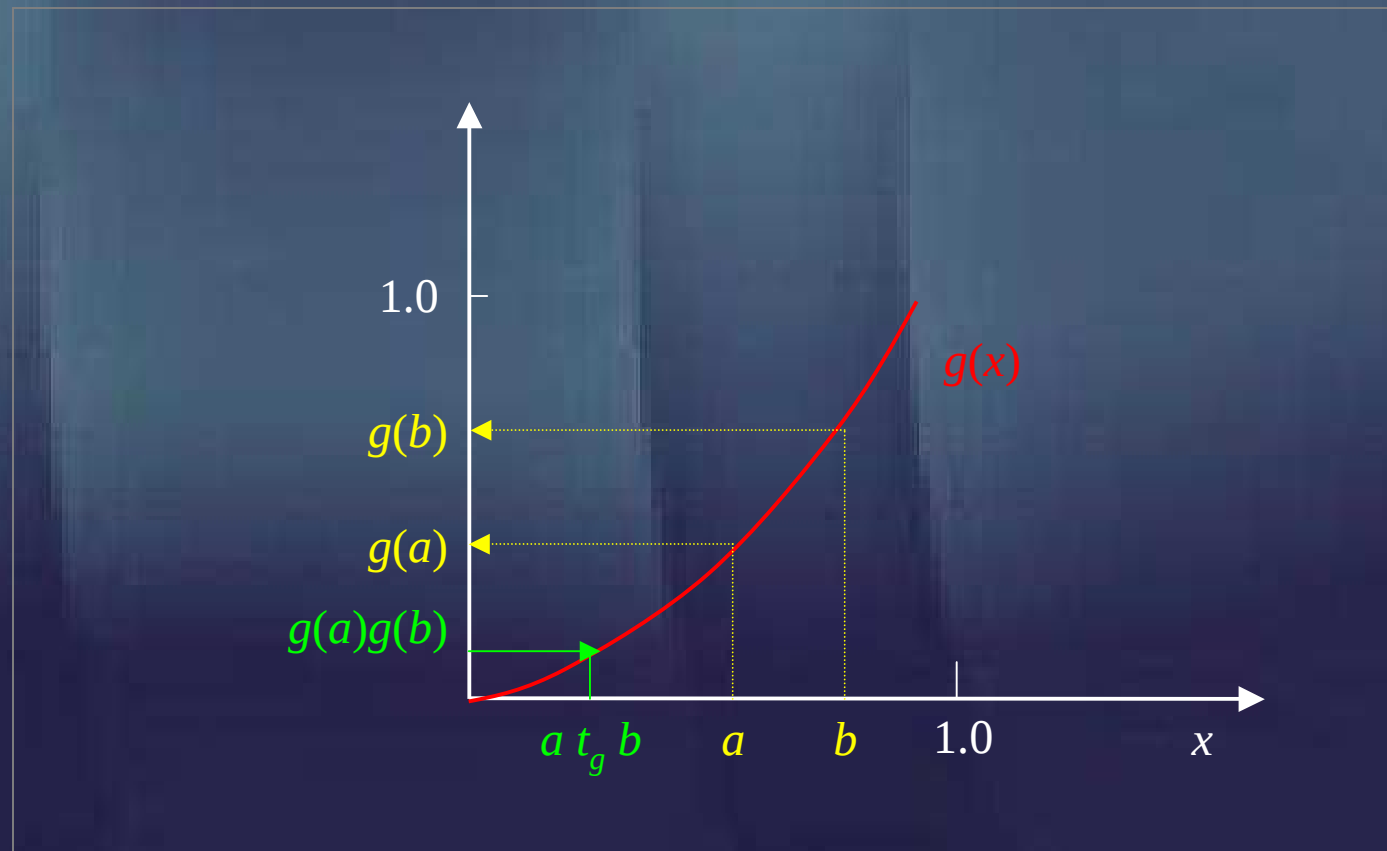
$$f^{-1}(f(a) + f(b)) = e^{\log(ab)} = ab$$

- $a \, t_f \, b = ab$ (Archimedean t-norm)

Multiplicative generators

- $g: [0,1] \rightarrow [0, 1], g(1) = 1$
 - continuous
 - strictly increasing
- $a \, t_g \, b = g^{-1}(g(a)g(b)) \Leftrightarrow$ is a Archimedean t-norm

Multiplicative generators of t-norms



Example

- $g = x^2$

$$a \, t_f \, b = ab \quad (\text{Archimedean t-norm})$$

- $g = e^{-f(x)}$

multiplicative and additive generators $\rightarrow$ same t-norm

Ordinal sums

$$t_o: [0,1] \rightarrow [0, 1]$$

denoted $t_o = (\langle \alpha_k, \beta_k, t_k \rangle, k \in K)$

$$t_o(a,b,I,\tau) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)t_k\left(\frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k}\right) & \text{if } a, b \in [\alpha_k, \beta_k] \\ \min(a, b) & \text{otherwise} \end{cases}$$

$$I = \{[\alpha_k, \beta_k], k \in K\}$$

- nonempty, countable family
- pairwise disjoint subintervals of $[0,1]$

$$\tau = \{t_k, k \in K\}$$

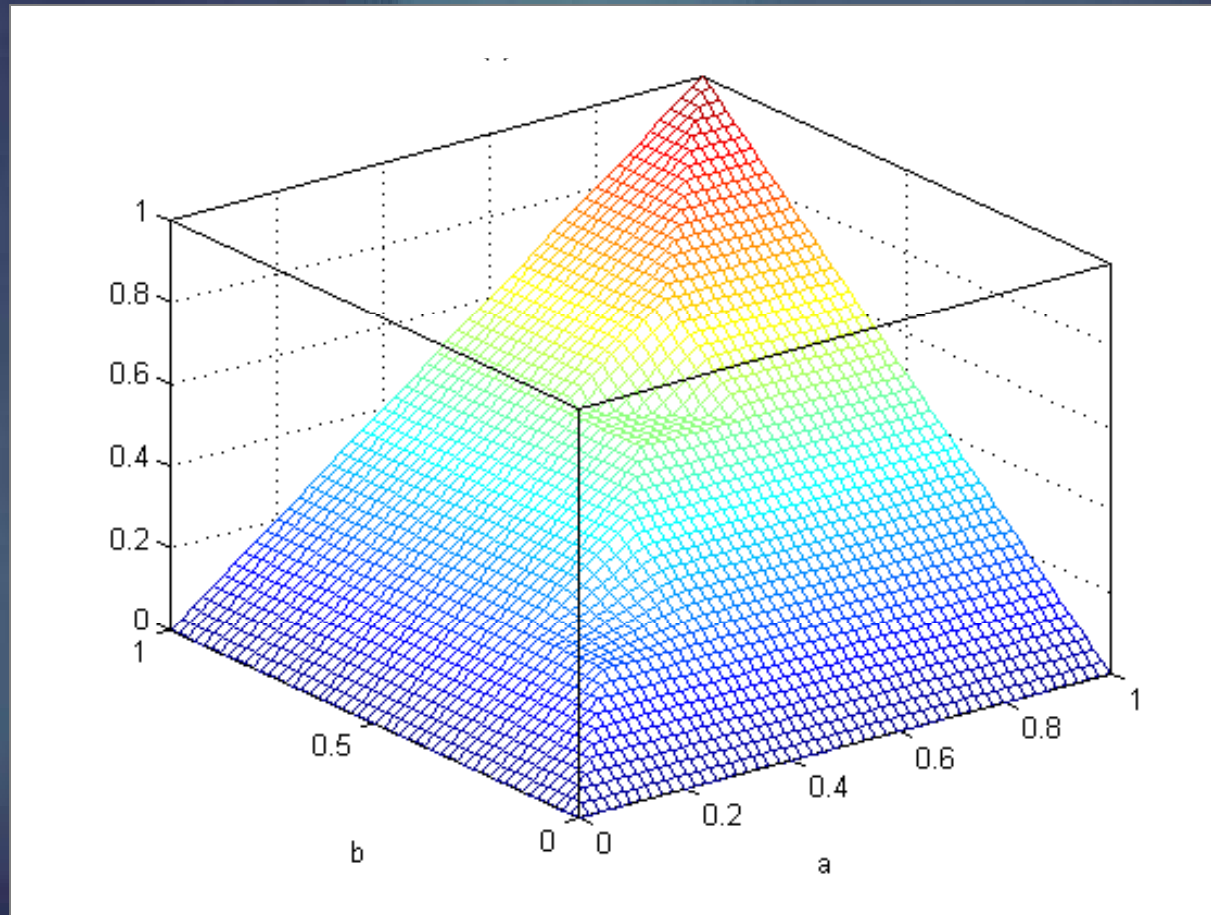
- family of t-norms

Example

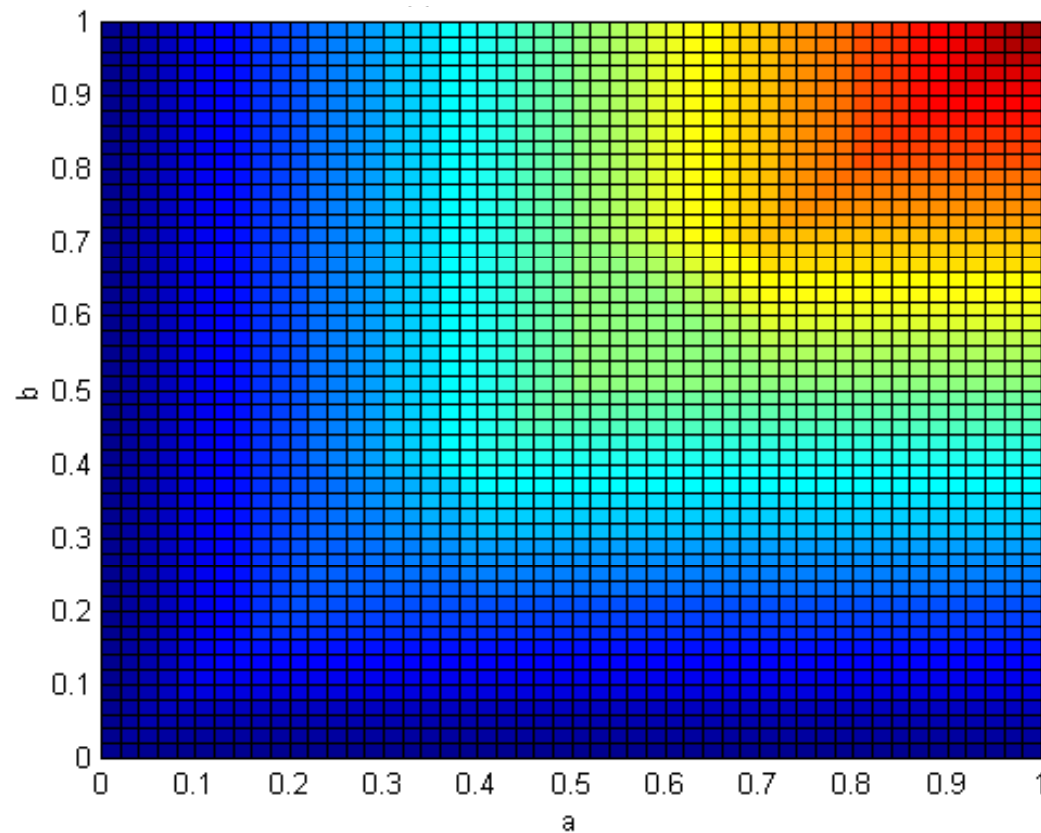
$$t_o(a,b,I,\tau) = \begin{cases} 0.2 + 5(a - 0.2)(b - 0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + \max(a + b - 1.2, 0) & \text{if } a,b \in [0.5,0.7] \\ \min(a,b) & \text{otherwise} \end{cases}$$

$$I = \{[0.2, 0.4], [0.5, 0.7]\} \quad K = \{1,2\}$$

$$\tau = \{t_p, t_1\}, t_1 = t_p, t_2 = t_1$$



$$t_0(a, b, I, \tau) = \begin{cases} 0.2 + 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + \max(a + b - 1.2, 0) & \text{if } a, b \in [0.5, 0.7] \\ \min(a, b) & \text{otherwise} \end{cases}$$



$$t_o(a,b,I,\tau) = \begin{cases} 0.2 + 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + \max(a + b - 1.2, 0) & \text{if } a, b \in [0.5, 0.7] \\ \min(a, b) & \text{otherwise} \end{cases}$$

5.4 Triangular conorms

Definition

$$s : [0,1] \times [0,1] \rightarrow [0,1]$$

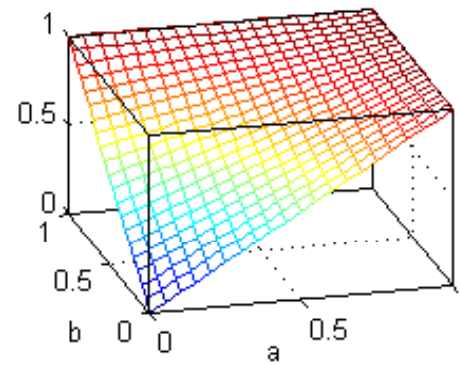
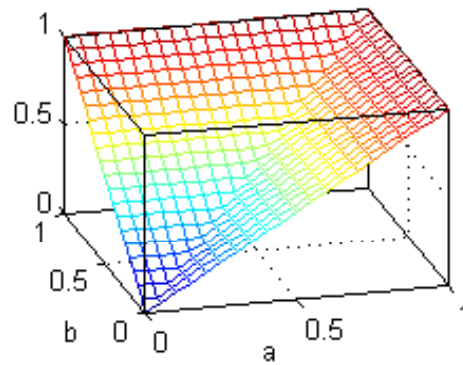
- Commutativity: $a \, s \, b = b \, s \, a$
- Associativity: $a \, s \, (b \, s \, c) = (a \, s \, b) \, s \, c$
- Monotonicity: if $b \leq c$ then $a \, s \, b \leq a \, s \, c$
- Boundary conditions: $a \, s \, 1 = 1$
 $a \, s \, 0 = a$

Examples

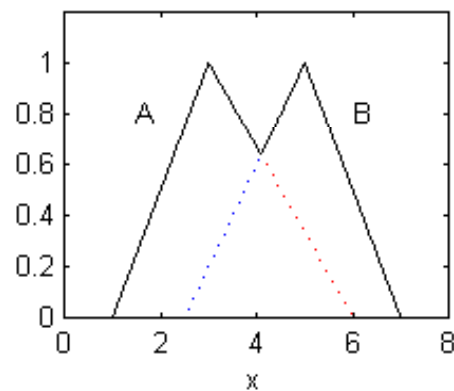
$$a \text{ s}_m b = \max(a, b)$$

$$a \text{ s}_p b = a + b - ab$$

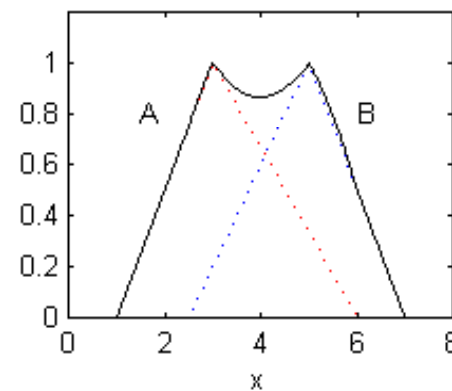
maximum



probabilistic
sum



(a)

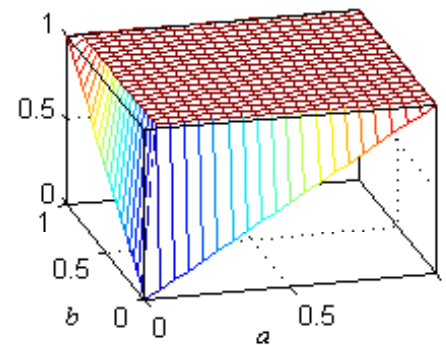
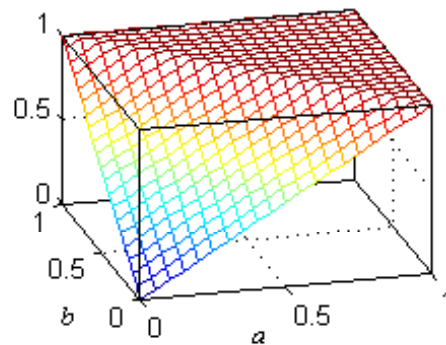


(b)

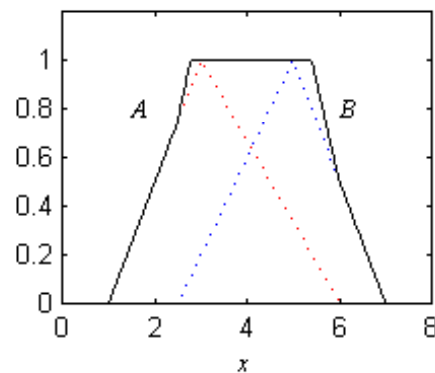
Lukasiewicz

$$a \text{ s}_l b = \max(a+b, 1)$$

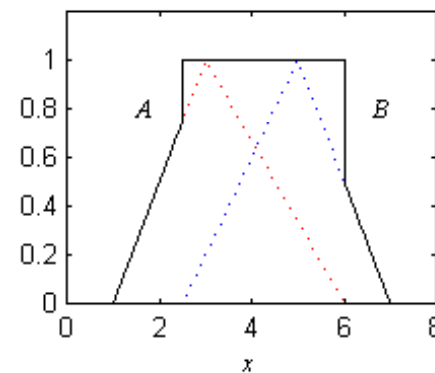
$$a \text{ s}_d b = \begin{cases} a & \text{if } b=0 \\ b & \text{if } a=0 \\ 1 & \text{otherwise} \end{cases}$$



drastic
sum



(c)



(d)

$$\max(a, b) \leq asb \leq as_d b$$

lower and upper bounds

$$asa \geq a$$

Archimedean (if t is continuous)

$$a^n = 1$$

nilpotent (e.g. Lukasiewicz)

$$a s a = a^2, \dots, a^{n-1} s a = a^n$$

Dual norms and De Morgan laws

$$a \, s \, b = 1 - (1 - a) \, t \, (1 - b)$$

$$a \, t \, b = 1 - (1 - a) \, s \, (1 - b)$$

Dual triangular norms

$$(1 - a) \, t \, (1 - b) = 1 - a \, s \, b$$

$$(1 - a) \, s \, (1 - b) = 1 - a \, t \, b$$

De Morgan

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Constructors of s-norms

- Monotonic function transformation
- Additive and multiplicative generators
- Ordinal sums

Monotonic function transformation

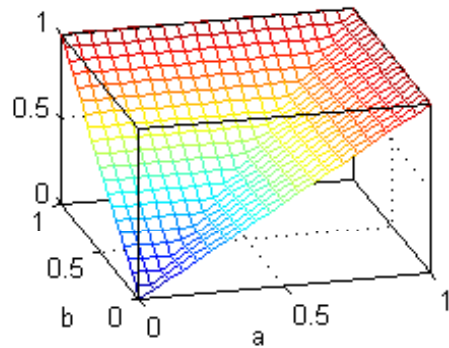
- If $h: [0,1] \rightarrow [0,1]$ is a strictly increasing bijection
then $s_h(a,b) = h^{-1}(s(h(a),h(b)))$ is a t-norm
- h is a scaling transformation

Examples

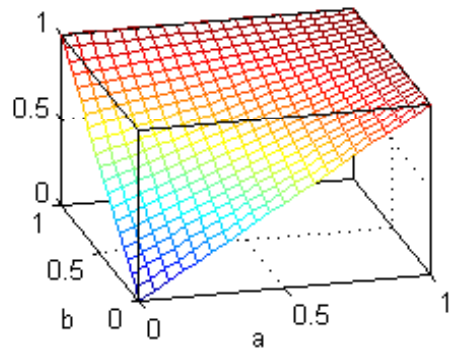
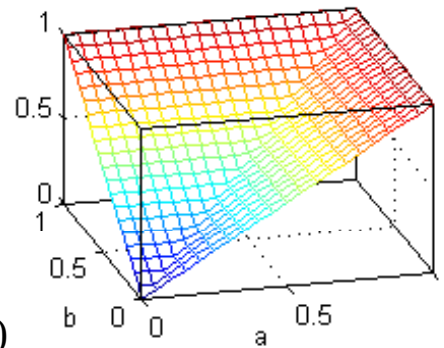
$$h = x^2, x \in [0,1]$$

S

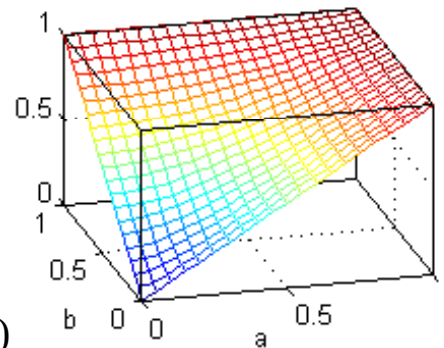
S_h



(a)



(b)



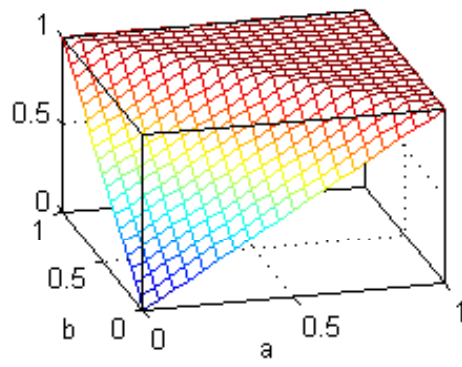
maximum

probabilistic
sum

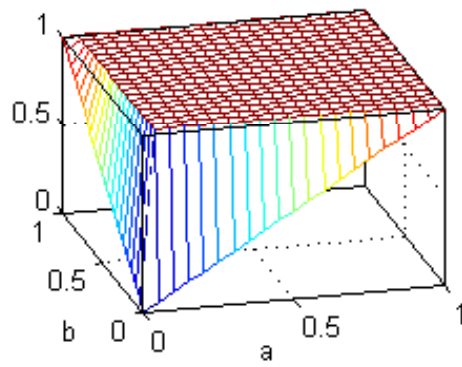
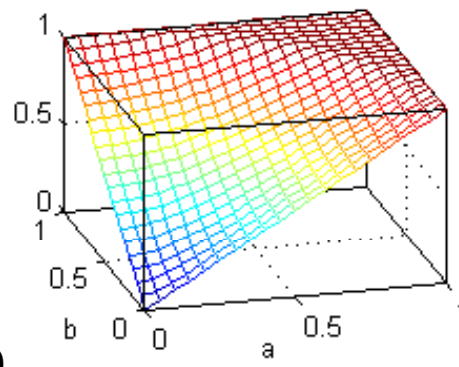
$$h = x^2, x \in [0,1]$$

S

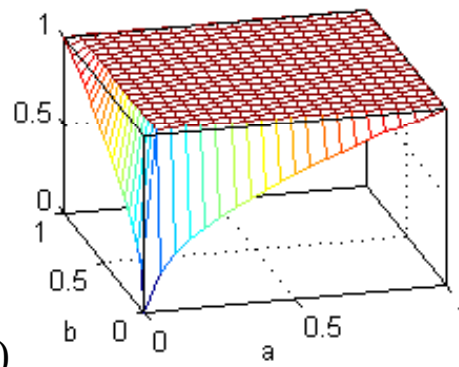
S_h



(c)



(d)



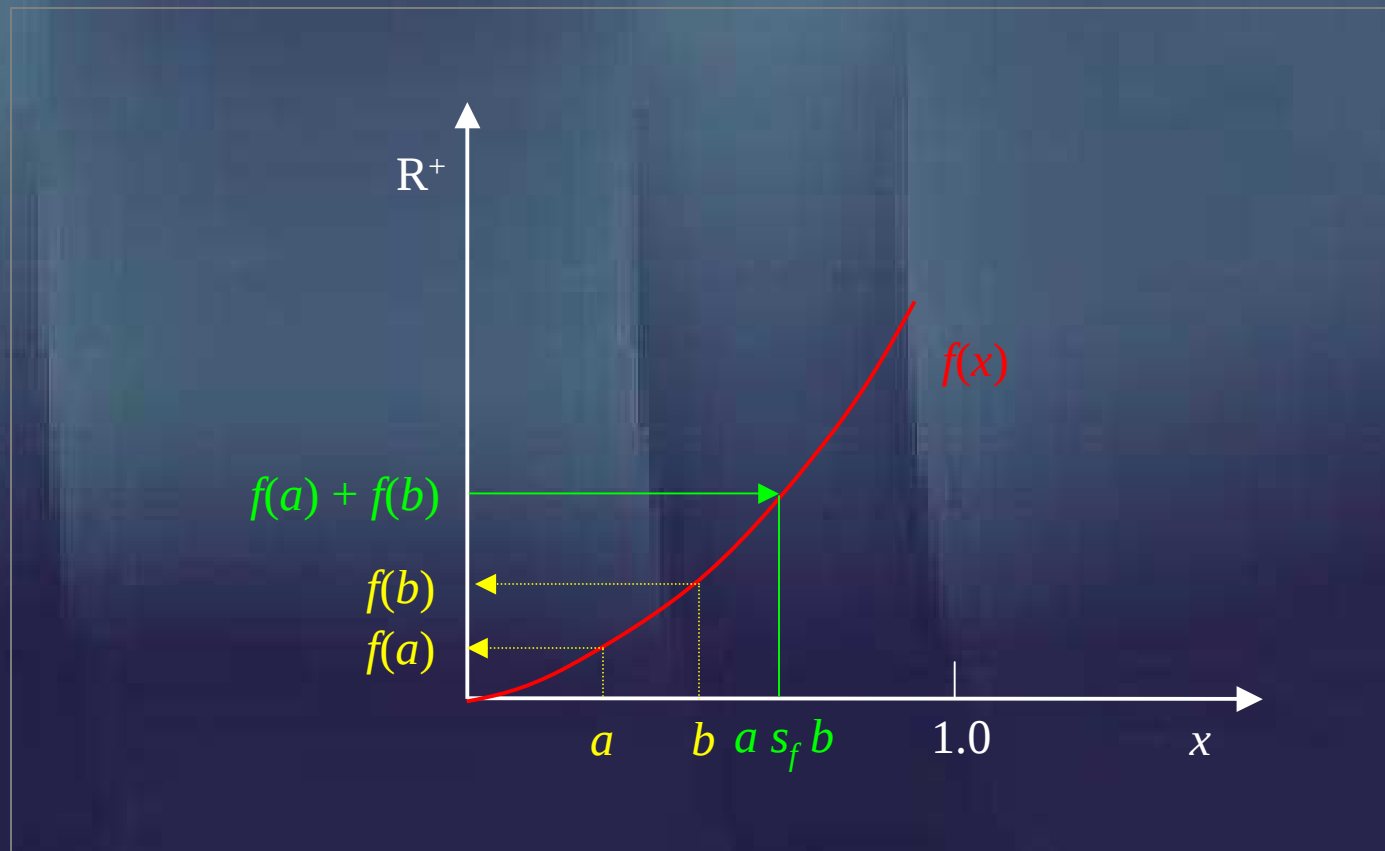
Lukasiewicz

drastic sum

Additive generators

- $f: [0,1] \rightarrow [0, \infty), f(0) = 0$
 - continuous
 - strictly increasing
- $a s_f b = f^{-1}(f(a) + f(b)) \Leftrightarrow$ is a Archimedean t-conorm

Additive generators of t-conorms



Example

$$f = -\log(1 - x)$$

$$\begin{aligned} f(a) + f(b) &= -\log(1 - a) - \log(1 - b) \\ &= -(\log(1 - a) + \log(1 - b)) \\ &= -\log(1 - a)(1 - b) \end{aligned}$$

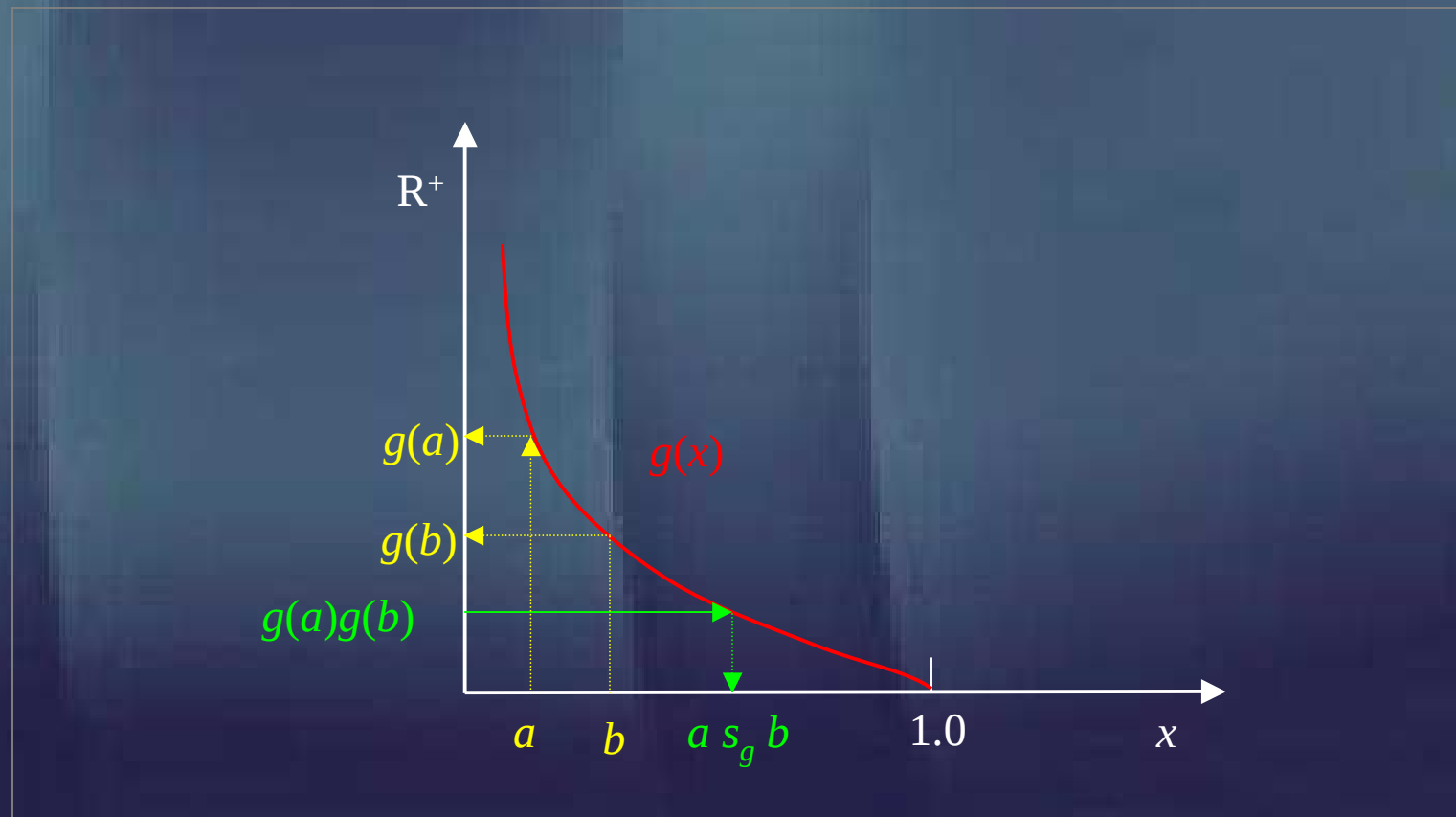
$$f^{-1}(f(a) + f(b)) = 1 - e^{\log(1 - a)(1 - b)} = a + b - ab$$

$$\blacksquare a \, s_f \, b = a + b - ab \quad (\text{Archimedean t-conorm})$$

Multiplicative generators

- $g: [0,1] \rightarrow [0, 1], g(0) = 1$
 - continuous
 - strictly decreasing
- $a s_g b = g^{-1}(g(a)g(b)) \Leftrightarrow$ is a Archimedean t-norm

Multiplicative generators of t-conorms



$$\mathbb{R}^+ \equiv [0, \infty),$$

Example

- $g = 1 - x$

$$a \, s_g \, b = a + b - ab \quad (\text{Archimedean t-norm})$$

- $g = e^{-f(x)}$

multiplicative and additive generators $\rightarrow$ same t-conorm

Ordinal sums

$$s_o: [0,1] \rightarrow [0, 1]$$

$$\text{denoted } s_o = (\langle \alpha_k, \beta_k, s_k \rangle, k \in K)$$

$$s_o(a,b,I,\sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) s_k \left(\frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } a, b \in [\alpha_k, \beta_k] \\ \max(a, b) & \text{otherwise} \end{cases}$$

$$I = \{[\alpha_k, \beta_k], k \in K\}$$

- nonempty, countable family
- pairwise disjoint subintervals of $[0,1]$

$$\sigma = \{s_k, k \in K\}$$

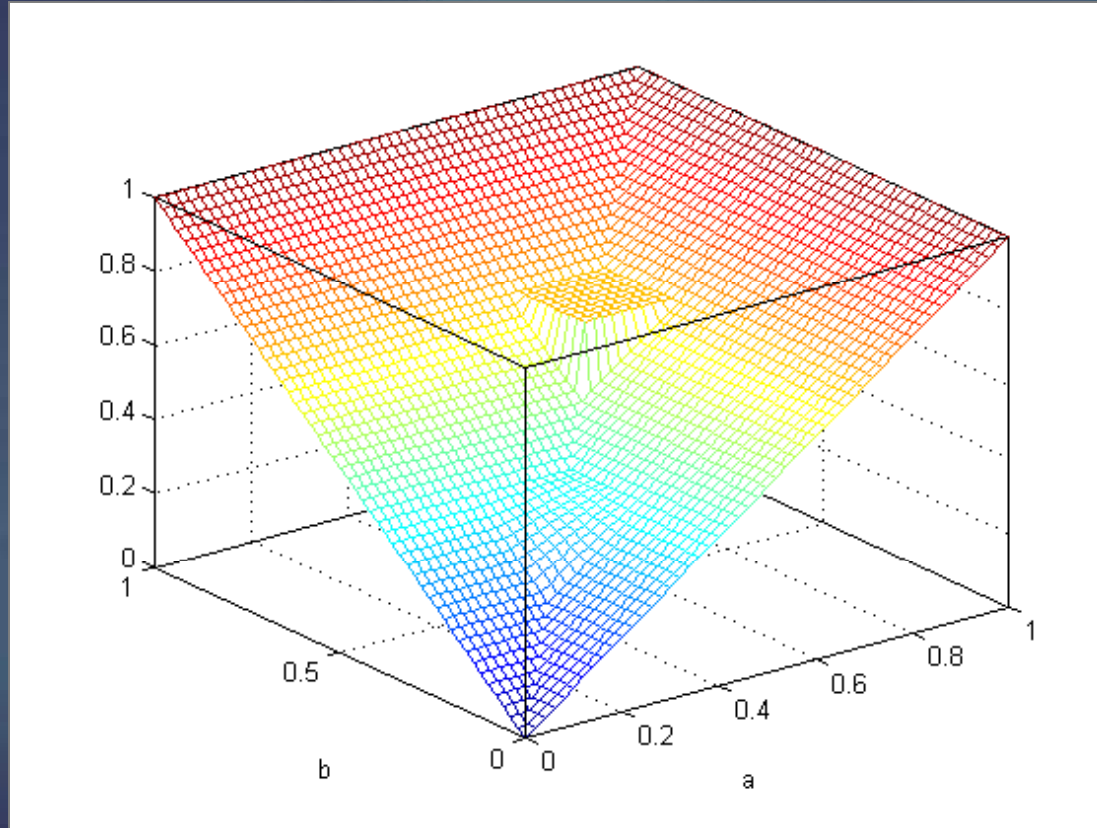
- family of t-conorms

Example

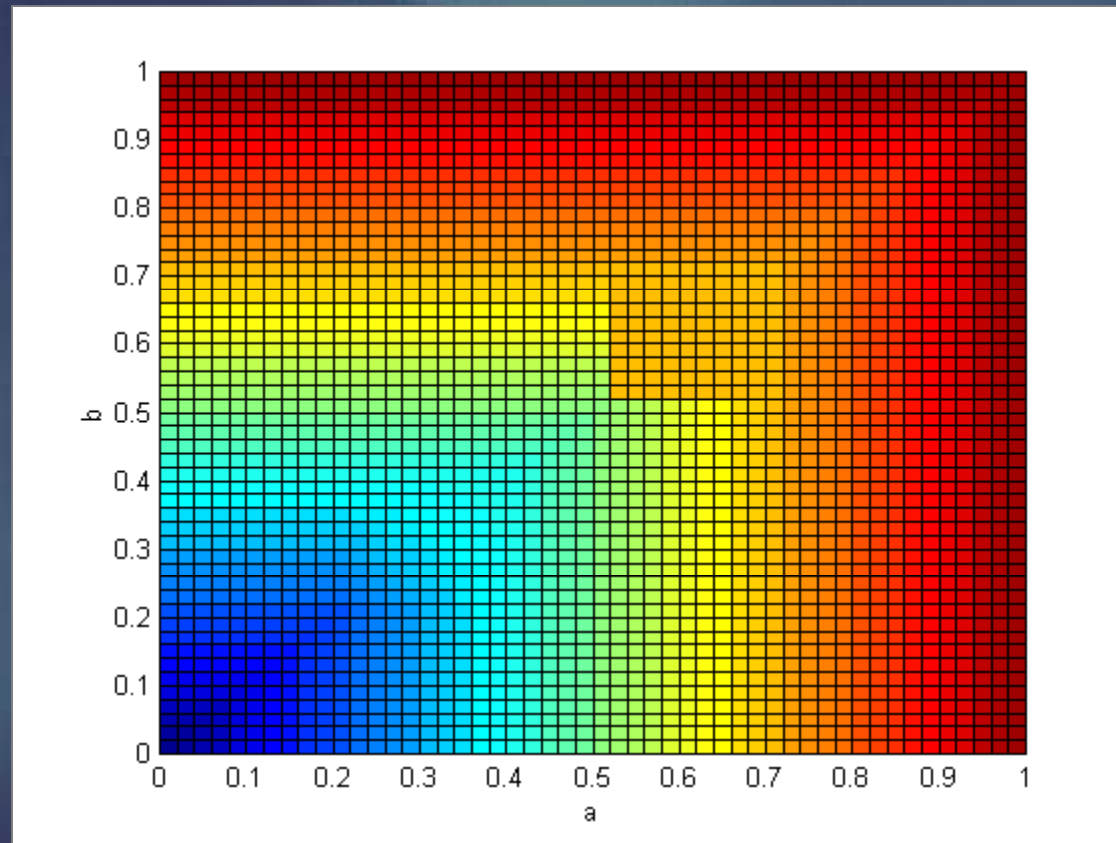
$$s_o(a,b,I,\sigma) = \begin{cases} 0.2 + (a - 0.2) + (b - 0.2) - 5(a-0.2)(b-0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + 0.2\min(5(a - 0.2) + 5(b - 0.2), 1) & \text{if } a,b \in [0.5,0.7] \\ \max(a,b) & \text{otherwise} \end{cases}$$

$$I = \{[0.2, 0.4], [0.5, 0.7]\} \quad K = \{1,2\}$$

$$\sigma = \{s_p, s_l\}, s_1 = s_p, s_2 = s_l$$



$$s_o(a,b,I,\sigma) = \begin{cases} 0.2 + (a - 0.2) + (b - 0.2) - 5(a-0.2)(b-0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + 0.2\min(5(a - 0.2) + 5(b - 0.2), 1) & \text{if } a,b \in [0.5,0.7] \\ \max(a,b) & \text{otherwise} \end{cases}$$



$$s_o(a, b, I, \sigma) = \begin{cases} 0.2 + (a - 0.2) + (b - 0.2) - 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + 0.2 \min(5(a - 0.2) + 5(b - 0.2), 0) & \text{if } a, b \in [0.5, 0.7] \\ \max(a, b) & \text{otherwise} \end{cases}$$

5.5 Triangular norms as general category of logical operators

Motivation

- **Fuzzy propositions involves linguistic statements:**
 - temperature is low *and* humidity is high
 - velocity is high *or* noise level is low
- **Logical operations:**
 - and ($\wedge$)
 - or ($\vee$)

Truth value assignment

$L = \{P, Q, \dots\}$ $P, Q, \dots$ atomic statements

$\text{truth}: L \rightarrow [0, 1]$ $p, q, \dots \in [0, 1]$

$\text{truth}(P \text{ and } Q) = \text{truth}(P \wedge Q) \rightarrow p \wedge q = p \text{ } t \text{ } q$

$\text{truth}(P \text{ or } Q) = \text{truth}(P \vee Q) \rightarrow p \vee q = p \text{ } s \text{ } q$

Examples

p	q	$\min(p, q)$	$\max(p, q)$	pq	$p + q - pq$
1	1	1	1	1	1
1	0	0	1	0	1
0	1	0	1	0	1
0	0	0	0	0	0
0.2	0.5	0.2	0.5	0.1	0.6
0.5	0.8	0.5	0.8	0.4	0.9
0.8	0.7	0.7	0.8	0.56	0.94

Implication induced by a t-norm

$$a \varphi b \equiv a \Rightarrow b$$

$$a \varphi b = \sup \{ c \in [0, 1] \mid a t c \leq b \} \quad \forall a, b \in [0, 1] \quad \text{residuation}$$

φ operator or Boolean values of its arguments

a	b	$a \Rightarrow b$	$a \varphi b$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

5.6 Aggregation operations

Definition

$$g : [0,1]^n \rightarrow [0,1]$$

1. Monotonicity:

$$g(x_1, x_2, \dots, x_n) \geq g(y_1, y_2, \dots, y_n) \text{ if } x_i \geq y_i, i = 1, \dots, n$$

2. Boundary conditions:

$$g(0, 0, \dots, 0) = 0$$

$$g(1, 1, \dots, 1) = 1$$

1. Neutral element (e):

$$g(x_1, x_2, \dots, x_{i-1}, e, x_{i+1}, \dots, x_n) = g(x_1, x_2, \dots, x_{i-1}, e, x_{i+1}, \dots, x_n) \quad n \geq 2$$

2. Annihilator (l):

$$g(x_1, x_2, \dots, x_{i-1}, l, x_{i+1}, \dots, x_n) = l$$

Observation: Annihilator $\equiv$ absorbing element

Averaging operations

$$g(x_1, x_2, \dots, x_n) = \sqrt[p]{\frac{1}{n} \sum_{i=1}^n (x_i)^p}, \quad p \in \mathbf{R}, \quad p \neq 0$$

$$p = 1 \quad g(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i$$

arithmetic mean

$$p \rightarrow 0 \quad g(x_1, x_2, \dots, x_n) = \sqrt[n]{\prod_{i=1}^n x_i}$$

geometric mean

$$p = -1 \quad g(x_1, x_2, \dots, x_n) = \frac{n}{\sum_{i=1}^n 1/x_i}$$

harmonic mean

$$p \rightarrow -\infty \quad g(x_1, x_2, \dots, x_n) = \min(x_1, x_2, \dots, x_n)$$

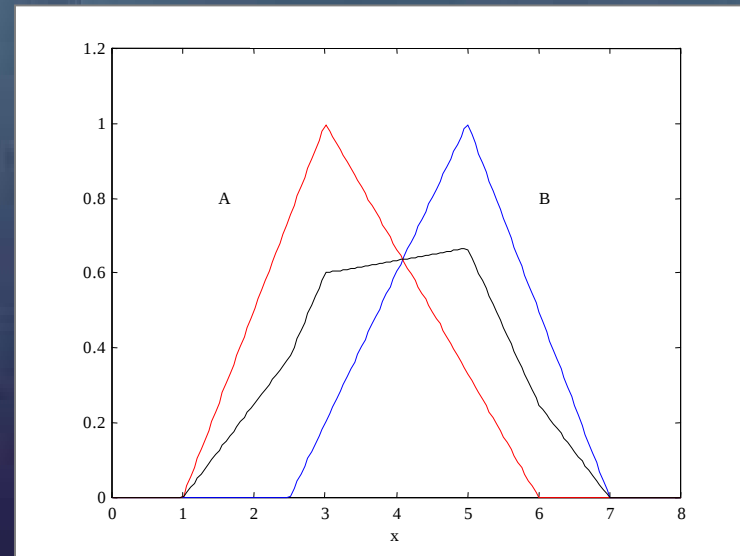
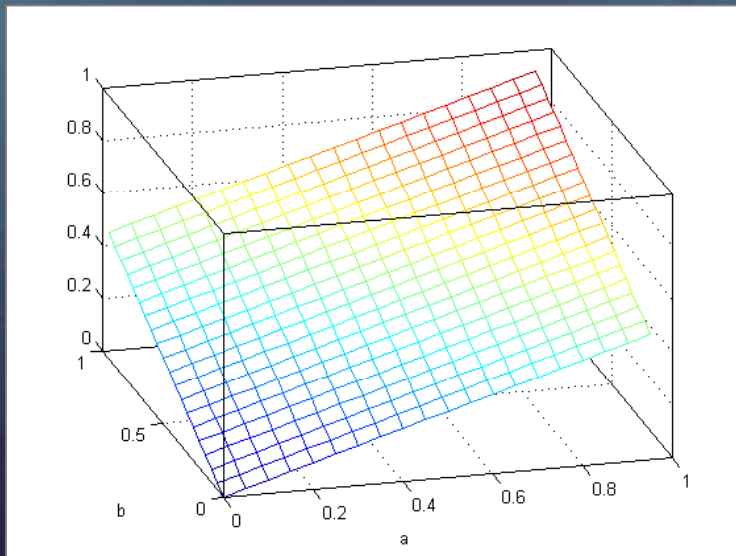
$$p \rightarrow \infty \quad g(x_1, x_2, \dots, x_n) = \max(x_1, x_2, \dots, x_n)$$

Bounds

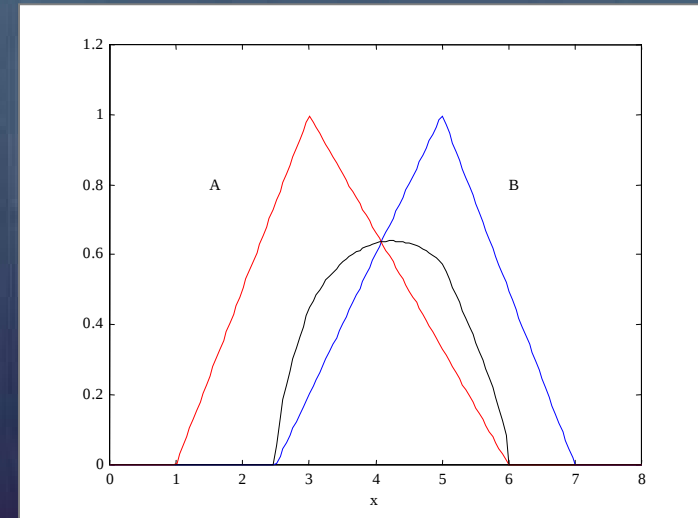
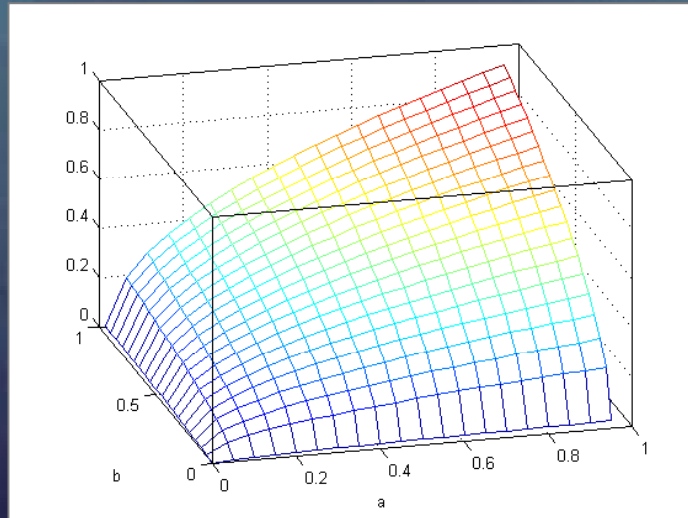
$$\min(x_1, x_2, \dots, x_n) \leq g(x_1, x_2, \dots, x_n) \leq \max(x_1, x_2, \dots, x_n)$$

Examples

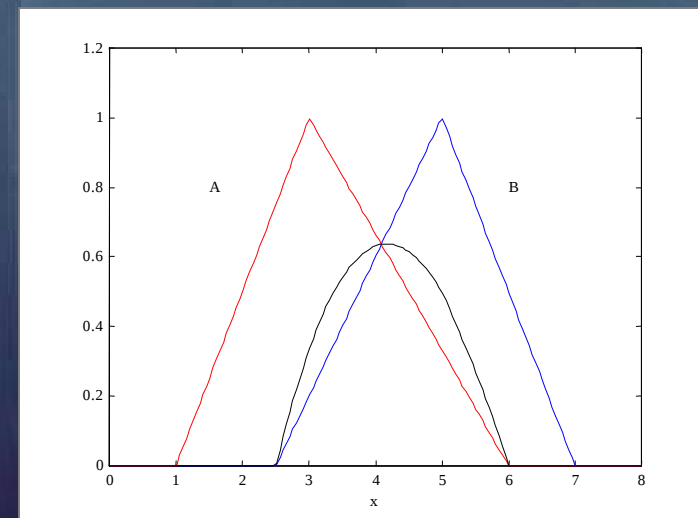
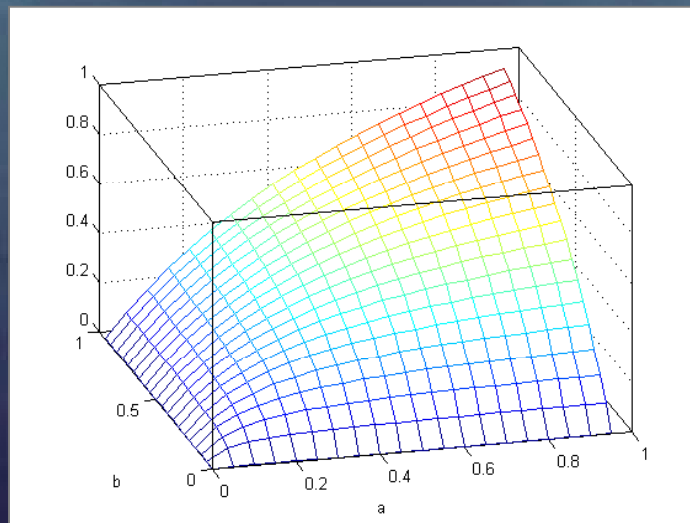
Arithmetic mean



Geometric mean



Harmonic mean



Ordered Weighted Averaging (OWA)

$$\text{OWA}(A, \mathbf{w}) = \sum_{i=1}^n w_i A(x_i)$$

$$\sum w_i = 1, \quad w_i \in [0, 1]$$

$$A(x_1) \leq A(x_2) \leq \dots \leq A(x_n)$$

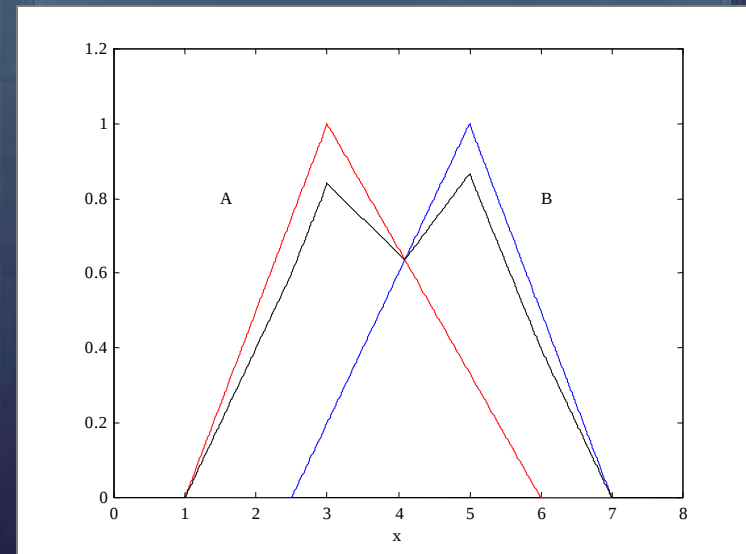
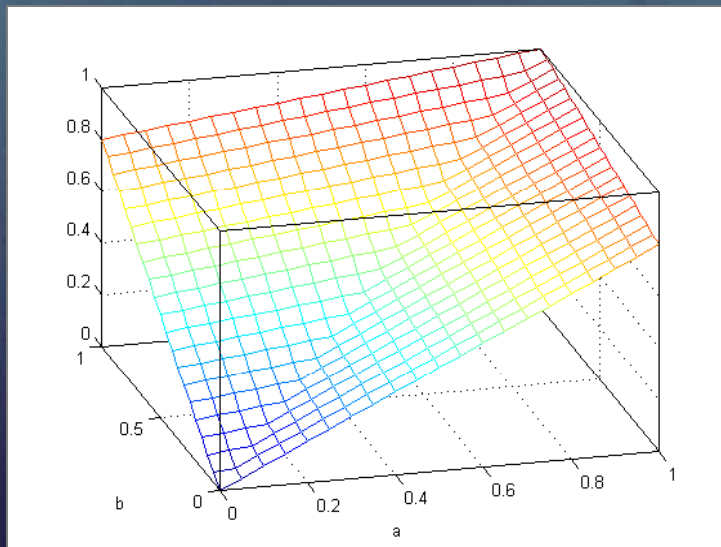
Examples

1. $\mathbf{w} = [1, 0, \dots, 0]$ $\text{OWA}(A, \mathbf{w}) = \min (A(x_1), A(x_2), \dots, A(x_n))$
2. $\mathbf{w} = [0, 0, \dots, 1]$ $\text{OWA}(A, \mathbf{w}) = \max (A(x_1), A(x_2), \dots, A(x_n))$
3. $\mathbf{w} = [1/n, 1/n, \dots, 1/n]$ $\text{OWA}(A, \mathbf{w}) = \text{arithmetic mean}$

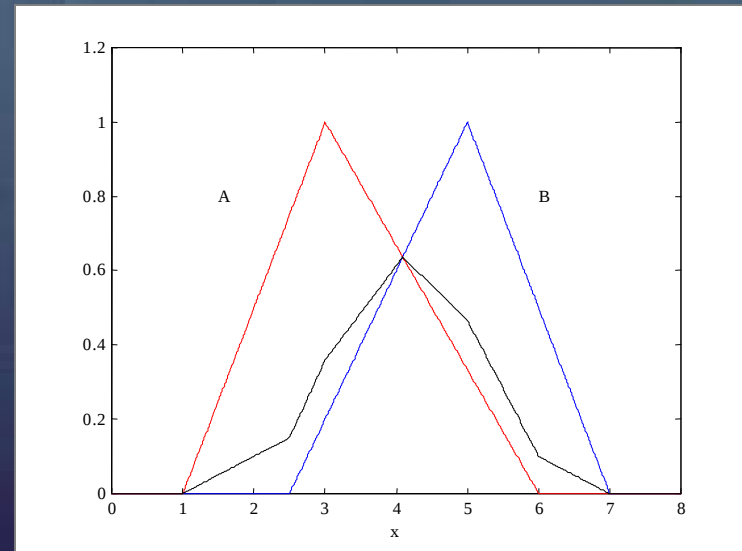
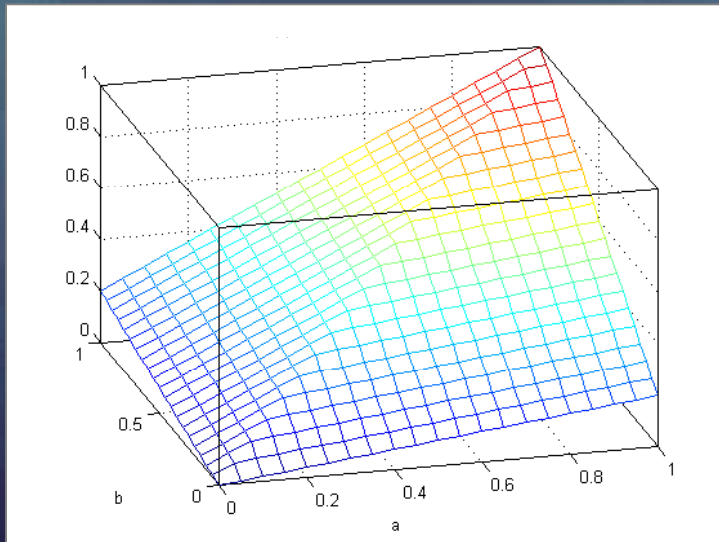
$$\min (A(x_1), A(x_2), \dots, A(x_n)) \leq \text{OWA}(A, \mathbf{w}) \leq \max (A(x_1), A(x_2), \dots, A(x_n))$$

Examples

$$\mathbf{w} = [0.8, 0.2]$$



$$\mathbf{w} = [0.2, 0.8]$$



Uninorms

$$u : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity:

$$a \, u \, b = b \, u \, a$$

- Associativity:

$$a \, u \, (b \, u \, c) = (a \, u \, b) \, u \, c$$

- Monotonicity:

$$\text{if } b \leq c \text{ then } a \, u \, b \leq a \, u \, c$$

- Identity:

$$a \, u \, e = a \quad \forall a \in [0, 1]$$

$e \in [0, 1]$, $e = 1$ u is a t-norm, $e = 0$ u is a t-conorm

Results on uninorms

1. $t_u, s_u: [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $\forall e \in [0, 1]$

$$at_ub = \frac{(ea)u(eb)}{e}$$

$$as_ub = \frac{(e + (1-e)a)u(e + (1-e)b) - e}{1-e}$$

t_u and s_u are t-norm and t-conorm

2. If $a \leq e \leq b$ or $a \geq e \geq b$ then

If $a \leq e \leq b$ or $a \geq e \geq b$ then
 $\min(a, b) \leq a \cup b \leq \max(a, b)$

3. For any u with $e \in [0, 1]$

$$au_w b \leq aub \leq au_s b$$

$$au_w b = \begin{cases} 0 & \text{if } 0 \leq a, b \leq e \\ \max(a, b) & \text{if } e \leq a, b \leq 1 \\ \min(a, b) & \text{otherwise} \end{cases}$$

$$au_s b = \begin{cases} \min(a, b) & \text{if } 0 \leq a, b \leq e \\ 1 & \text{if } e \leq a, b \leq 1 \\ \max(a, b) & \text{otherwise} \end{cases}$$

4. Conjunctive and disjunctive uninorm

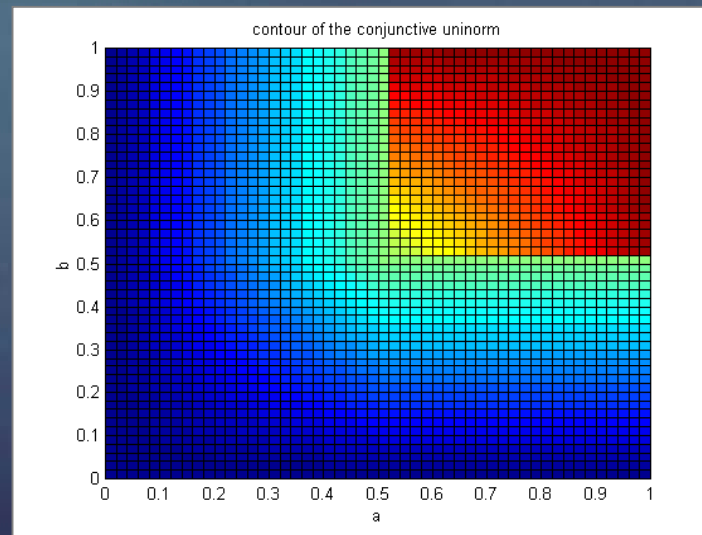
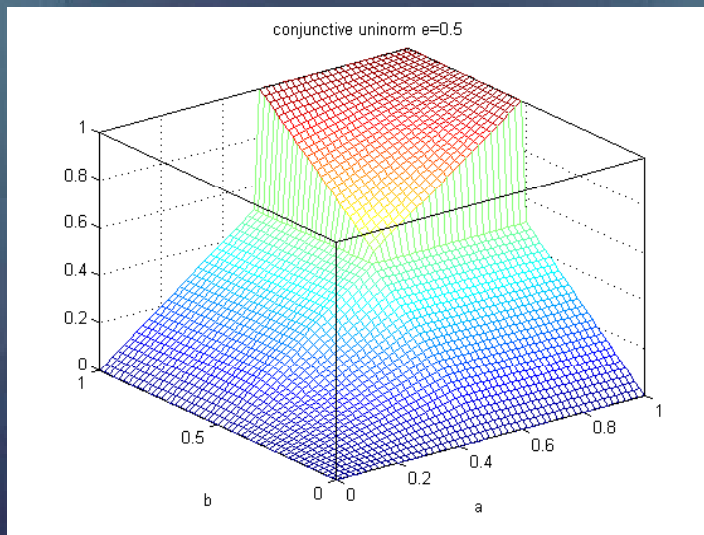
if $(0u1) = 0$ then

$$au_c b = \begin{cases} e(\frac{a}{e})t(\frac{b}{e}) & \text{if } 0 \leq a, b \leq e \\ e + (1-e)(\frac{a-e}{1-e})s(\frac{b-e}{1-e}) & \text{if } e \leq a, b \leq 1 \\ \min(a, b) & \text{otherwise} \end{cases}$$

if $(0u1) = 1$ then

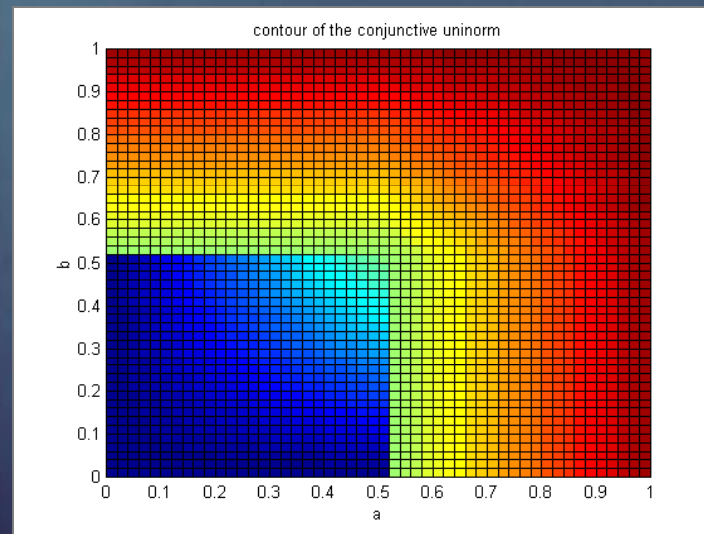
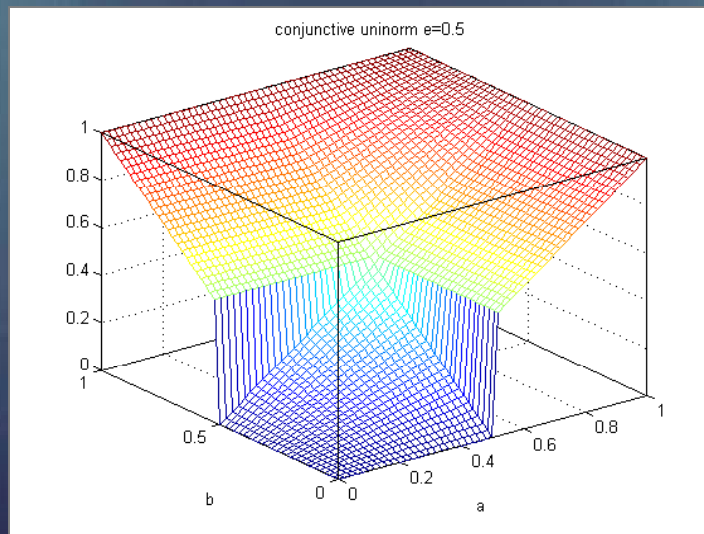
$$au_d b = \begin{cases} e(\frac{a}{e})t(\frac{b}{e}) & \text{if } 0 \leq a, b \leq e \\ e + (1-e)(\frac{a-e}{1-e})s(\frac{b-e}{1-e}) & \text{if } e \leq a, b \leq 1 \\ \max(a, b) & \text{otherwise} \end{cases}$$

Conjunctive uninorm



$$e = 0.5$$

Disjunctive uninorm



$$e = 0.5$$

5. Almost continuous Archimedean uninorms

$$a \mathbin{u} a < a \text{ for } 0 < a < e$$

$$a \mathbin{u} a > a \text{ for } e < a < 1$$

6. Additive and multiplicative generators of almost continuous uninorms

$$a \mathbin{u}_f b = f^{-1}(f(a) + f(b))$$

$$a \mathbin{u}_g b = g^{-1}(g(a)g(b))$$

$$g(x) = e^{-f(x)}$$

f strictly increasing

g strictly decreasing

7. Ordinal sum

$$u_{\text{co}}(a, b, l, \tau, \sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) t_k \left(\frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a, b] \in \iota_1 \\ \alpha_k + (\beta_k - \alpha_k) s_k \left(\frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a, b] \in \iota_2 \\ \max(a, b) & \text{if } a, b \notin [\alpha_k, \beta_k] \\ \min(a, b) & \text{and } a, b \geq e \\ & \text{otherwise} \end{cases}$$

$$l = \{[\alpha_k, \beta_k], k \in K\}$$

$$l_1 = \{[\alpha_k, \beta_k] \in l \mid \beta_k \leq e\}$$

$$l_2 = \{[\alpha_k, \beta_k] \in l \mid \alpha_k \geq e\}$$

$$\tau = \{t_k, k \in K\}, \quad \sigma = \{s_k, k \in K\}$$

$$u_{\text{co}}(a,b,l,\tau,\sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)t_k \left(\frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a,b] \in \mathfrak{I}_1 \\ \alpha_k + (\beta_k - \alpha_k)s_k \left(\frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a,b] \in \mathfrak{I}_2 \\ \min(a,b) & \text{if } a,b \notin [\alpha_k, \beta_k] \\ \max(a,b) & \text{and } a,b \leq e \\ & \text{otherwise} \end{cases}$$

Nullnorms

$$\nu : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity:

$$a \nu b = b \nu a$$

- Associativity:

$$a \nu (b \nu c) = (a \nu b) \nu c$$

- Monotonicity:

$$\text{if } b \leq c \text{ then } a \nu b \leq a \nu c$$

- Absorbing element:

$$a \nu e = e \quad \forall a \in [0, 1]$$

- Boundary conditions:

$$a \nu 0 = a \quad \forall a \in [0, e]$$

$$a \nu 1 = a \quad \forall a \in [e, 1]$$

$$as_v b = \frac{(ea)v(eb)}{e}$$

$$at_v b = \frac{(e + (1 - e)a)v(e + (1 - e)b) - e}{1 - e}$$

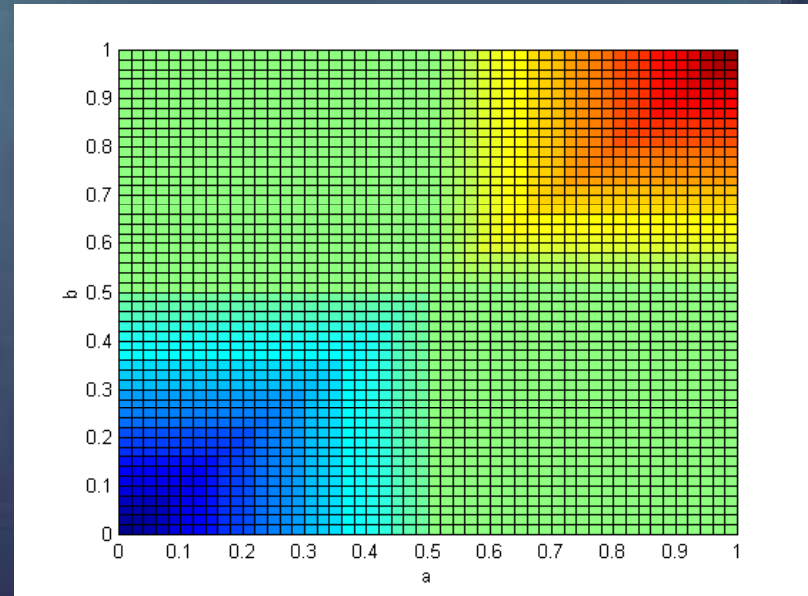
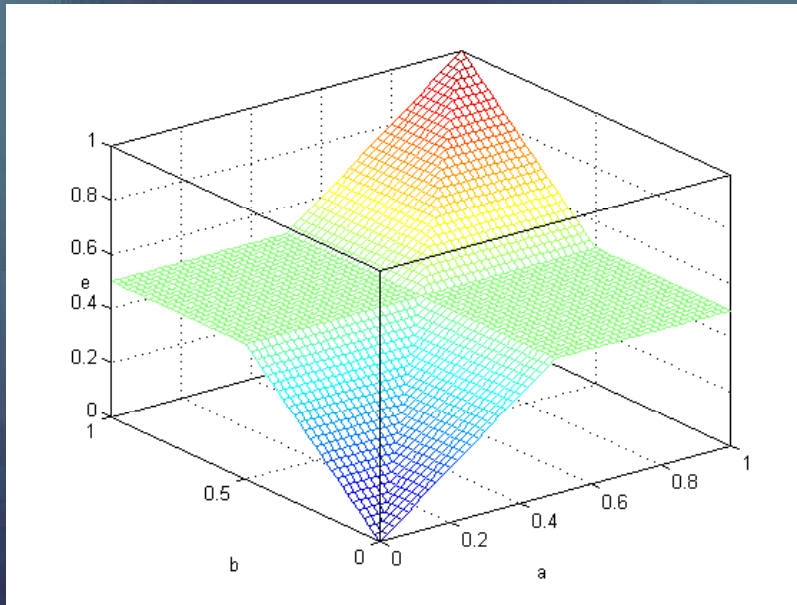
$$e \in [0, 1]$$

v behaves as a t-norm in $[0, e] \times [0, e]$

v behaves as a t-conorm in $[e, 1] \times [e, 1]$

$v = e$ in the rest of the unit square

Example



$e = 0.5$, t-norm = min, t-conorm = max

Symmetric sums

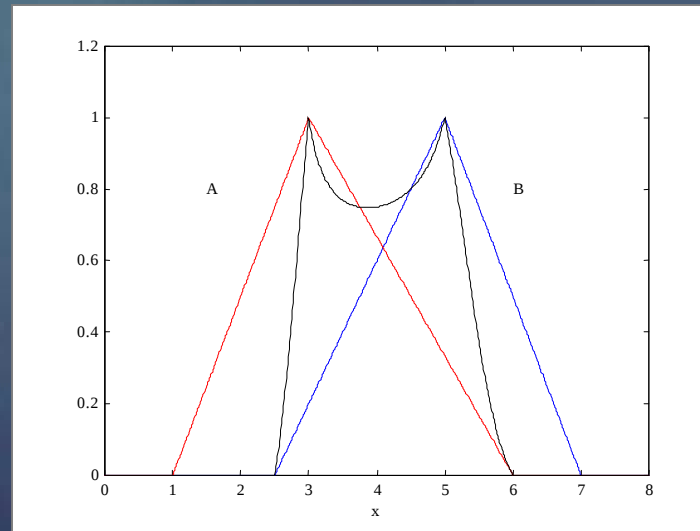
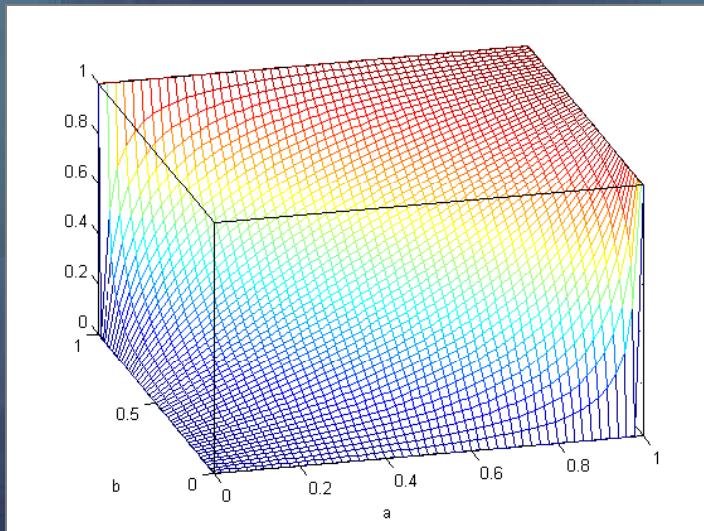
$$\sigma_s(a_1, a_2, \dots, a_n) = 1 - \sigma_s(1 - a_1, 1 - a_2, \dots, 1 - a_n)$$

$$\sigma_s(a_1, a_2, \dots, a_n) = \left[1 + \frac{f(1 - a_1, 1 - a_2, \dots, a_n)}{f(a_1, a_2, \dots, a_n)} \right]^{-1}$$

f increasing, continuous

$$f(0, 0, \dots, 0) = 0$$

Example



$$f(a,b) = a^2 + b^2$$

Compensatory operations

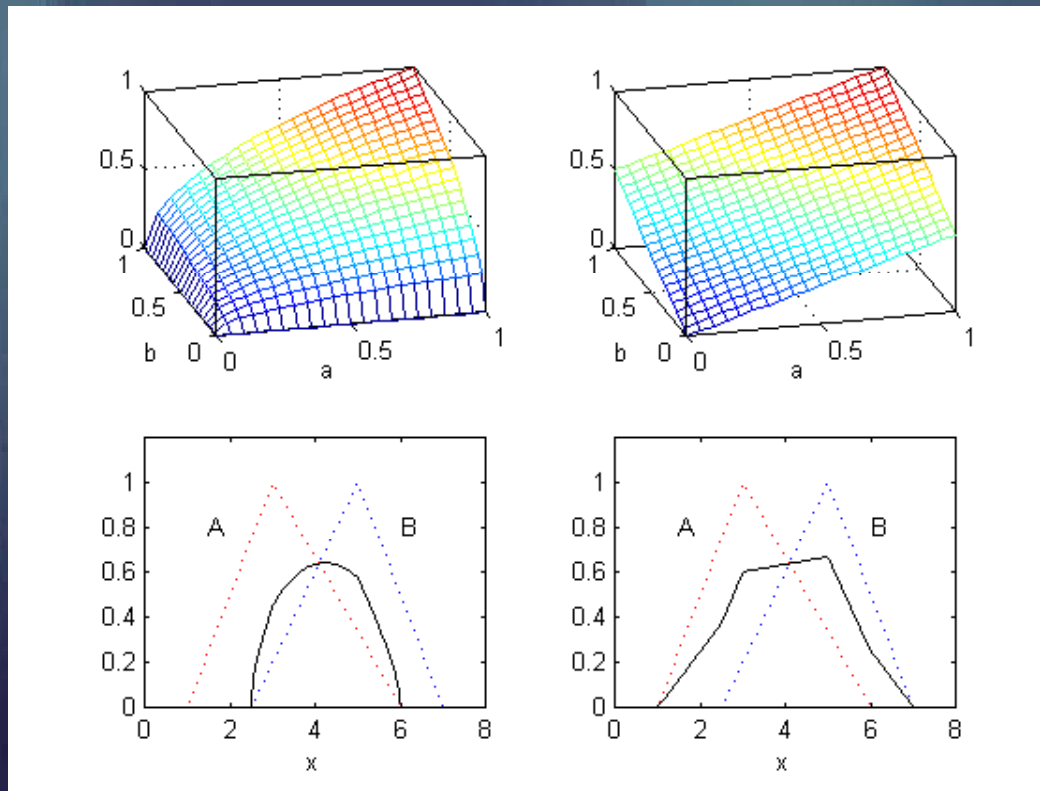
$$a \odot b = (a \text{ t } b)^{1-\gamma} (a \text{ s } b)^{\gamma}$$

compensatory product

$$a \oplus b = (1 - \gamma)(a \text{ t } b) + \gamma(a \text{ s } b)$$

compensatory sum

Example



(a)

(b)

$\gamma = 0.5$, t-norm = min, t-conorm = max

5.7 Fuzzy measure and integral

Fuzzy measure

$$g : \Omega \rightarrow [0,1]$$

- Boundary conditions: $g(\emptyset) = 0$
 $g(\mathbf{X}) = 1$
- Monotonicity: if $A \subset B$ then $g(A) \leq g(B)$

λ -fuzzy measure

$$g(A \cup B) = g(A) + g(B) + \lambda g(A) g(B), \quad \lambda > -1$$

- $\lambda = 0$ $g(A \cup B) = g(A) + g(B)$ additive
- $\lambda > 0$ $g(A \cup B) \geq g(A) + g(B)$ super-additive
- $\lambda < 0$ $g(A \cup B) \leq g(A) + g(B)$ sub-additive

Fuzzy integral

$h : X \rightarrow [0,1]$ Ω measurable

fuzzy integral of h with respect to g over A

$$\int_A h(x) \circ g() = \sup_{\alpha \in [0,1]} \{ \min[\alpha, g(A \cap H_\alpha)] \}$$

$$H_\alpha = \{x \mid h(x) \geq \alpha\}$$

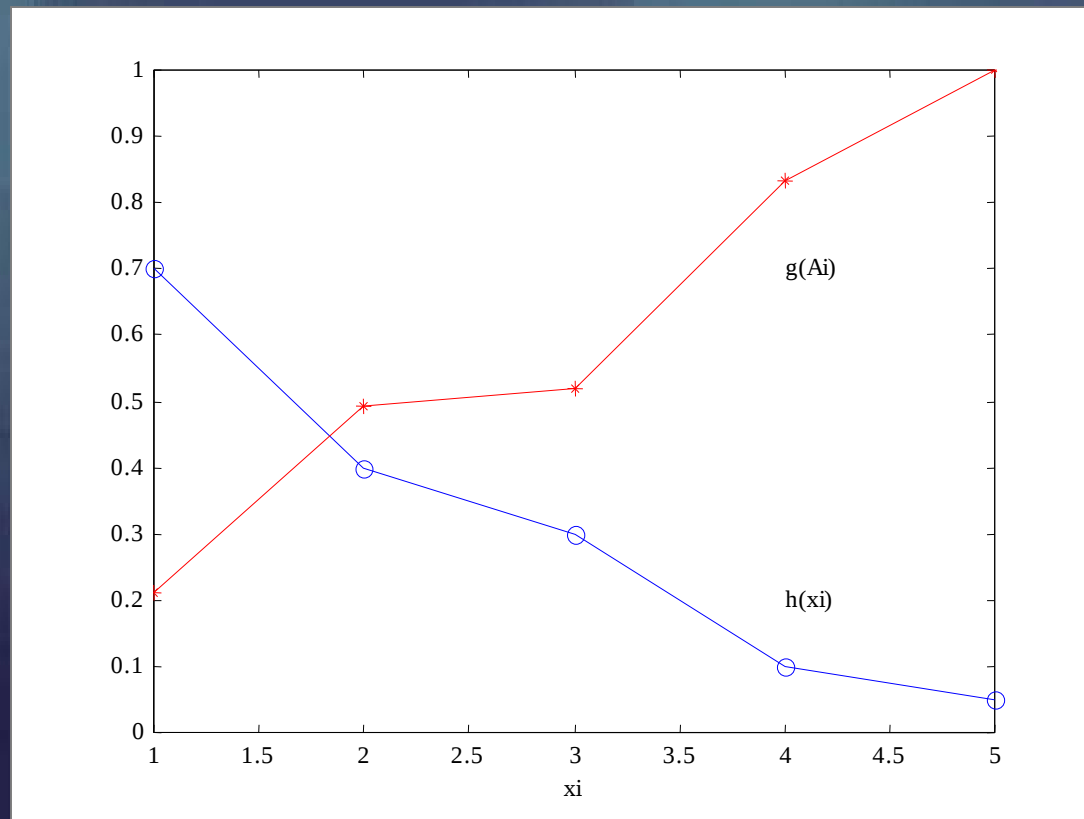
$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}$$

$$h(x_1) \geq h(x_2) \geq \dots \geq h(x_n)$$

$$A_1 = \{x_1\}, A_2 = \{x_1, x_2\}, \dots, A_n = \{x_1, x_2, \dots, x_n\} = \mathbf{X}$$

$$\int_A h(x) \circ g() = \max_{i=1, \dots, n} \{\min[h(x_i), g(A_i)]\}$$

Example



Choquet integral

$$(Ch)\int f \circ g = \sum_{i=1}^n [h(x_i) - h(x_{i+1})]g(A_i)$$

$$h(x_{n+1}) = 0$$

5.8 Negations

Definition

$$N : [0,1] \rightarrow [0,1]$$

1. Monotonicity:

N is nonincreasing

2. Boundary conditions:

$$N(0) = 1$$

$$N(1) = 0$$

3. Continuity:

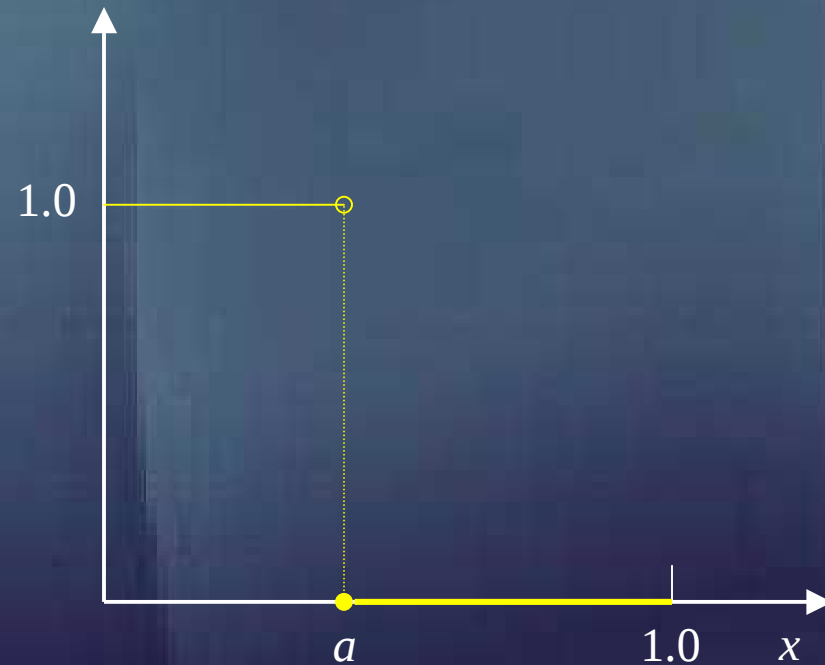
N is a continuous function

4. Involution:

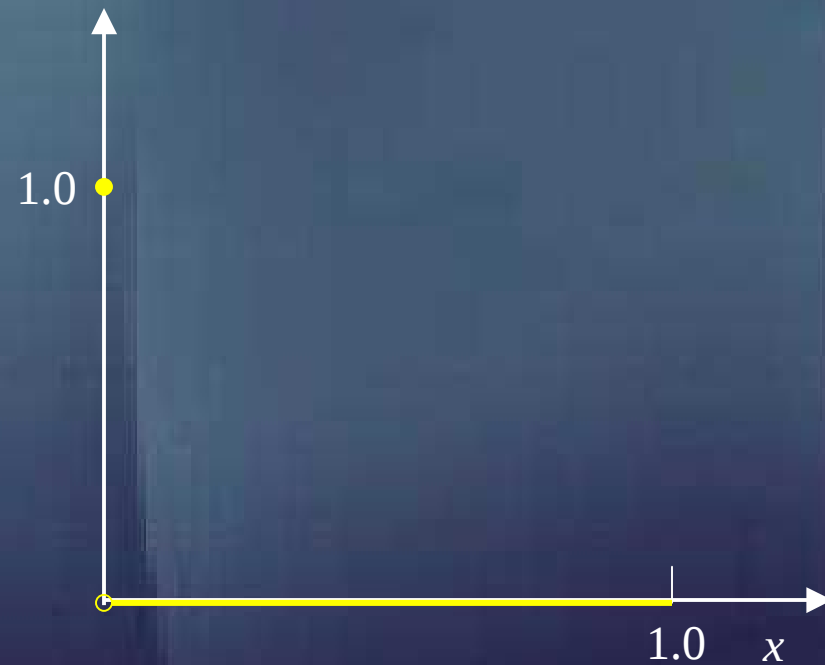
$$N(N(x)) = x \quad \forall x \in [0, 1]$$

Examples

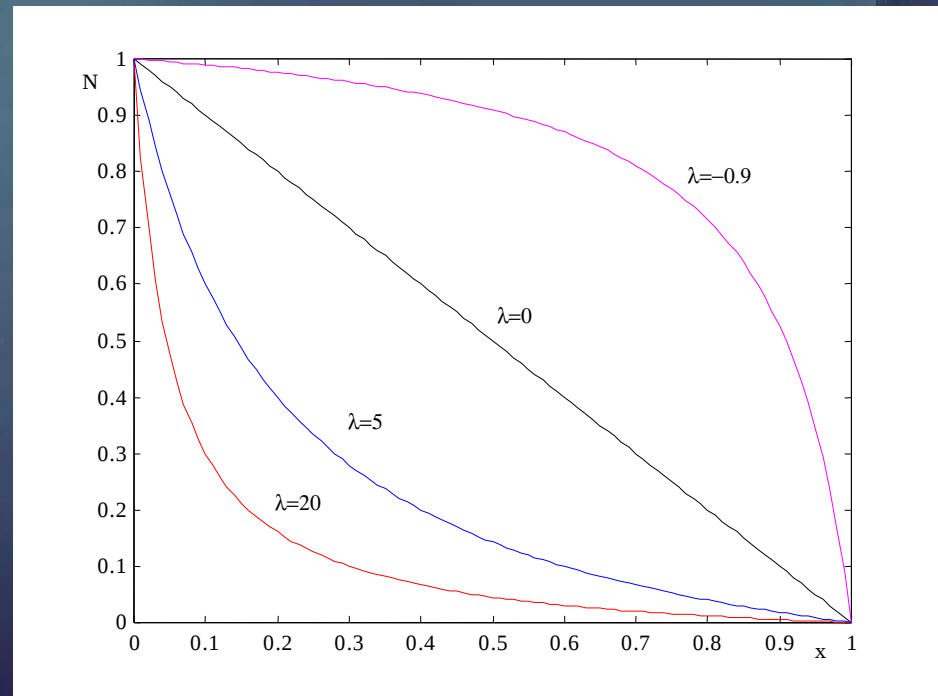
$$N(x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } x \geq a \end{cases}$$



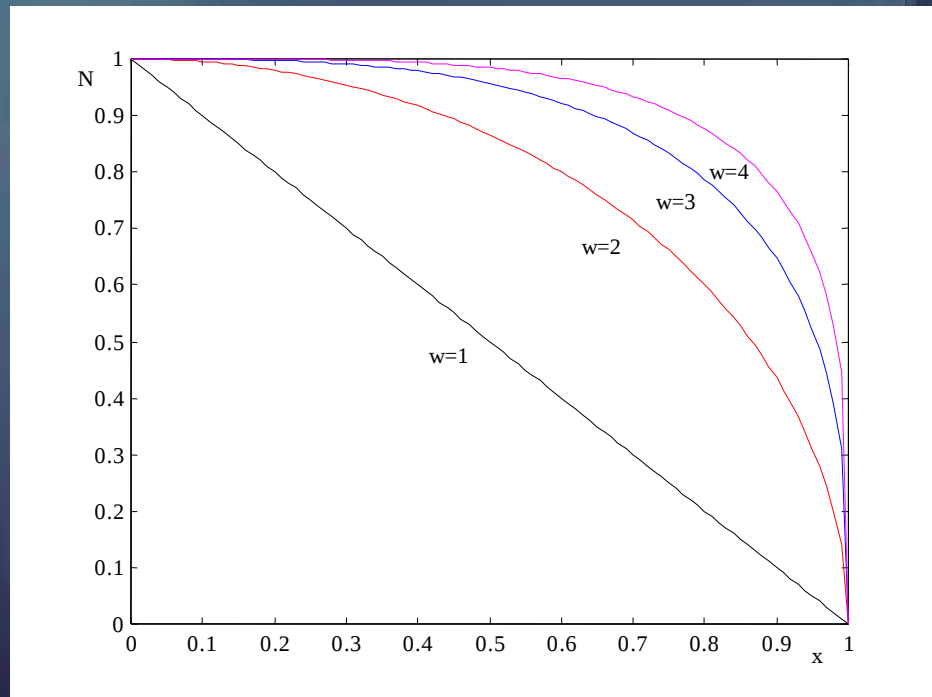
$$N(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$



$$N(x) = \frac{1-x}{1+\lambda x} \quad \lambda \in (-1, \infty)$$



$$N(x) = \sqrt[w]{1-x^w} \quad w \in (0, \infty)$$



(t, s, N) system

$$x \, s \, y = N (N(x) \, t \, N(y))$$

$$x \, t \, y = N (N(x) \, s \, N(y))$$

$$\forall x, y \in [0, 1]$$

Examples

$$xty = \max\left(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right)$$

$$xsy = \min(1, x + y - 1 + \lambda xy)$$

$$N(x) = \frac{1 - x}{1 + \lambda x} \quad \lambda \in (-1, \infty)$$

$$xty = \min(x, y)$$

$$xsy = \max(x, y)$$

$$N(x) = 1 - x$$