An Overview of Fuzzy Numbers and Fuzzy Arithmetic

Fernando Gomide
Unicamp-FEEC-DCA

Purpose

To provide a tutorial review of operations with uncertain quantities from the point of view of the theory of fuzzy sets.

Outline

1- Introduction

2- Fuzzy Numbers and Arithmetic

3- New Approaches for Fuzzy Arithmetic

4- Analysis and Discussion

5- Conclusions

1- Introduction

- **Exact values (e.g. parameters) are rare in practice**
- **Reason: incomplete or imprecise information**
- **How to model e.g. imprecise system parameters ?**
- **How to compute with imprecise parameters ?**
- **Fuzzy numbers and fuzzy arithmetic provide an answer**
  - L. Zadeh, Calculus of fuzzy restrictions, in L. Zadeh, K. Fu, K. Tanaka M. Shimura (Eds.), Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, NY, 1975, pp. 1-39.
  - D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications Academic Press, NY, 1980.

- **Issues with standard fuzzy arithmetic:**
  - Overestimation: accumulation of fuzziness skews membership functions
  - Shape preservation: membership function forms are not preserved
  - Properties: commutative, associative, sub-distributive
- **Consequences for system modeling and applications:**
  - Analysis, validation and interpretation of imprecise models more complex
  - Overestimation and shape preserving mean non-intuitive results
- **Mathematically speaking: fuzzy arithmetic is well developed**

2- Fuzzy Numbers and Arithmetic

- **Fuzzy quantity** $A: R \rightarrow [0,1]$ is a fuzzy set A of R such that:

  1- A is a normal fuzzy set

  2-The support $\{ x: A(x) > 0 \}$ of A is bounded

  3-The α-cuts of A are closed intervals

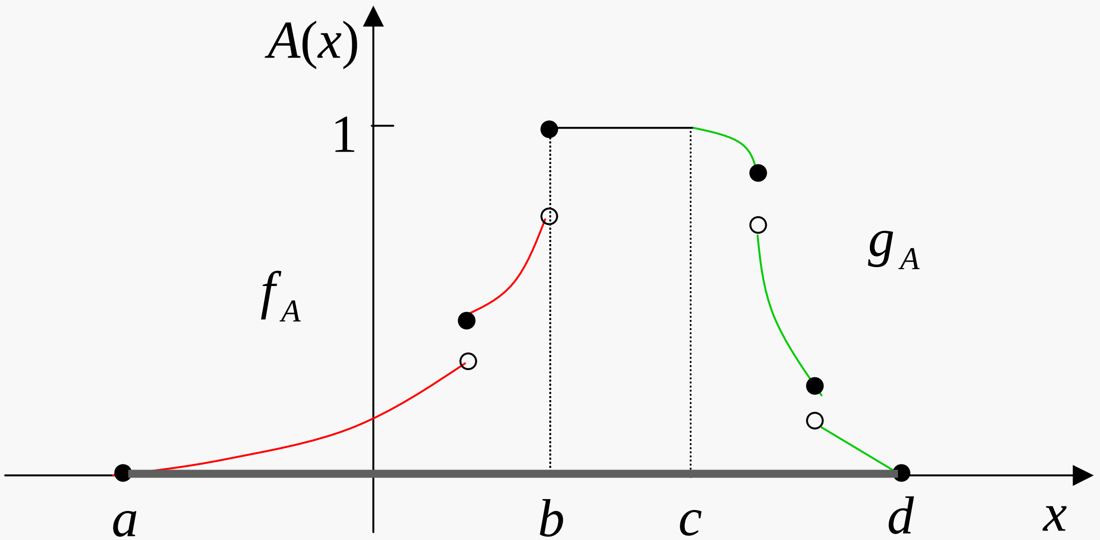
- **Fuzzy Number**: If $A(x)=1$ for exactly one x

- **Fuzzy Interval**: Otherwise

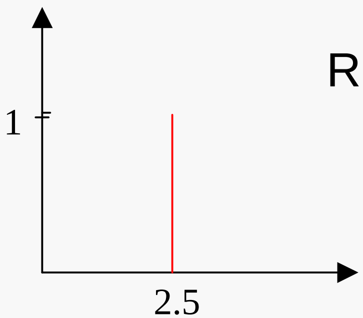
General form of a fuzzy quantity

$$A(x) = \begin{cases} f_A(x) & x \in [a,b] \\ 1 & x \in [b,c] \\ g_A(x) & x \in [c,d] \\ 0 & \text{otherwise} \end{cases}$$

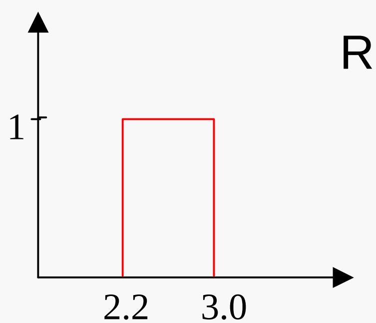
$$[b,c] \neq \phi$$



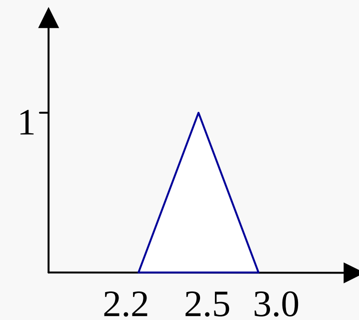
Examples of fuzzy quantities



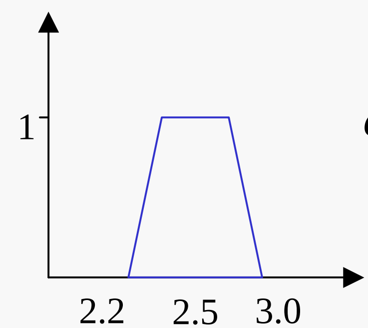
Real number
2.5



Real interval
[2.2, 3.0]



Fuzzy number
about 2.5

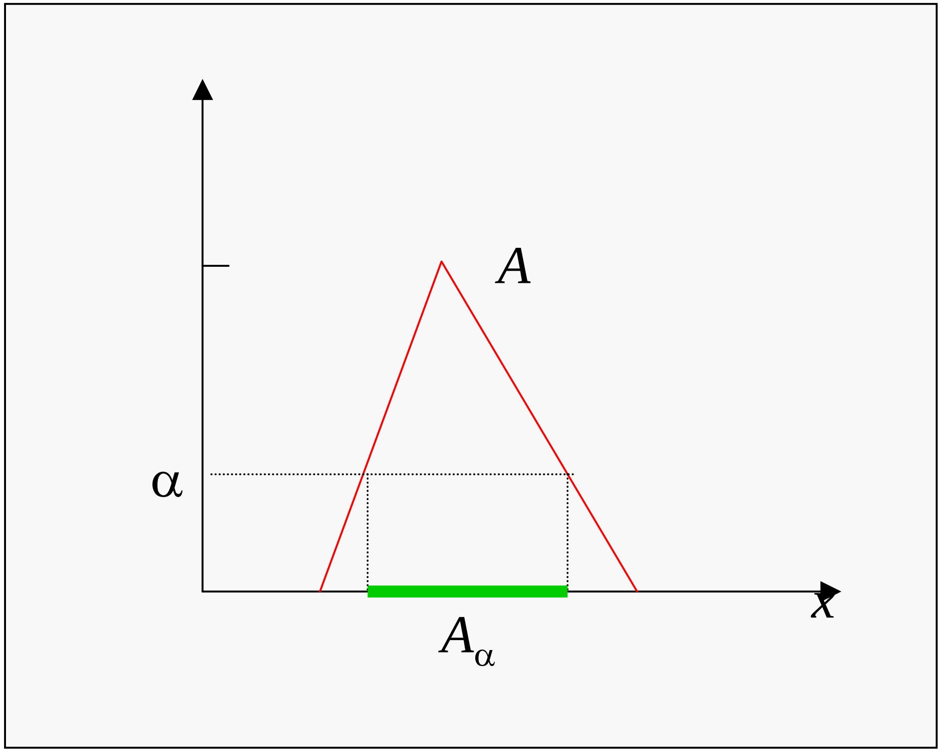


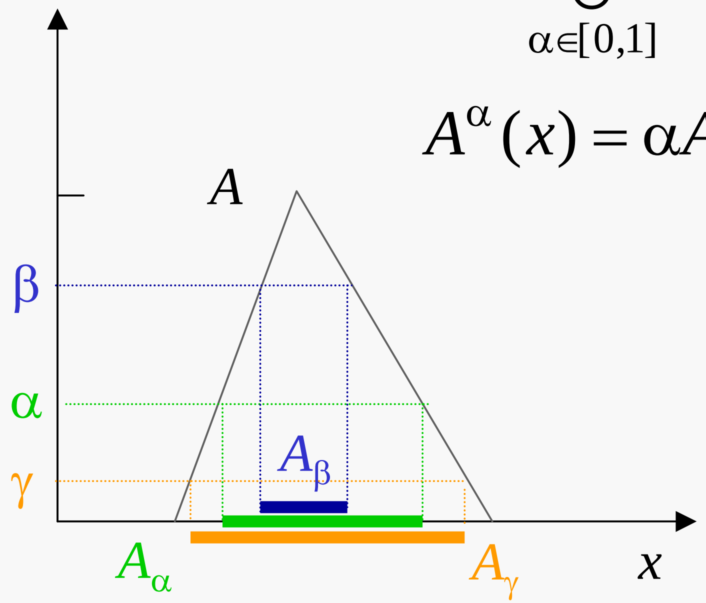
Fuzzy interval
around [2.2, 3.0]

Major approaches for fuzzy arithmetic

I- Extension of interval arithmetic

- α-cuts A^{α} of fuzzy sets
- representation theorem





$$A = \bigcup_{\alpha \in [0,1]} A^{\alpha}$$

$$A^{\alpha}(x) = \alpha A_{\alpha}(x)$$

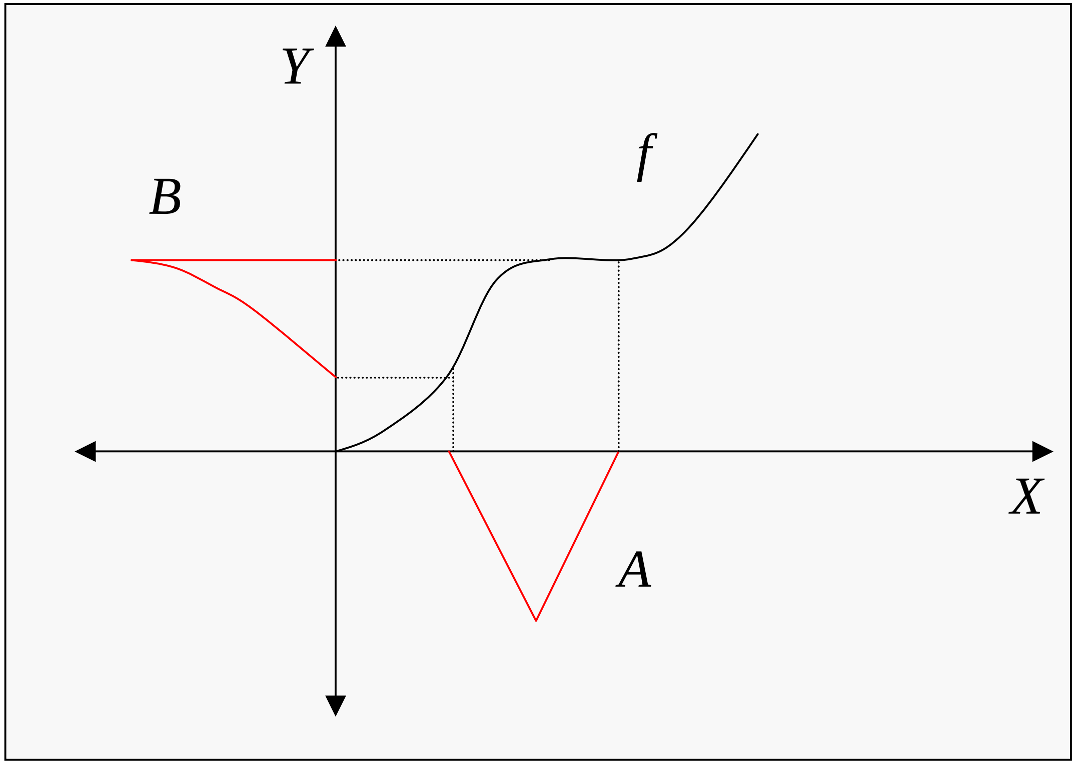
II- Extension Principle

$$f: X \rightarrow Y$$

$$f: F(X) \rightarrow F(Y)$$

$$B = f(A)$$

$$B(y) = \sup_{x / y = f(x)} A(x)$$



I- Extension of Interval arithmetic

- If * is any of the arithmetic operations: +, - , . , / , then:

$$[a, b] * [d, e] = \{ v * w \mid a \le v \le b,\ d \le w \le e\}$$ except when $0 \in [d, e]$

- Arithmetic operations with closed intervals are:

$$[a,b]+[d,e]=[a+d,b+e]$$

$$[a,b]-[d,e]=[a-e,b-d]$$

$$[a,b].[d,e]=[\min(ad,ae,bd,be),\max(ad,ae,bd,be)]$$

$$[a,b]/[d,e]=[a,b].[1/e,1/d]=[\min(a/d,a/e,b/d,b/e),\max(a/d,a/e,b/d,b/e)]$$

$$0 \notin [d,e]$$

- Examples

$$[2,5]+[1,3]=[3,8]$$

$$[2,5]-[1,3]=[-1,4]$$

$$[-1,1].[-2,-0.5]=[-2,2]$$

$$[-1,1]/[-2,-0.5]=[-2,2]$$

- Let: $A=[a_1,a_2],\ B=[b_1,b_2],\ C=[c_1,c_2],\ \mathbf{1}=[1,1],\ \mathbf{0}=[0,0]$

• Properties

$1-\ A+B=B+A$

$A.B=B.A$ Commutativity

$2-\ (A+B)+C=A+(B+C)$

$(A.B).C=A.(B.C)$ Associativity

$3-\ A=\mathbf{0}+A=A+\mathbf{0}$

$A=\mathbf{1}.A=A.\mathbf{1}$ Identity

$4-\ A.(B+C)\subseteq A.B+A.C$ Subdistributivity

$5-$ If $b.c\geq 0\ \forall b\in B$ and $\forall c\in C$ then $A.(B+C)=A.B+A.C$ Distributivity

Fuzzy arithmetic

$$(A * B)_\alpha = A_\alpha * B_\alpha = \{x * y \mid (x, y) \in A_\alpha \times B_\alpha\}, \quad \alpha \in (0,1]$$

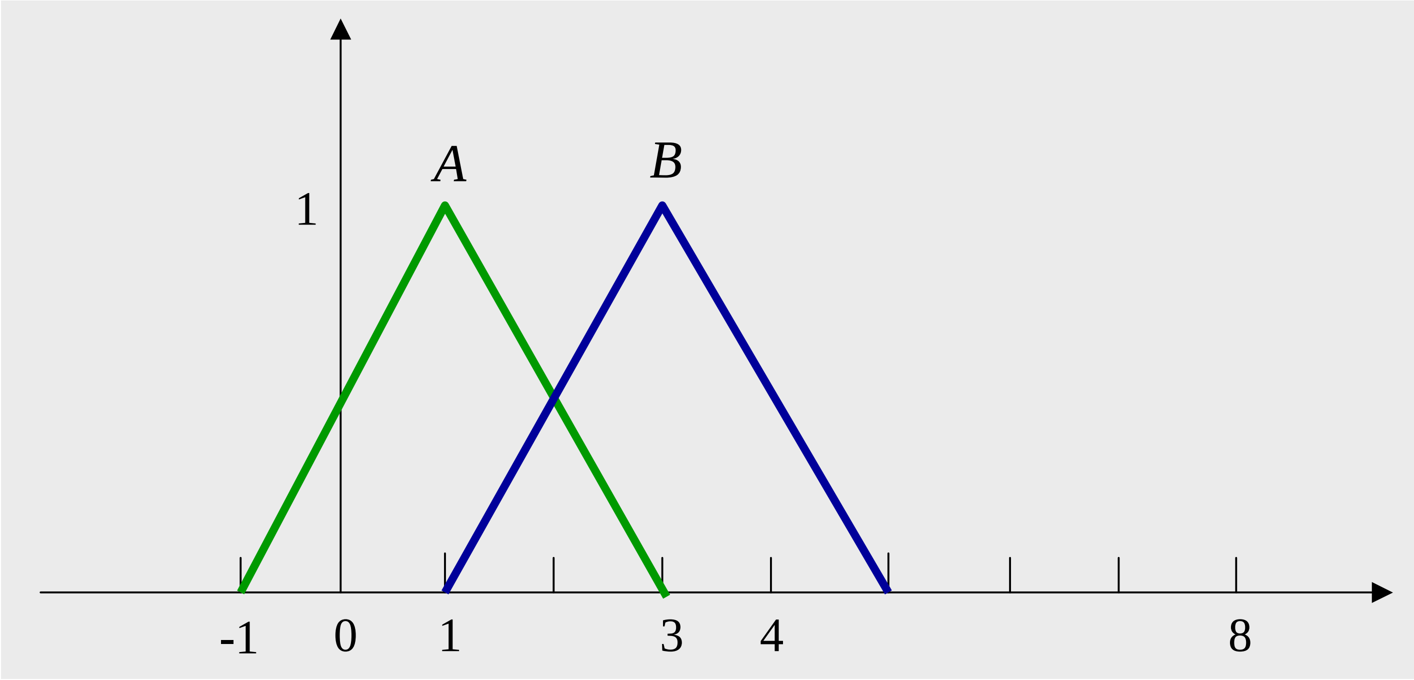
$$A * B = \bigcup_{\alpha \in [0,1]} (A * B)_\alpha$$

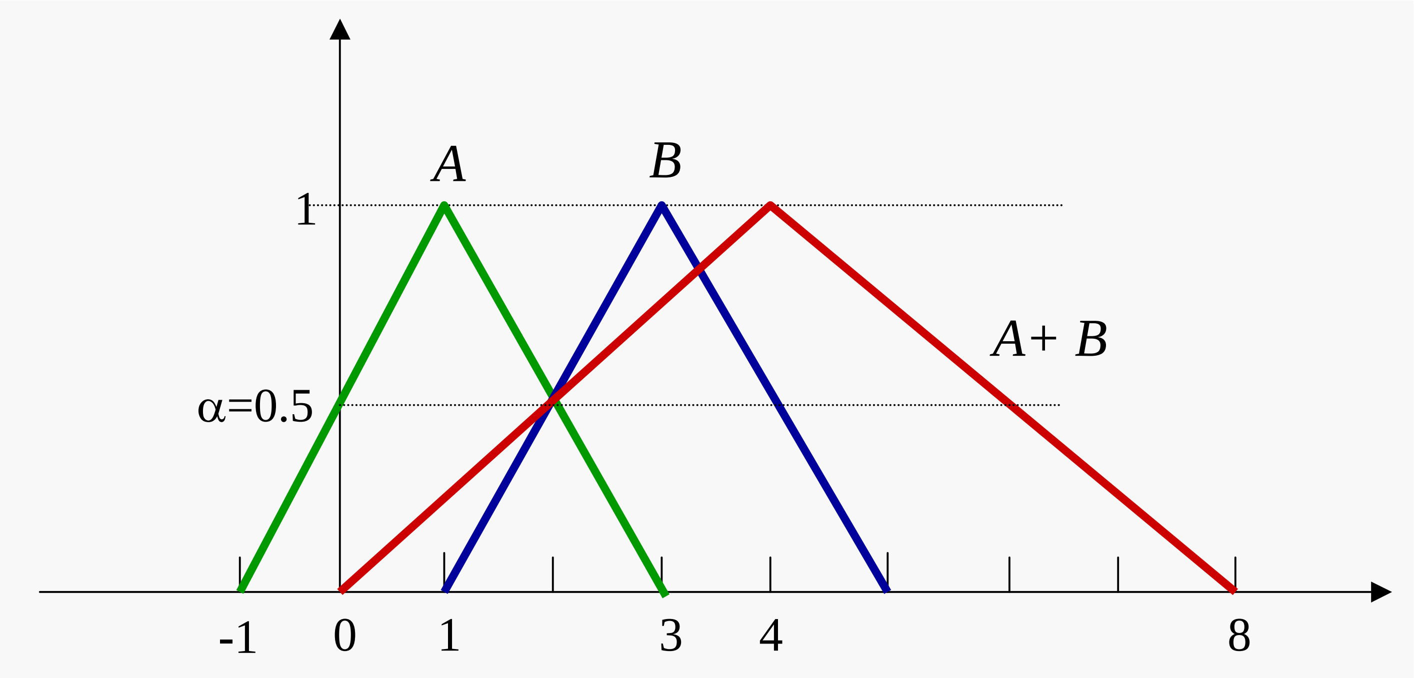
- assuming continuous membership functions
- when $* = /$ assume $0 \notin B_\alpha \ \forall \alpha \in (0,1]$

Example

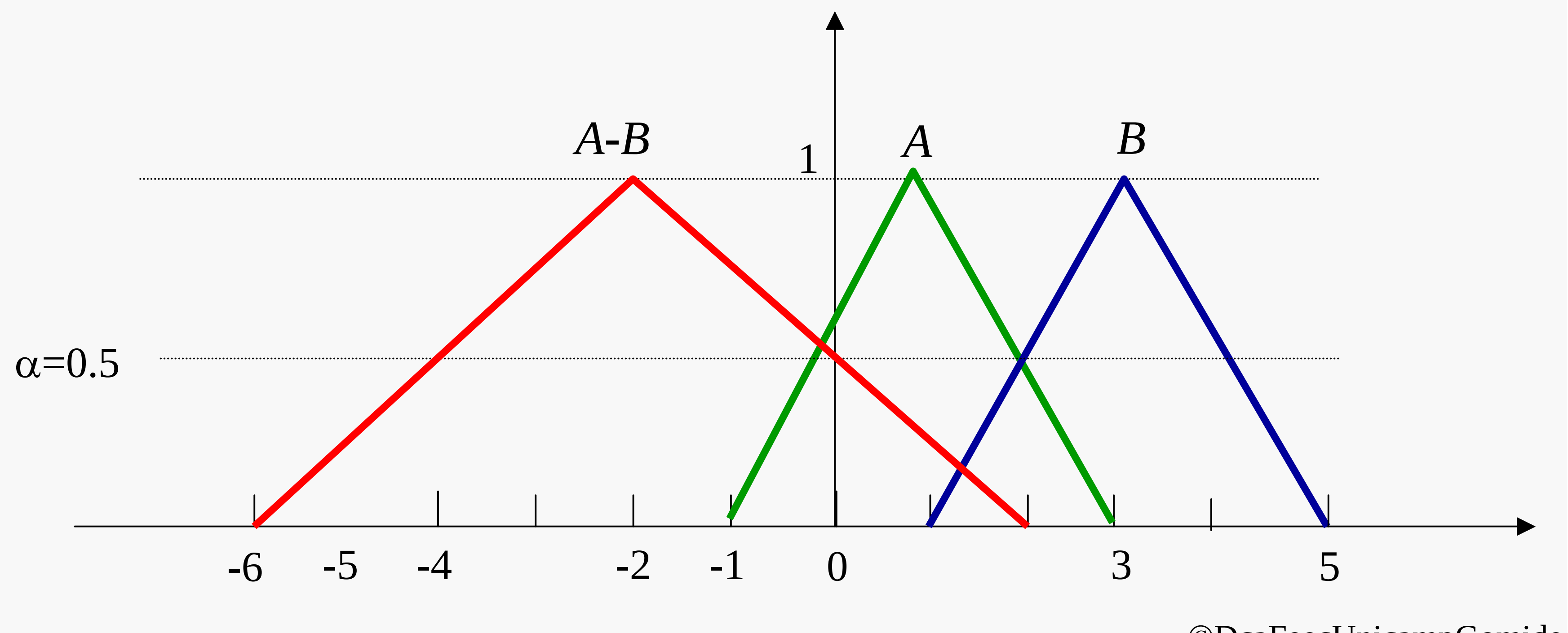
$$A(x)=\begin{cases}0 & x\le -1\,;\ x>3\\ (x+1)/2 & -1<x\le 1\\ (3-x)/2 & 1<x\le 3\end{cases}\qquad B(x)=\begin{cases}0 & x\le 1\,;\ x>5\\ (x-1)/2 & 1<x\le 3\\ (5-x)/2 & 3<x\le 3\end{cases}$$

$$A^{\alpha}=[2\alpha-1,\,3-2\alpha]\qquad B^{\alpha}=[2\alpha+1,\ 5-2\alpha]$$

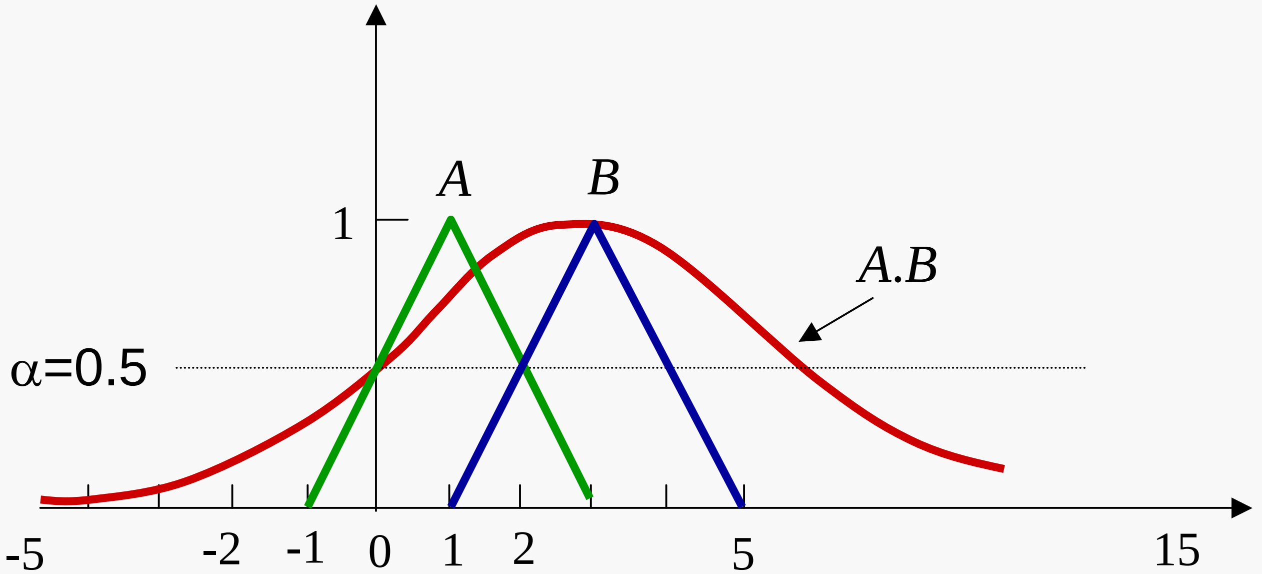




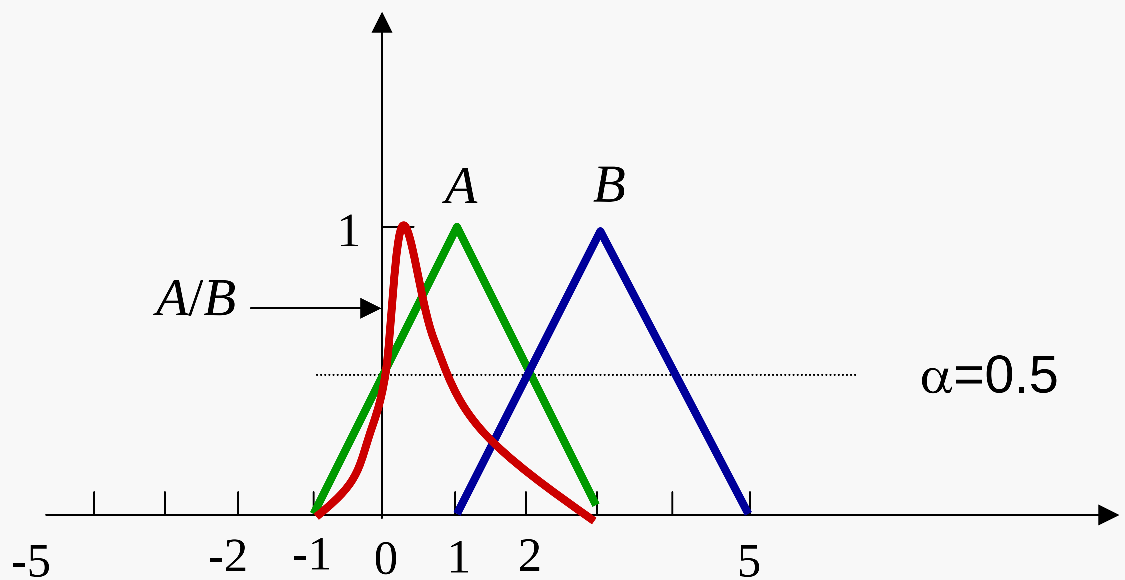
A
B
1
A+ B
α=0.5
-1
0
1
3
4
8



A-B
1
A
B
α=0.5
-6
-5
-4
-2
-1
0
3
5



A
B
1
A.B
α=0.5
-5
-2
-1
0
1
2
5
15



A
B
1
A/B
α=0.5
-5
-2
-1
0
1
2
5

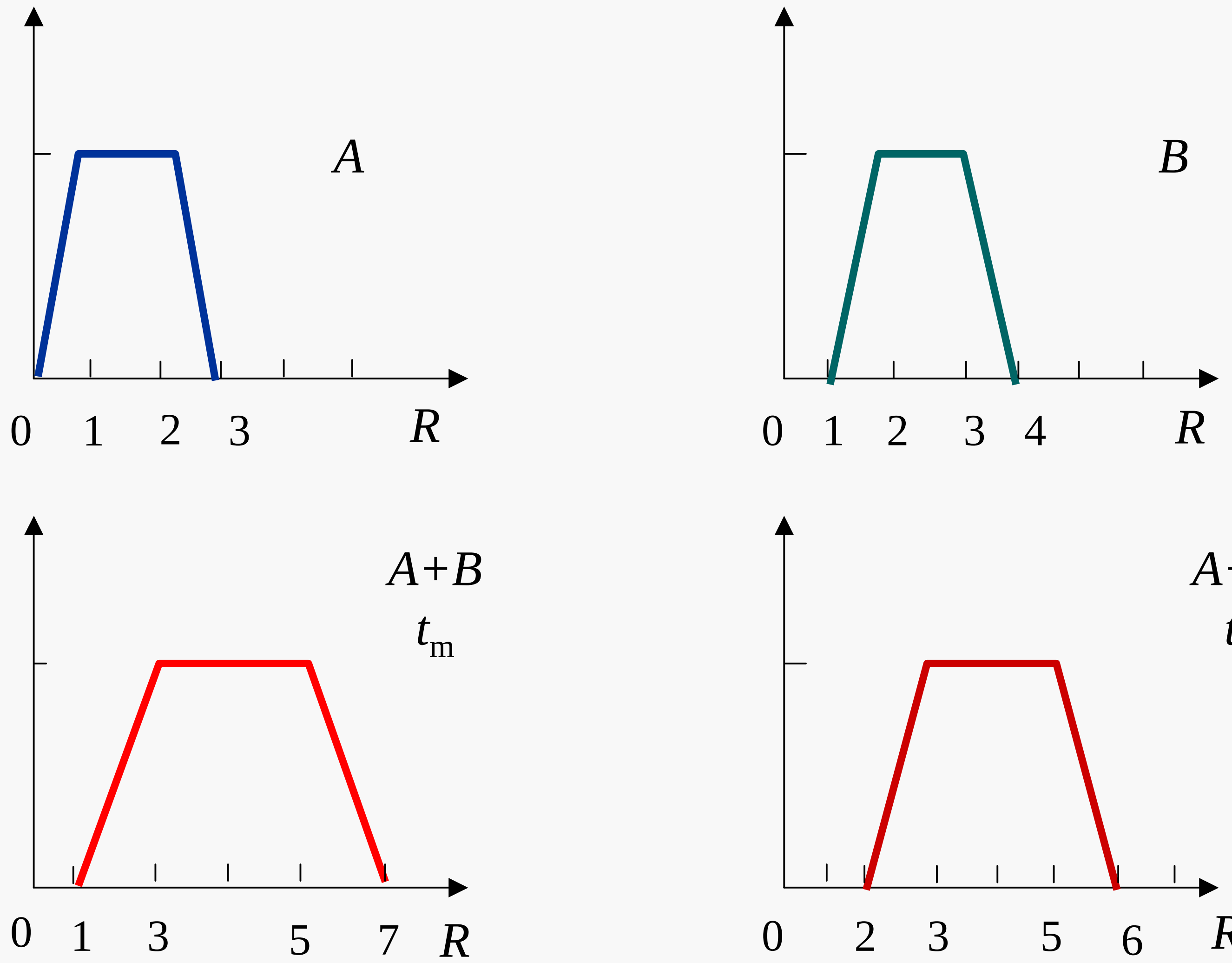
II- Extension Principle

$$(A * B)(x) = \sup_{z = x * y} \min[A(x), B(y)], \quad \forall z \in R$$

$$(A * B)(x) = \sup_{z = x * y} [A(x) \text{ t } B(y)], \quad \forall z \in R$$

t is a t-norm

Example



3- New Approaches for Fuzzy Arithmetic

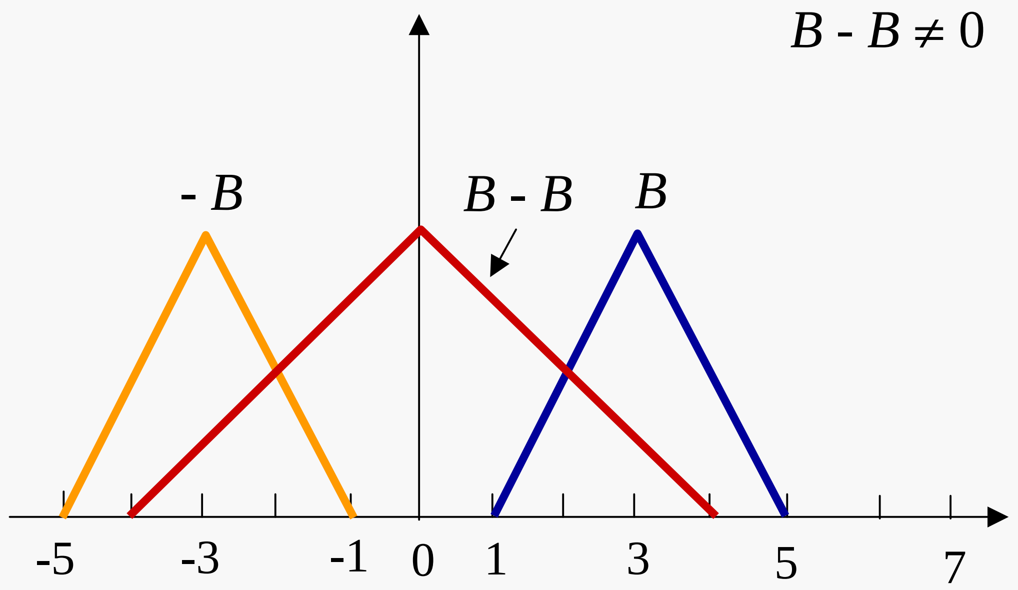
I - Requisite constraints (Klir, 1997)

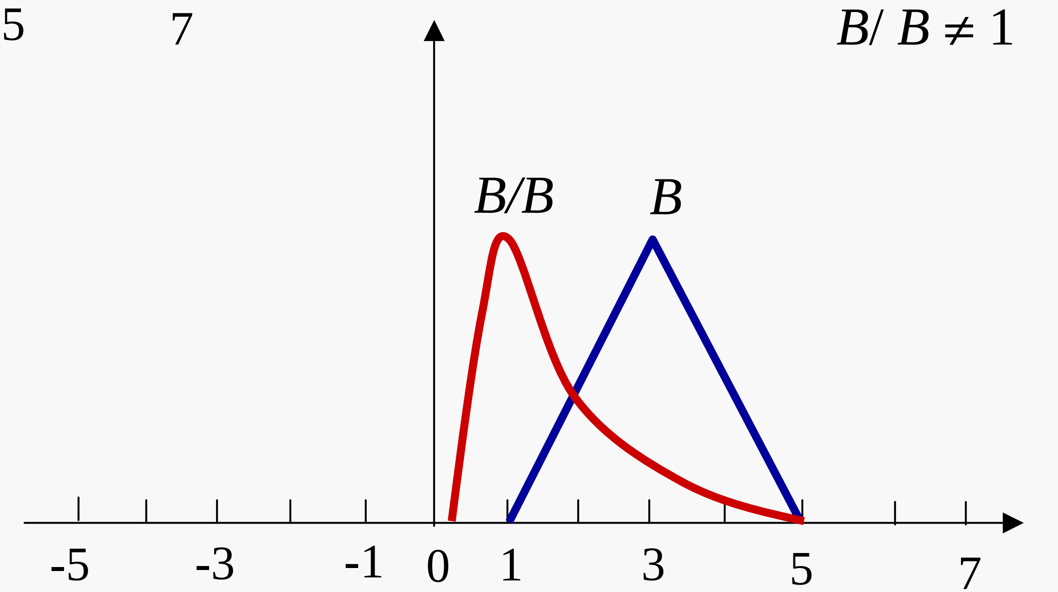
II - Discrete fuzzy arithmetic (Hanss, 1999, 2000)

III - Specialized approaches

- Kreinovich and Pedrycz, 2001
- Filev and Yager, 1997

I - Requisite constraints





choices x in A_α and y in B_α are not constrained by the equality $x = y$

- constraints: supplementary information not contained in operands

- constraints: result from the meaning of operands rather than the operands themselves

- constraints: are not taken into account by standard fuzzy arithmetic

Greater imprecision than justifiable in all computations that involve the requisite equality constraint (Klir, 1997)

- **Fuzzy interval arithmetic with requisite constraint**

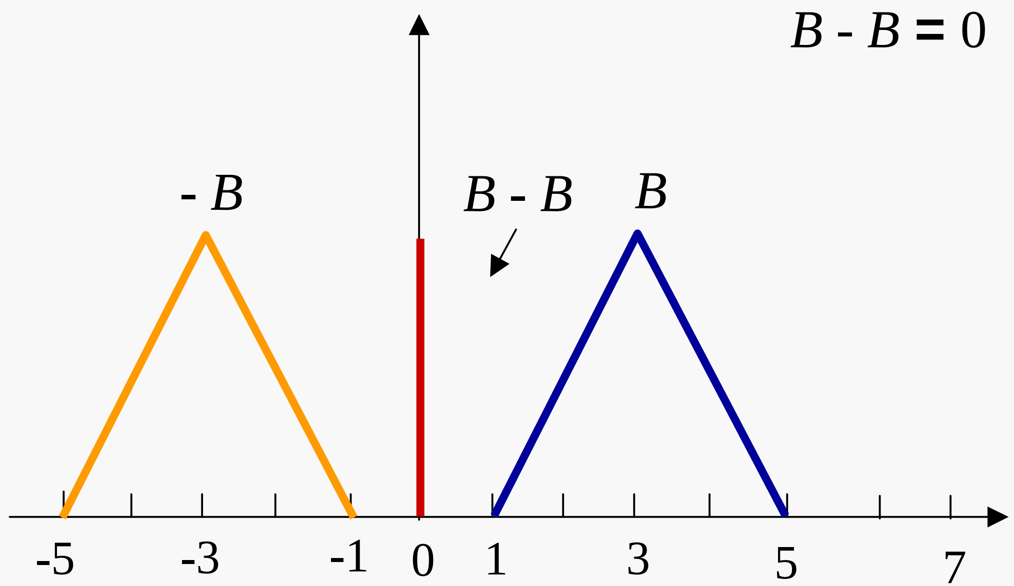
$$(A*B)^R_\alpha = \{x*y \mid (x,y) \in (A_\alpha \times B_\alpha) \cap \Re_\alpha, \quad \alpha \in (0,1]$$

$$(A*B)^R = \bigcup_{\alpha\in[0,1]} (A*B)^\alpha_R$$

- **Extension principle with requisite constraint**

$$(A*B)^R(x) = \sup_{z=x*y} \min[A(x),\ B(y), \Re(x,y)]$$

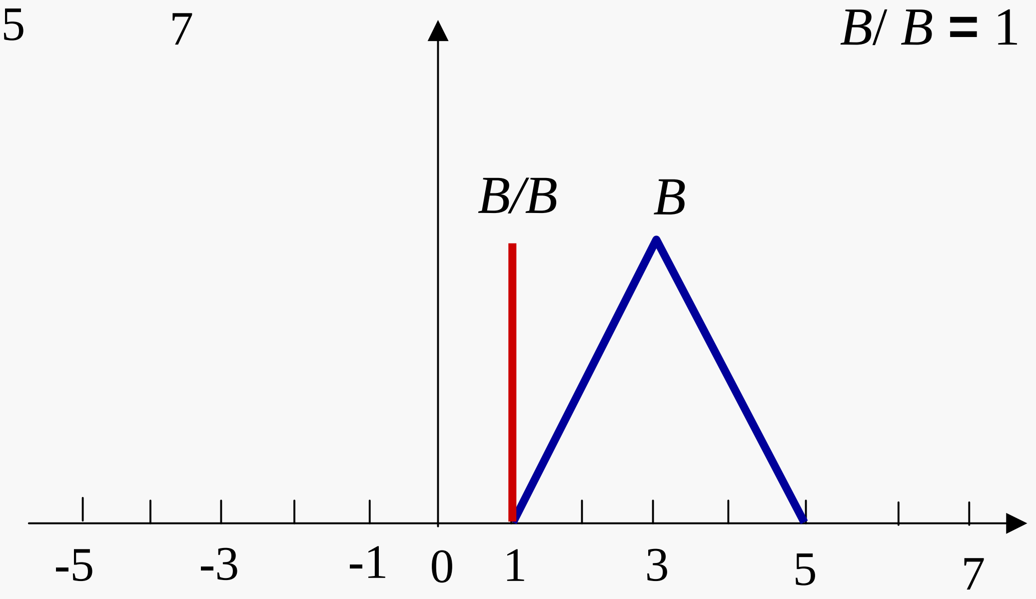
$$(A*B)^R(x) = \sup_{z=x*y} \{[A(x)\ \mathrm{t}\ B(y)] \wedge \Re(x,y)\}$$



$$(B - B)^E_\alpha = \{b - b \mid b \in B_\alpha\} = 0$$

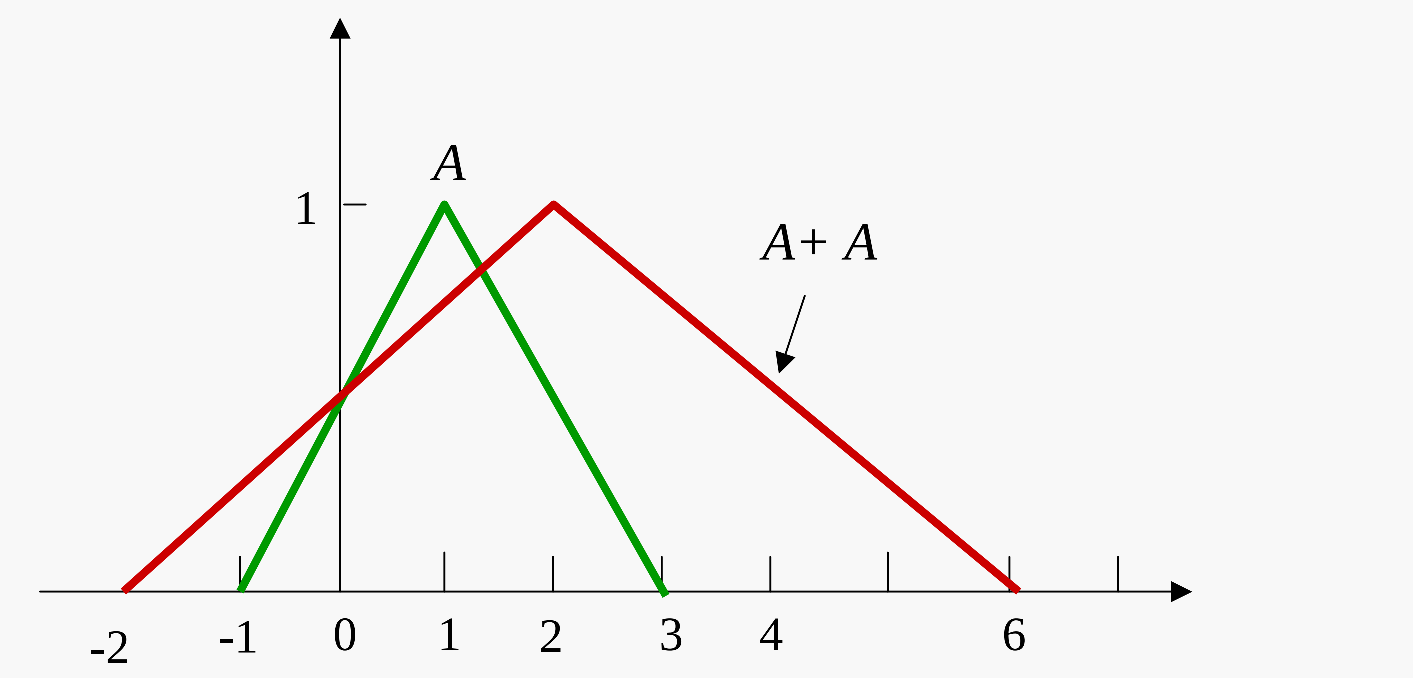
$$(B / B)^R_\alpha = \{b / b \mid b \in B_\alpha,\ 0 \notin B_\alpha\}$$

$$= 1$$

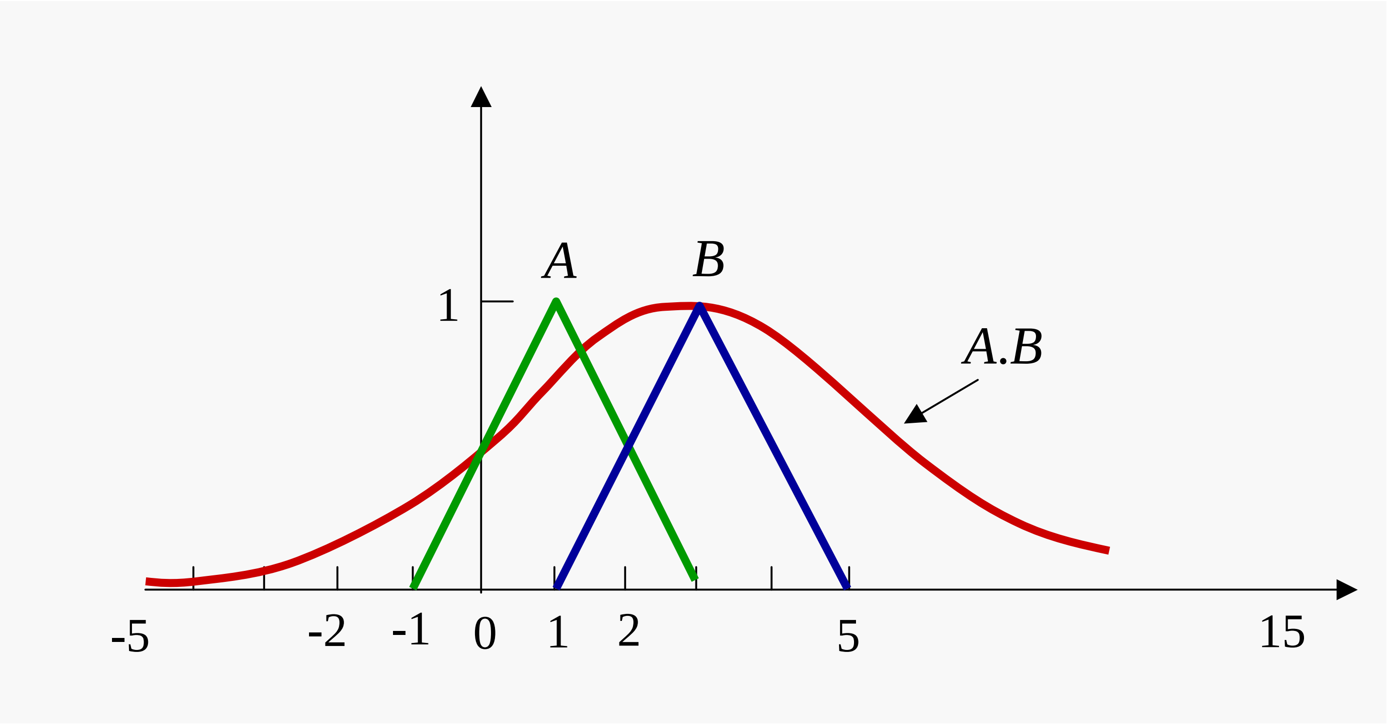


4- Analysis and Discussion

- Overestimation



• Shape preservation



• Properties: still to be proved for all three approaches

5- Conclusions

- Fuzzy quantities is important in many fields
- Mathematically OK but not always intuitive
- Requisite constraints seem to be a key
- Heuristics and approximate reasoning as a link

References

1-D. Dubois and H. Prade, Fuzzy Numbers: An Overview. In: J. Bezdek (ed.) Analysis of Fuzzy Information, CRC Press, Boca Raton, 1988, vol. 2, pp. 3-39.

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3-M. Hanss, On Implementation of Fuzzy Arithmetical Operations for Engineering Problems. *Proceedings of NAFIPS 1999* (1999) 462-466, New York.

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5-V. Kreinovich and W. Pedrycz, How to Make Sure that "≈100"+1 is ≈100 in Fuzzy Arithmetic: Solution and its (Inevitable) Drawbacks. *Proceedings of the FUZZ-IEEE* (2001), 1653-1658, Melbourne.

6-D. Filev and R. Yager, Operations on Fuzzy Numbers via Fuzzy Reasoning. *Fuzzy sets and Systems*, **91** (1997) 137-142.