2 Notions and Concepts of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

2.1 Sets and fuzzy sets: A departure from the principle of dichotomy

Dichotomy

Set and the principle of dichotomy

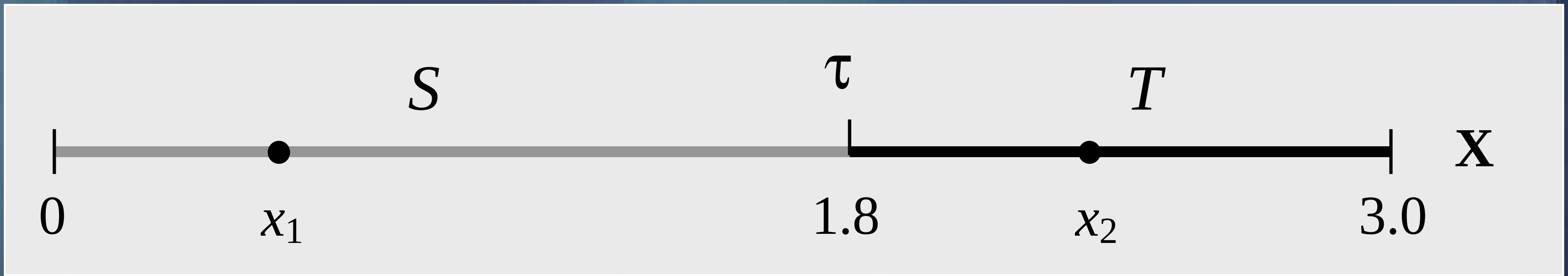
Relaxation of complete inclusion and exclusion

Inherent problems of dichotomization

"One seed does not constitute a pile nor two or three. From the other side, everybody will agree that 100 million seeds constitutes a pile. What is therefore the appropriate limit?"

E. Borel, 1950

Sets



Threshold $\tau = 1.8$

$S = \{x \in \mathbf{X} \mid 0 \le x \le 1.8\}$

$T = \{x \in \mathbf{X} \mid 1.8 < x \le 3.0\}$

Dichotomy

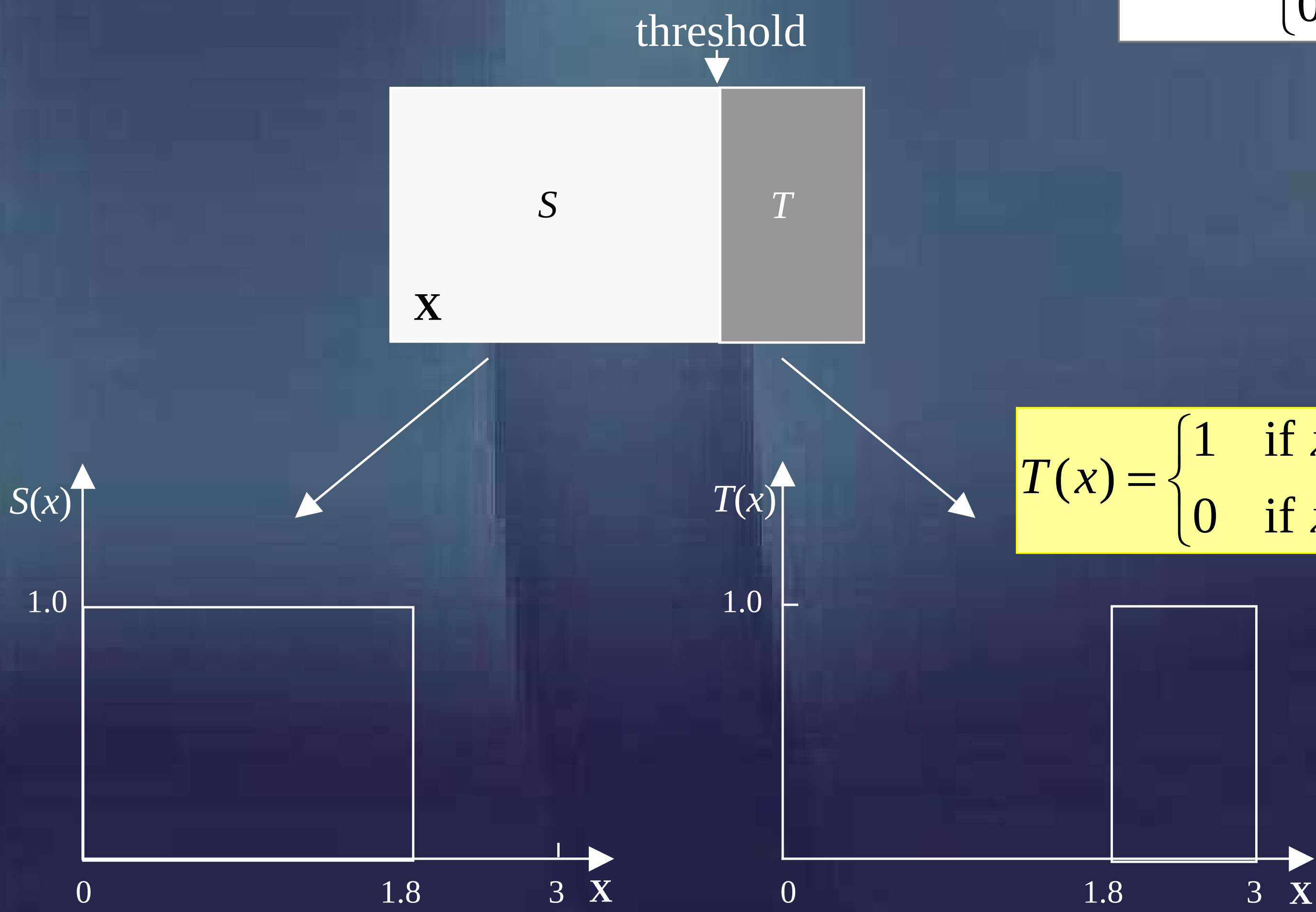
$x_1 \in S, \quad x_1 \notin T$

$x_2 \in T, \quad x_2 \notin S$

Characteristic function

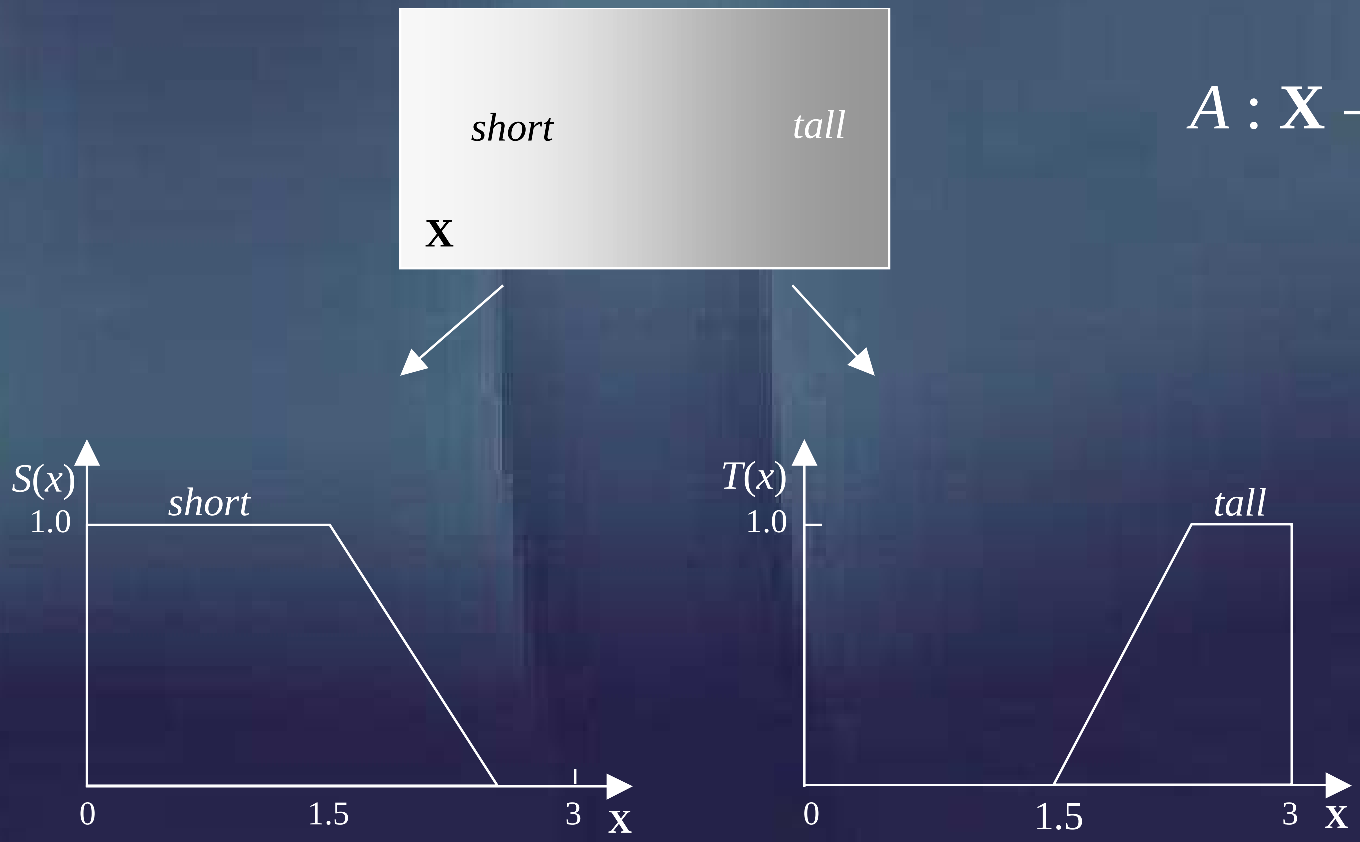
$$A : \mathbf{X} \rightarrow \{0,1\}$$

$$A(x) = \begin{cases} 1, \; if \; x \in A \\ 0, \; if \; x \notin A \end{cases}$$



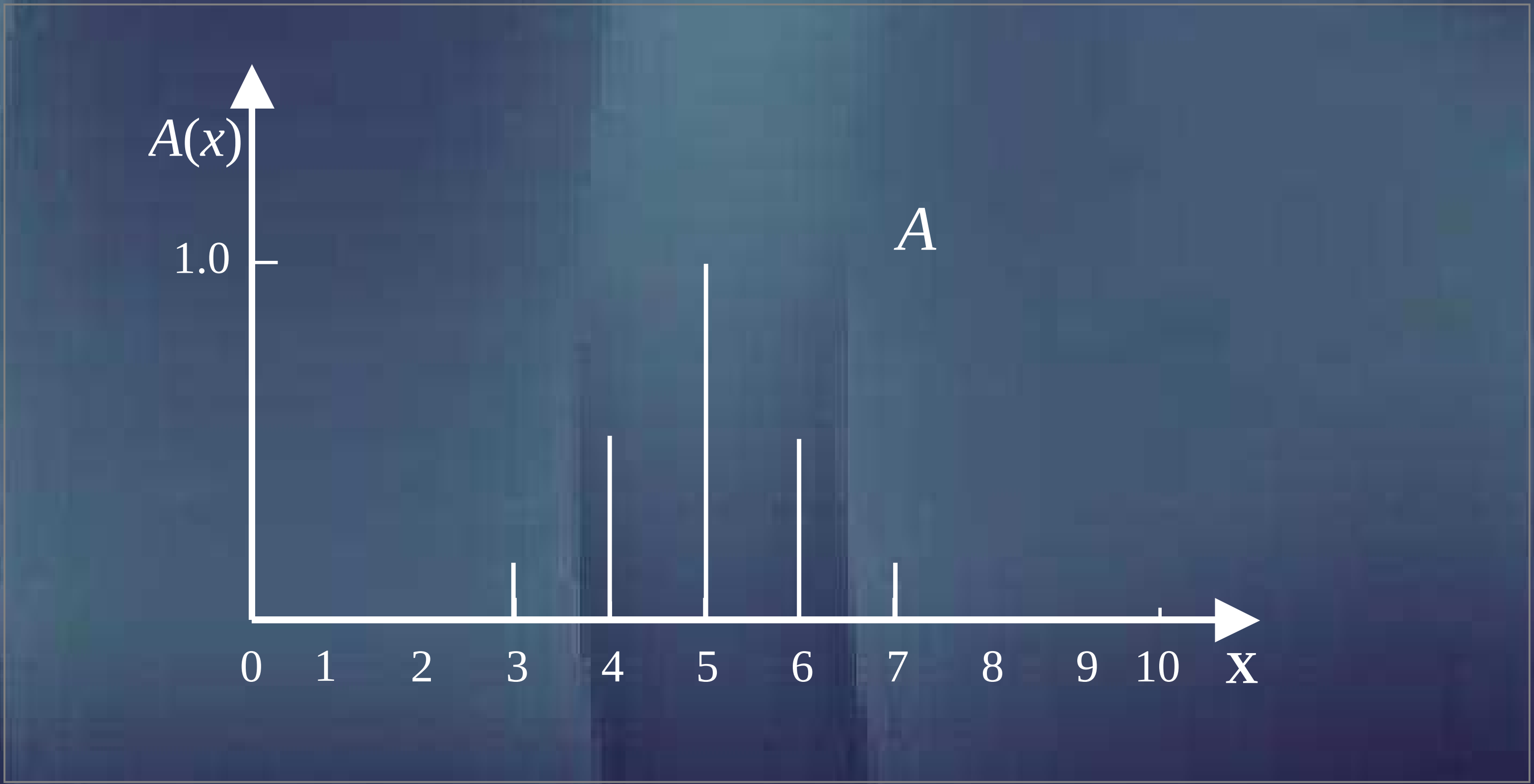
$$T(x) = \begin{cases} 1 \quad \text{if } x \in [1.8,\, 3.0] \\ 0 \quad \text{if } x \notin [1.8,\, 3.0] \end{cases}$$

Fuzzy set: Membership function



$$A : \mathbf{X} \rightarrow [0,1]$$

Fuzzy sets in discrete universes



$\mathbf{X} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\boxed{A = \{(A(x),x)\}}$

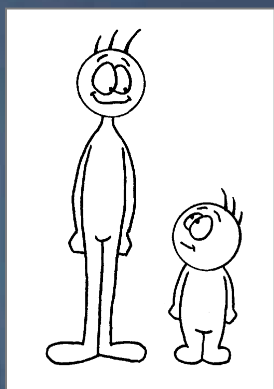
$A = \{0/0, 0/1, 0/2, 0.2/3, 0.5/4, 1.0/5, 0.5/6, 0.2/7, 0/8, 0/9, 0/10\}$

$A = [0, 0, 0, 0.2, 0.5, 1.0, 0.5, 0.2, 0, 0, 0]$

2.2 Interpretation of fuzzy sets

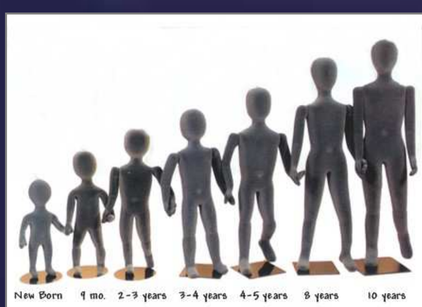
Fuzziness $\neq$ Probability

John is tall



Head or tail ?

Height of people



Fuzziness

$A : \mathbf{X} \rightarrow [0,1]$

$\mathbf{X}$: universe (set)

A: membership function

Probability

$P(A) : \mathbf{F} \rightarrow [0,1]$

P: probability (set) function

A: set

$\mathbf{X}$: universe (set)

$\mathbf{F}$: σ-algebra, a set of subsets of $\mathbf{X}$

Membership grades: semantics

- Similarity: degree of compatibility
  (data analysis and processing)
- Uncertainty: possibility
  (reasoning under uncertainty)
- Preference: degree of satisfaction
  (decision-making, optimization)

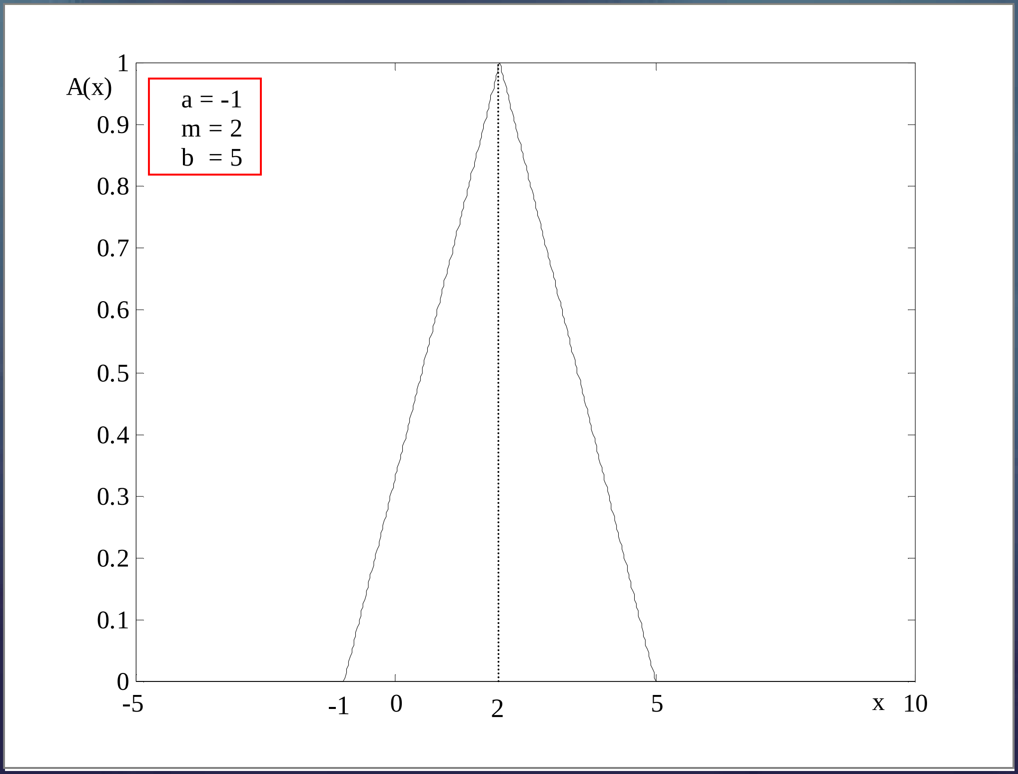
2.3 Membership functions and their motivation

Choosing membership functions

Criteria should reflect:

- Nature of the problem at hand
- Perception of the concept to represent
- Level of details to be captured
- Context of application
- Suitability for design and optimization

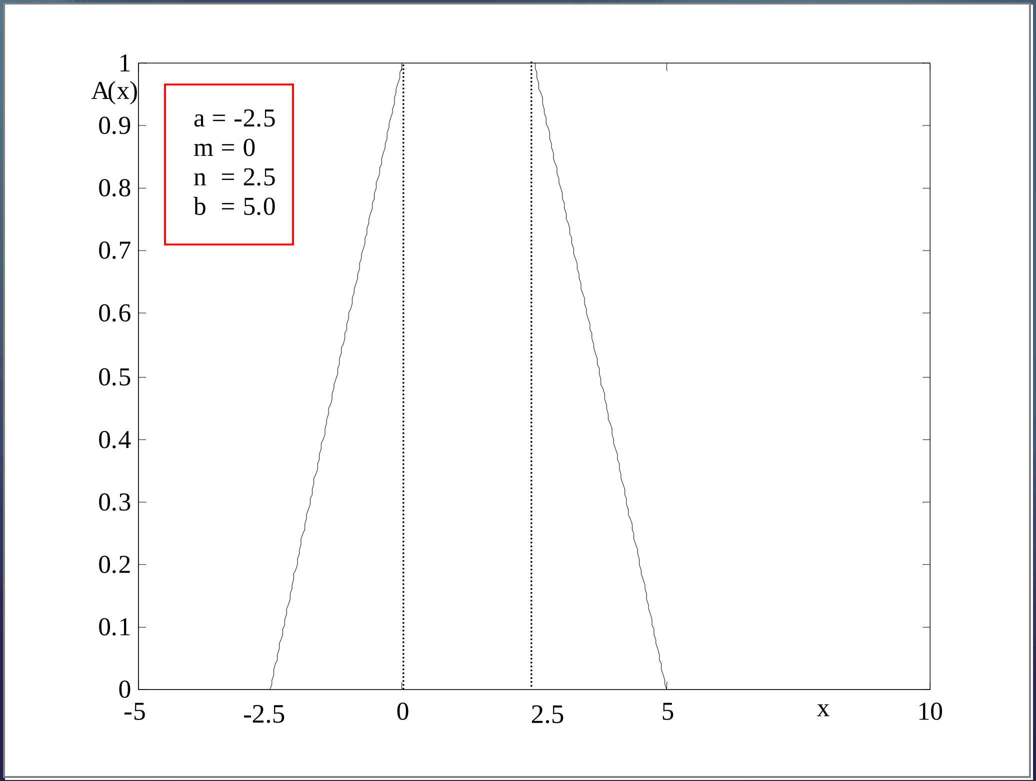
Triangular membership function



$$A(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{m-a} & \text{if } x \in [a,m] \\ \dfrac{b-x}{b-m} & \text{if } x \in [m,b] \\ 0 & \text{if } x \ge b \end{cases}$$

$$A(x,a,m,b) = \max\{\min[(x-a\}/(m-a),(b-x)/(b-m)],0\}$$

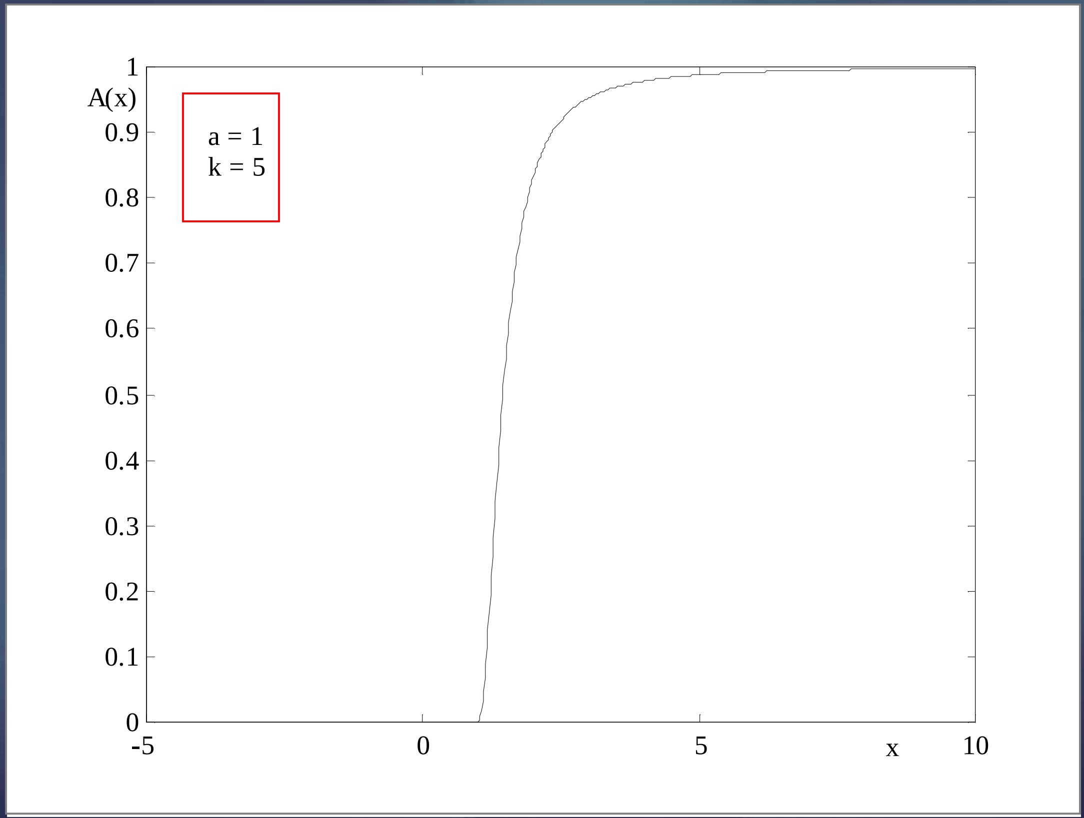
Trapezoidal membership function



$$A(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{m-a} & \text{if } x \in [a, m) \\ 1 & \text{if } x \in [m, n) \\ \dfrac{b-x}{b-n} & \text{if } x \in [n, b] \\ 0 & \text{if } x > b \end{cases}$$

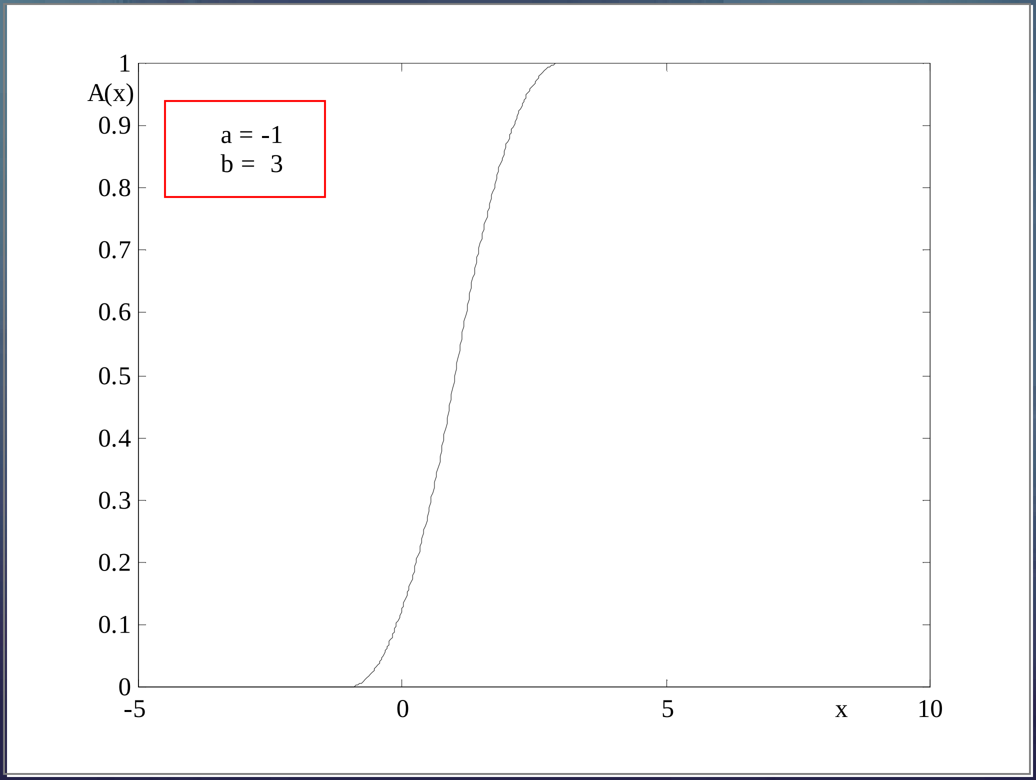
$$A(x, a, m, n, b) = \max\{\min[(x-a\}/(m-a), 1, (b-x)/(b-n)], 0\}$$

Γ-membership function



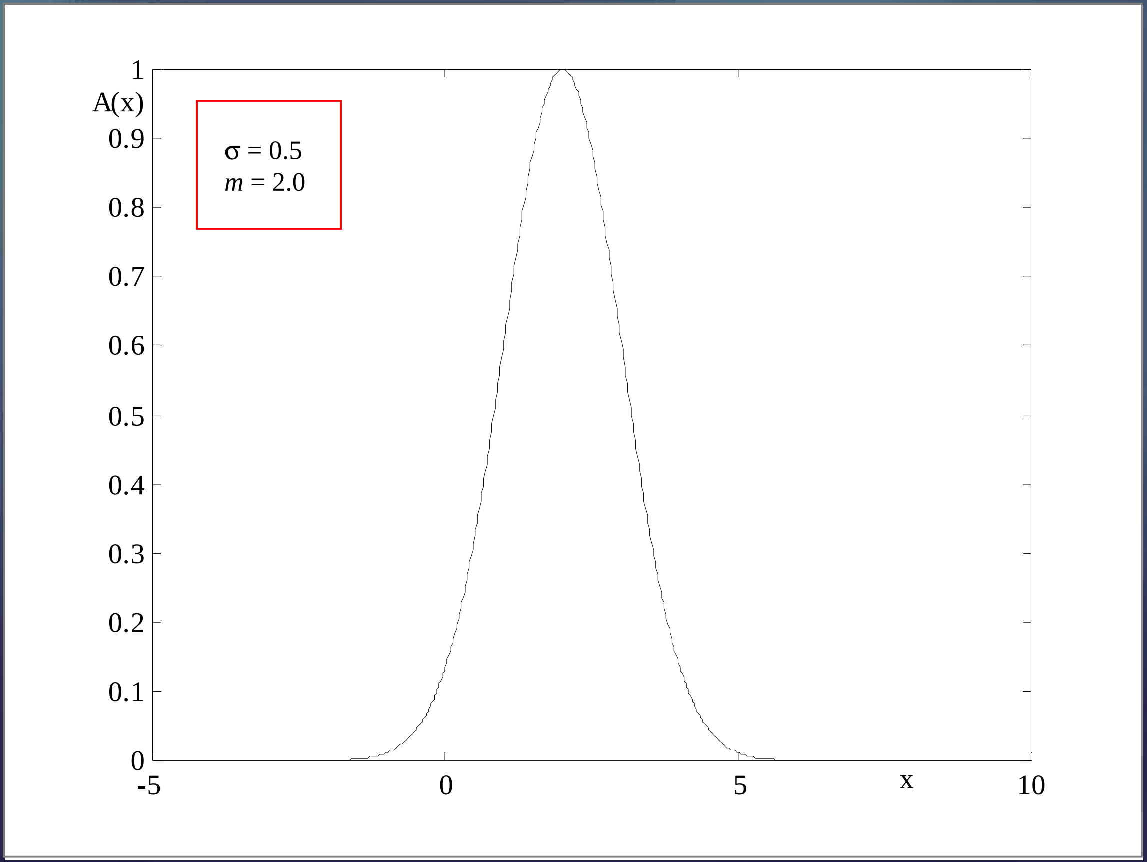
$$A(x) = \begin{cases} 0 & \text{if } x \le a \\ 1 - e^{-k(x-a)^2} & \text{if } x > a \end{cases} \quad \text{or} \quad A(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{k(x-a)^2}{1+k(x-a)^2} & \text{if } \mathrm{x} > \mathrm{a} \end{cases}$$

S-membership function



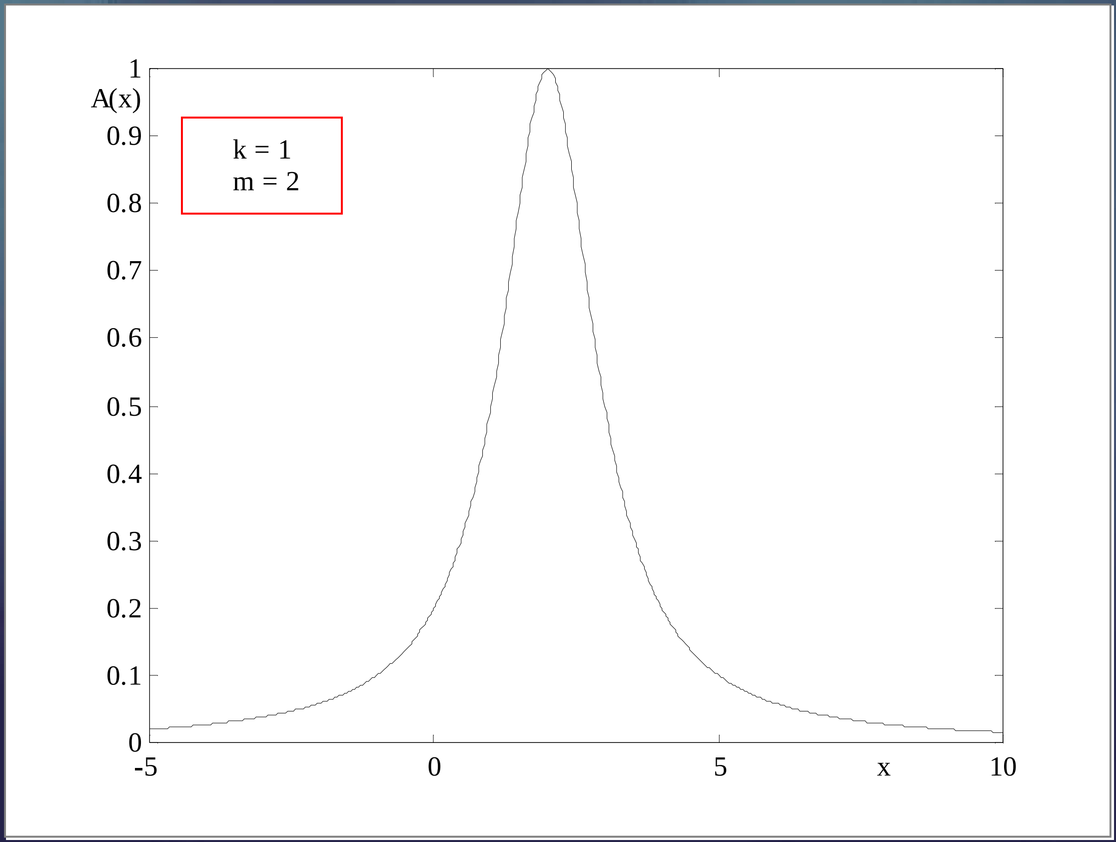
$$A(x)=\begin{cases}0 & \text{if } x\le a\\ 2\left(\dfrac{x-a}{b-a}\right)^2 & \text{if } x\in[a,m)\\ 1-2\left(\dfrac{x-b}{b-a}\right)^2 & \text{if } x\in(m,b]\\ 1 & \text{if } x>b\end{cases}$$

Gaussian membership function



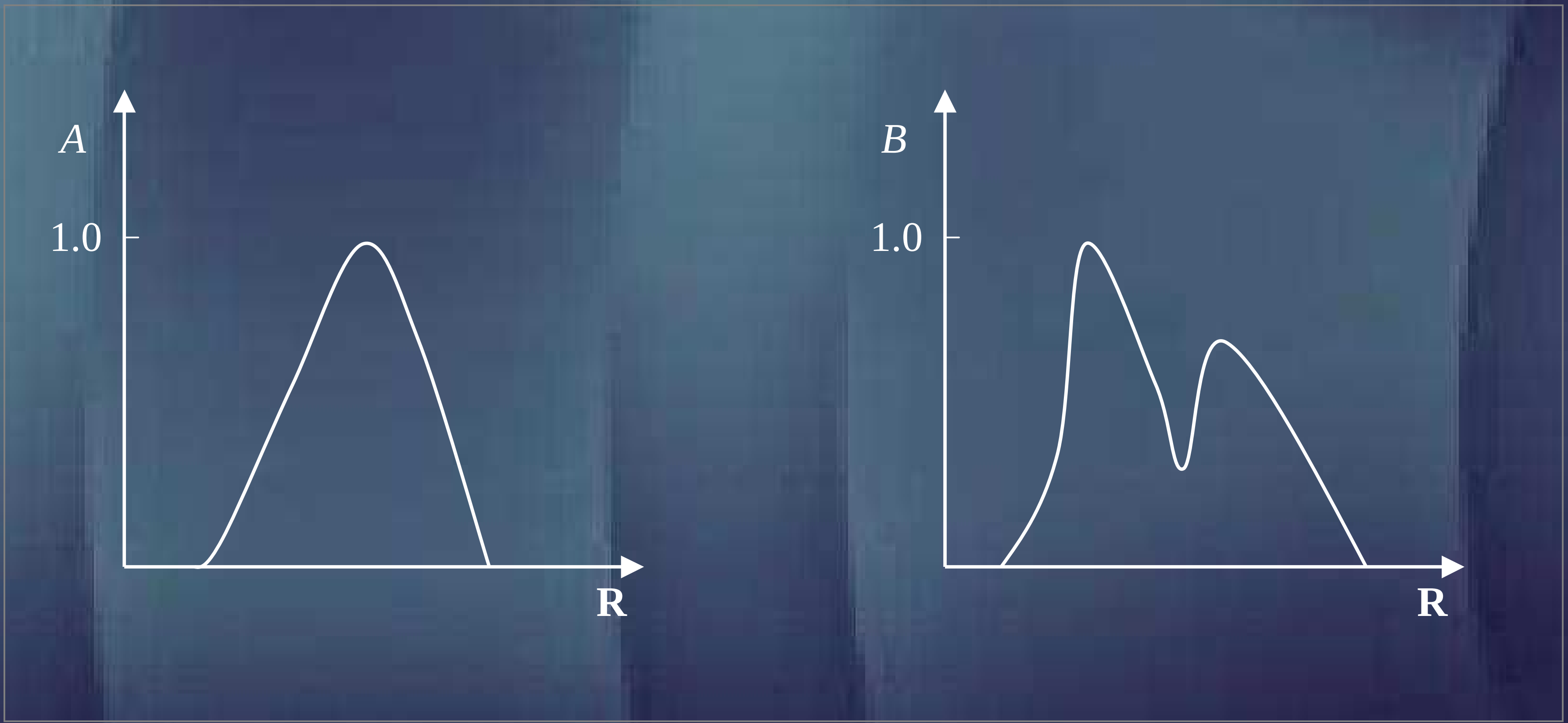
$$A(x) = \exp(-\frac{(x-m)^2}{\sigma^2})$$

Exponential-like membership function



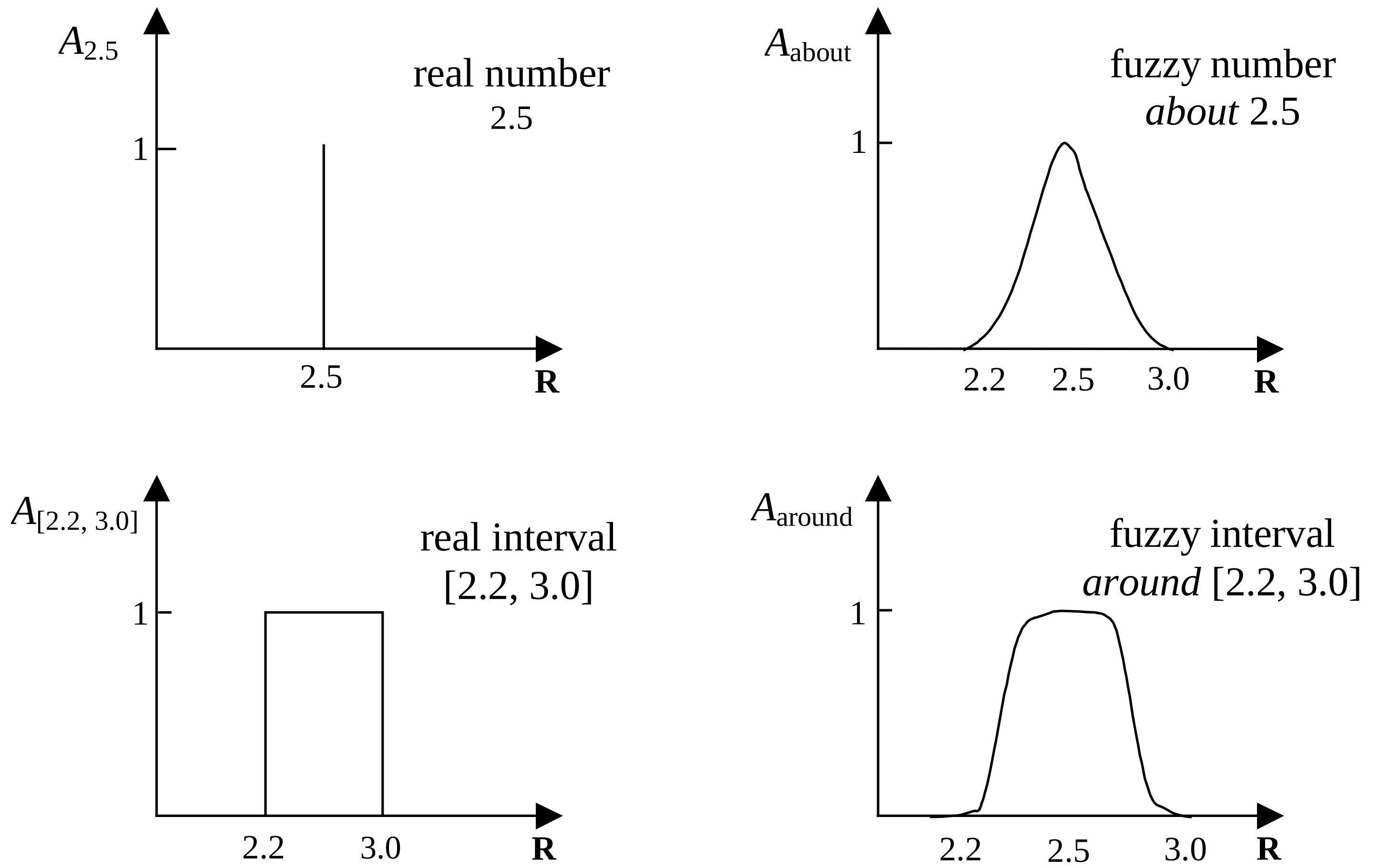
$$A(x) = \frac{1}{1 + k(x-m)^2} \qquad k > 0$$

2.4 Fuzzy numbers and intervals



A is a fuzzy number

B is not a fuzzy number


$A_{2.5}$
real number
2.5
1
2.5
R
A_{about}
fuzzy number
about 2.5
1
2.2
2.5
3.0
R
$A_{[2.2, 3.0]}$
real interval
[2.2, 3.0]
1
2.2
3.0
R
A_{around}
fuzzy interval
around [2.2, 3.0]
1
2.2
2.5
3.0
R

2.5 Linguistic variables

Linguistic variables

- A certain variable (attribute) can be quantified in terms of a small number of information granules
  - temperature is {*low*, *high*}
  - speed is { *low*, *medium*, *high*, *very high*}
- Each information granule comes with a well-defined meaning (semantics)

Linguistic variables: A definition

$\langle X, T(X), \mathbf{X}, G, M \rangle$

X : is the name of the variable

$T(X)$: is term set of X; elements of T are labels L of linguistic values of X

$\mathbf{X}$: universe

G : grammar that generates the names of X

M : semantic rule that assigns to each label $L \in T(X)$ a meaning whose realization is a fuzzy set on $\mathbf{X}$ with base variable x

Example

$\langle X, T(X), \mathbf{X}, G, M \rangle$

X : temperature

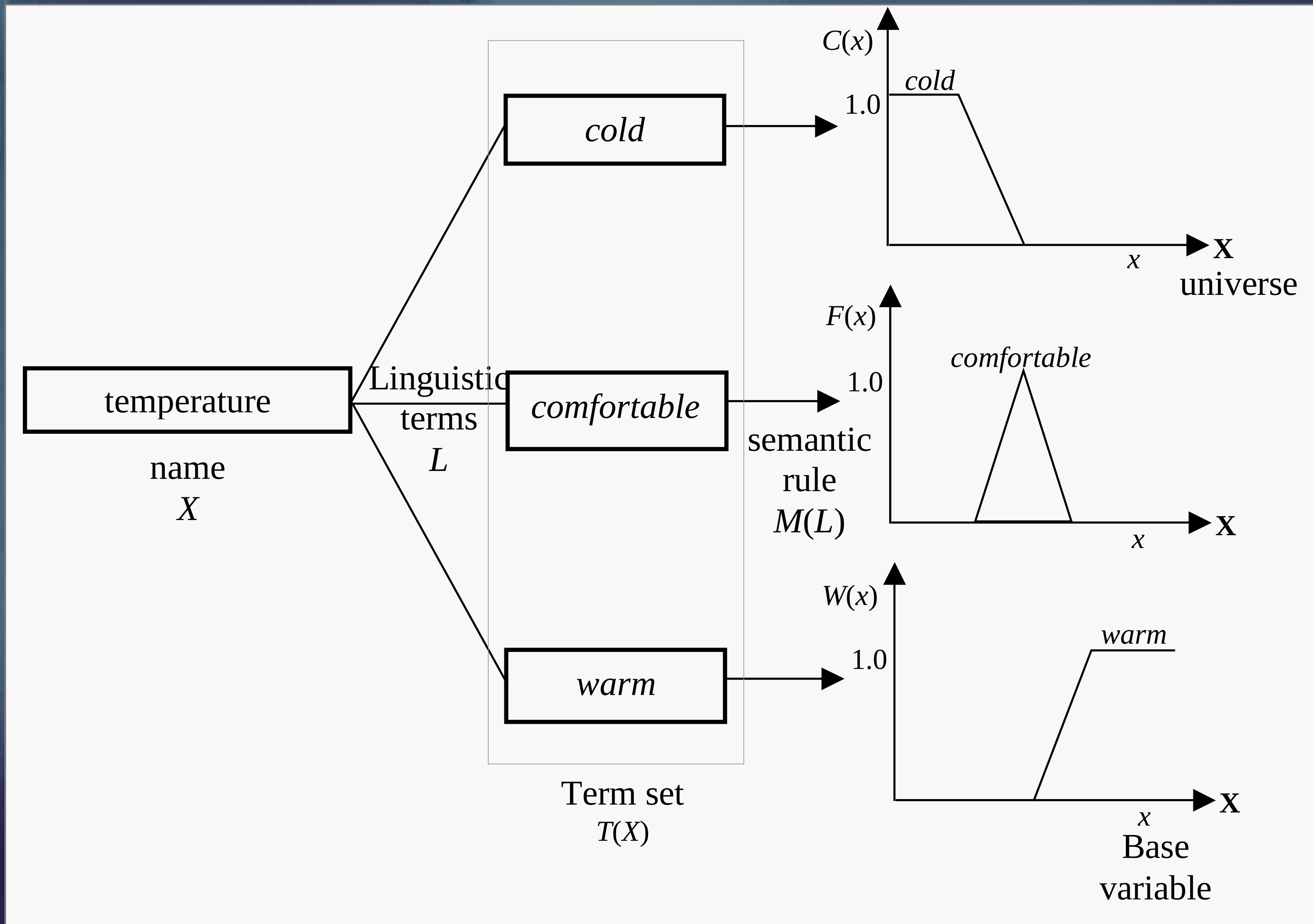
$\mathbf{X}$: [0, 40]

$T(X)$: $\{cold, comfortable, warm\}$

G : only terminal symbols, the terms of $T(X)$

$M\,(cold) \rightarrow C$
$M\,(comfortable) \rightarrow F$
$M\,(\text{warm}) \rightarrow W$

C, F and W are fuzzy sets in [0, 40]



$$\langle X, \mathrm{T}(X), \mathbf{X}, \mathrm{G}, \mathrm{M} \rangle$$