9 Interoperability Aspects of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

Fuzzy sets and a family of α -cuts

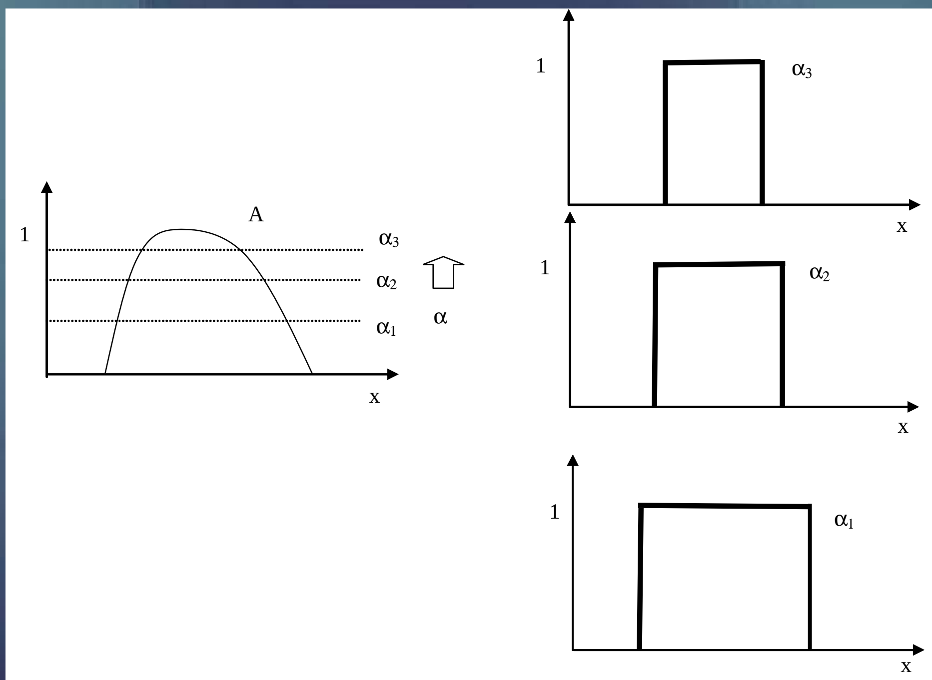
From fuzzy set to a family of sets

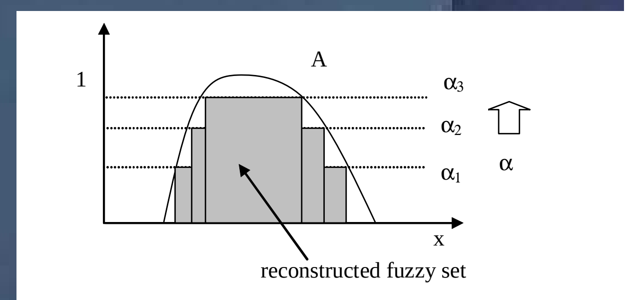
The representation theorem offers an important insight into links between a given fuzzy set and its α-cuts

In essence, any fuzzy set can be represented as an infinite family of α-cuts

$$A = \bigcup_{\alpha>0} \alpha A_{\alpha}$$

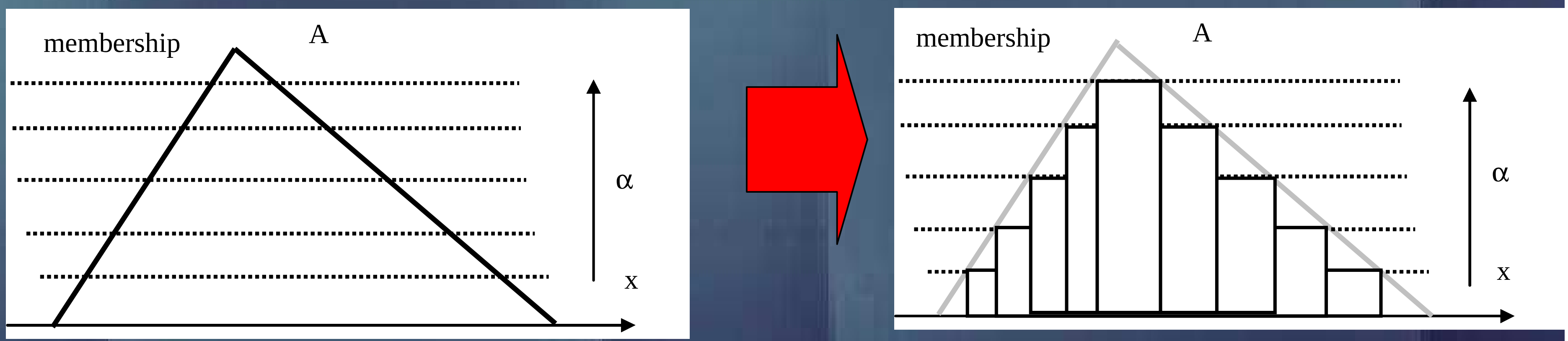
From fuzzy set to a family of sets





reconstruction

From fuzzy set to a family of sets



reconstruction

From fuzzy set to a family of sets
An optimization

Is there an optimal level a that optimizes a single α-cut of A so that A_α approximates A to the highest extent?

$\in$

$\geq$

Performance index

$$Q = \int_{x \notin A_\alpha} A(x)dx + \int_{x \in A_\alpha} (1 - A(x))dx$$

$$\min{}_\alpha Q = Q(\alpha_{opt}) \quad \text{viz. } \alpha_{opt} = \arg\ \mathrm{Min}_\alpha Q(\alpha($$

Triangular fuzzy sets - optimization

A(x) =max (1-x/b ,0) defined for positive values of x

Performance index

$$Q = \int_{b(1-\alpha)}^{b} (1-\frac{x}{b})dx + \int_{0}^{b(1-\alpha)} (1-1+\frac{x}{b})dx$$

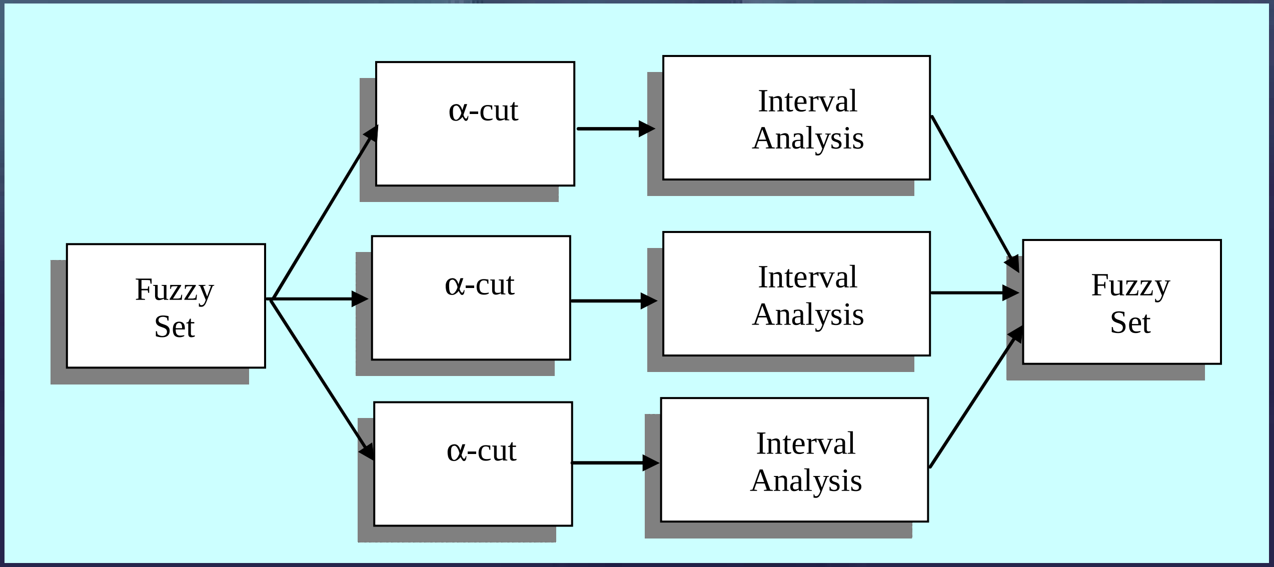
After integration

$$Q = b - b(1-\alpha) + b(1-\alpha)^2 - b/2$$

The minimum of Q, $dQ/d\alpha = 0$ is attained for $\alpha = ½$.

Set-based approximation of fuzzy sets

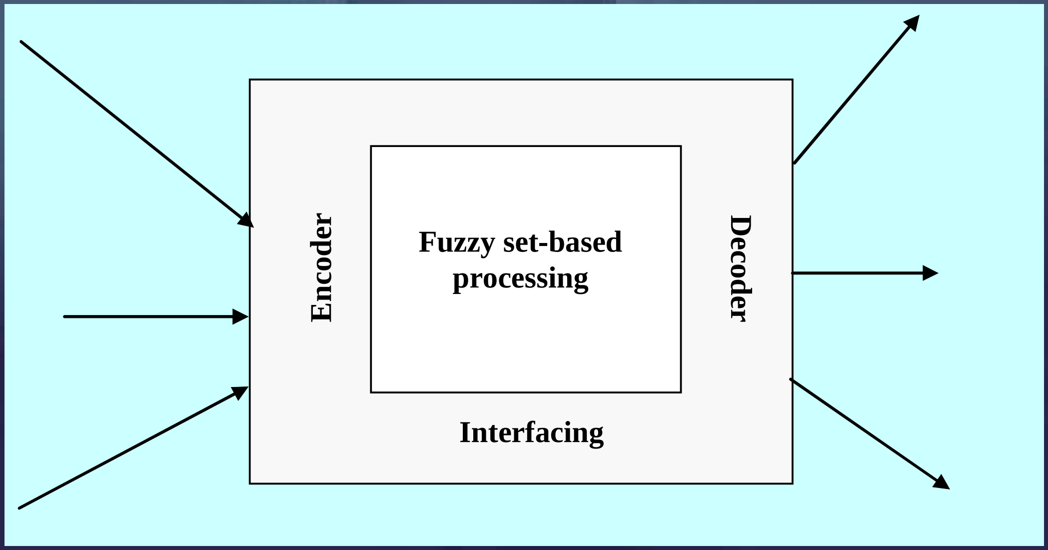
By approximating fuzzy sets by a finite family of sets we can directly exploit well-developed techniques of interval analysis and combine the partial results into a single fuzzy set (result).



Fuzzy sets and interfaces

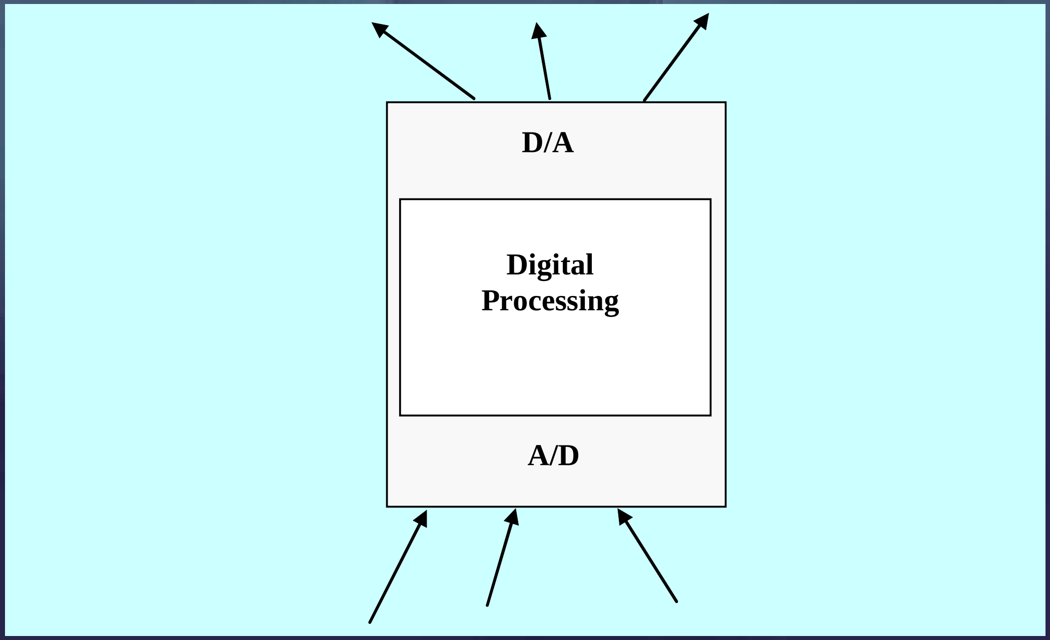
Fuzzy sets do not exist in real-world (sets do not as well).

To interact with the world one has to construct interfaces (encoders and decoders)



Fuzzy sets and interfaces

The need for building interfaces exists in case of sets (interval analysis); here we encounter well-known constructs of analog-to-digital (AD) and digital-to-analog (DA) converters.



Fuzzy sets and interfaces

There are two functional modules:

Encoders The objective is to translate input data into some internal format acceptable for processing at level of fuzzy sets

Decoders The objective is to convert the results of processing of fuzzy sets into some format acceptable by the external world (typically in the form of some numeric quantities)

For encoding and decoding we engage a collection of fuzzy sets – information granules

Encoding process

Given is a collection of fuzzy sets A_1, A_2, …, A_c; express some numeric input x in R in terms of these fuzzy sets

$$x \rightarrow [\ A_1(x) \quad A_2(x) \ldots\ A_c(x)]$$

Nonlinear mapping from **R** to c-dimensional unit hypercube

Decoding process

(a) decoding completed on a basis of a single fuzzy set

(b) Decoding realized on a basis of a certain finite family of fuzzy sets and levels of their activation.

Decoding process: a single fuzzy set

Single fuzzy set B → develop a single numeric representative

Single fuzzy set decoding: main strategies

Mean of maxima.Determine the arguments of **X** for which this membership function achieves its maximal values. Denote them by $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p$. The decoding is taken as the average of these values,

$$\hat{x} = \frac{\tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_p}{p}$$

Centre of area. Determine $\hat{x}$ such that it results in the equal areas below the membership function positioned on the left and on the right from this representative

$$\int_{-\infty}^{\hat{x}} B(x)dx = \int_{\hat{x}}^{\infty} B(x)dx$$

Centre of gravity Here the result of decoding is computed as follows

$$\hat{x} = \frac{\int_X B(x)xdx}{\int_X B(x)dx}$$

Single fuzzy set decoding: centre of gravity

Solution to the following optimization problem

$$V = \int_X B(x)[x - \hat{x}]^2 dx$$

$$\frac{dV}{d\hat{x}} = 0$$

$$2\int_X B(x)[x - \hat{x}]dx = 0$$

Single fuzzy set decoding: augmented strategies

Augmented centre of gravity

$$\hat{x} = \frac{\int_{x\in \mathbf{X}:B(x)\geq\beta} B(x)x\,dx}{\int_{x\in \mathbf{X}:B(x)\geq\beta} B(x)\,dx}$$

$$\hat{x} = \frac{\int_{x\in \mathbf{X}:B(x)\geq\beta} B^{\gamma}(x)x\,dx}{\int_{x\in \mathbf{X}:B(x)\geq\beta} B^{\gamma}(x)\,dx}$$

Single fuzzy set decoding: some general requirements

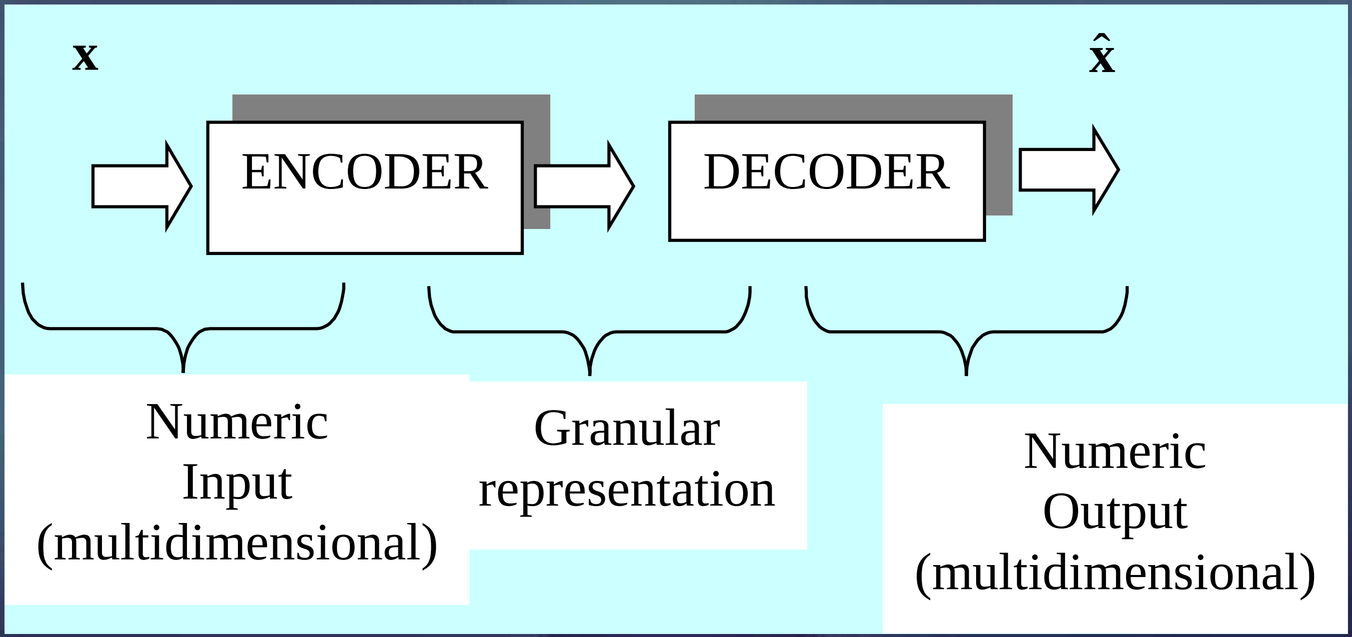
Requirements implied by :

•Monotonicity with respect to changeable membership functions

•Graphically motivated requirements (symmetry, translation, scaling…)

•The use of logic operations and logic modifiers

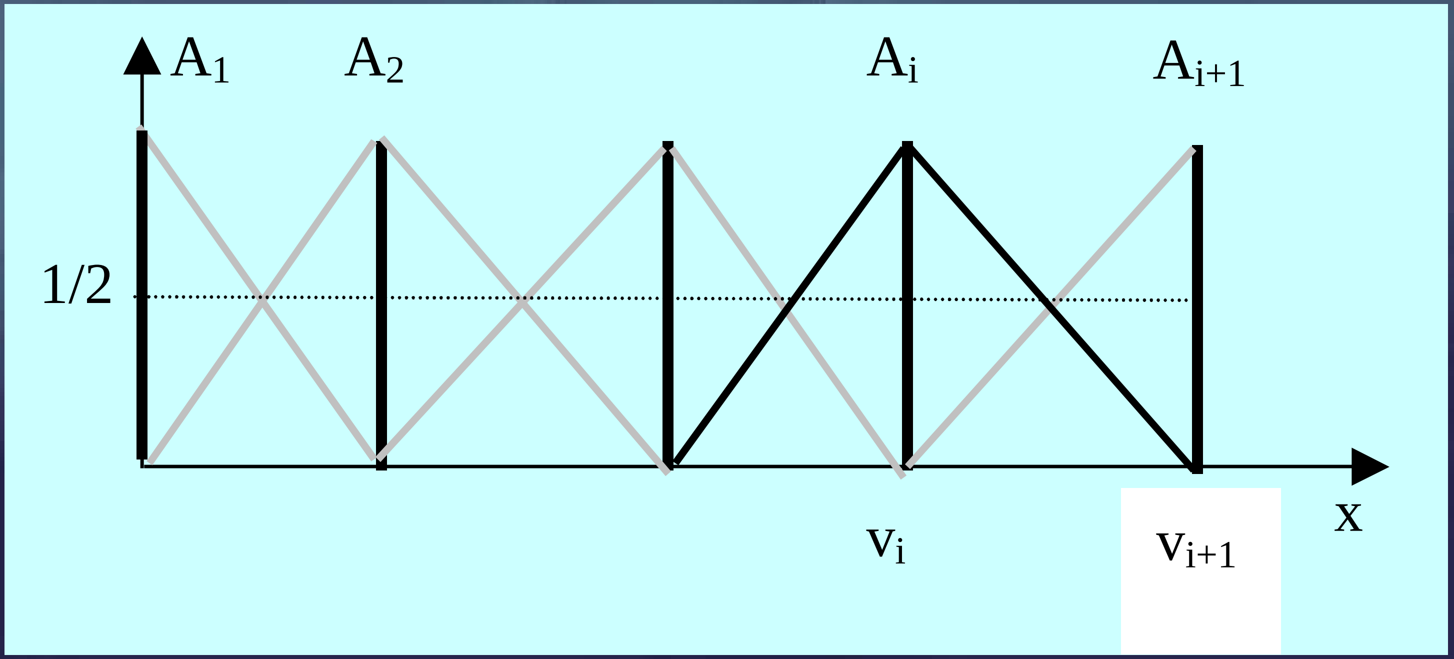
Decoding: a collection of fuzzy sets



- One-dimensional case
- Multivariable case

Decoding: one-dimensional (scalar) case

Codeboook – a finite family of fuzzy sets $\{A_1, A_2, \ldots, A_c\}$



One-dimensional decoding

Assume

(a) the fuzzy sets of the codebook $\{A_i\}$, i=1, 2,…,c form a fuzzy partition, $\sum_{i=1}^{c} A_i(x) = 1$, and for each x in **X** at least one element of the codebook is "activated", that is $A_i(x) > 0$

(b) for each x only two neighboring elements of the codebook are "activated" that is $A_1(x) = 0, \ldots, A_{i-1}(x) = 0$, $A_i(x) > 0$, $A_{i+1}(x) > 0$, $A_{i+2}(x) = \ldots = A_c(x) = 0$

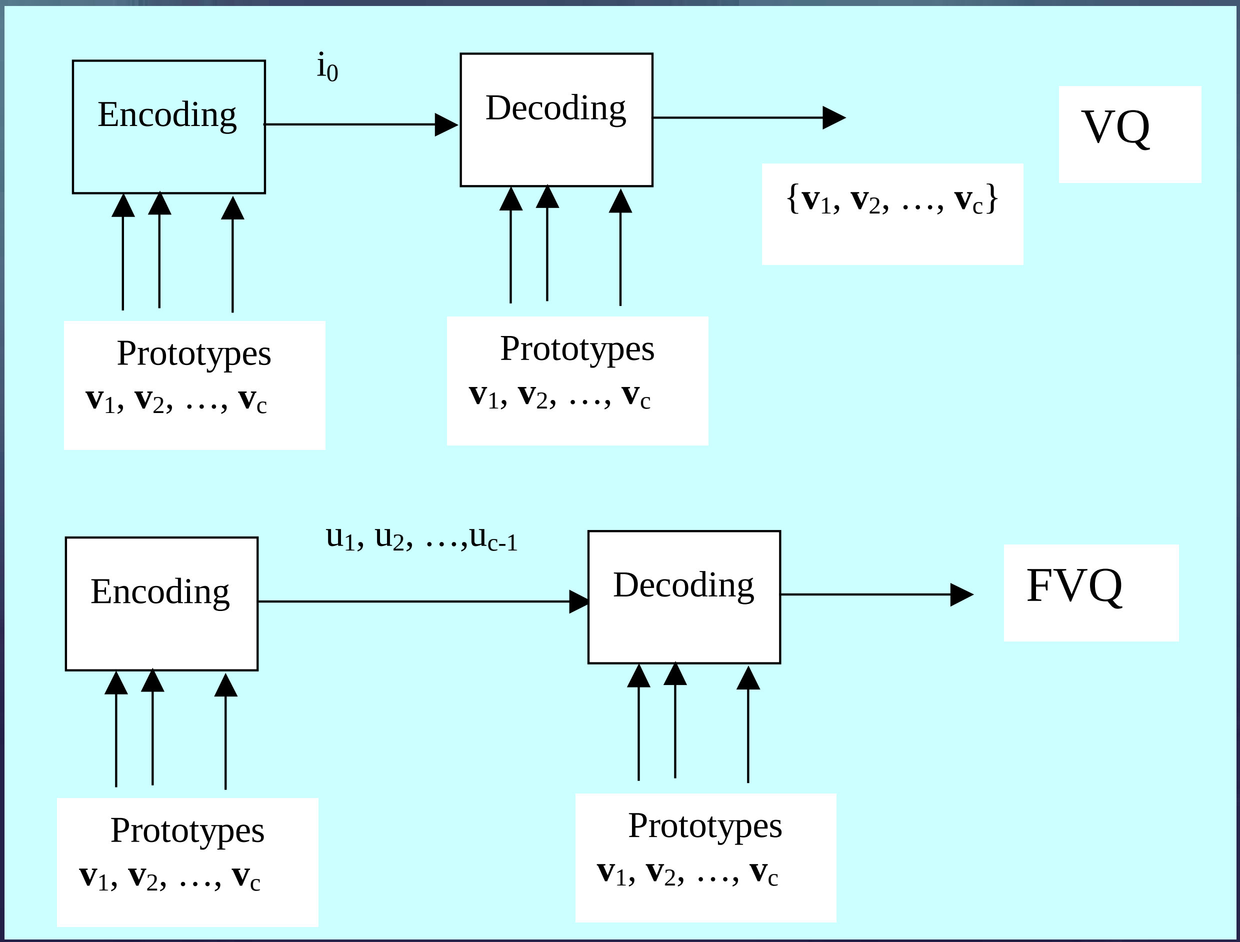
(c) the decoding is realized as a weighted sum of the activation levels and the prototypes of the fuzzy sets v_i , namely $\hat{x} = \sum_{i=1}^{c} A_i(x) v_i$

Then the elements of the codebook described by piecewise linear membership functions

$$A_i(x) = \begin{cases} \dfrac{x - v_{i-1}}{v_i - v_{i-1}} & \text{if} \quad x \in [v_{i-1}, v_i] \\ \dfrac{x - v_{i+1}}{v_i - v_{i+1}} & \text{if} \quad x \in [v_i, v_{i+1}] \end{cases}$$

lead to the zero decoding error (lossless compression) meaning that $\hat{x}$ = x.

Multivariable encoding and decoding: a global picture



use of sets –
Vector Quantization
(VQ)

use of fuzzy sets –
Fuzzy Vector Quantization
(FVQ)

Fuzzy Vector Quantization

The codebook formed through fuzzy clustering (FCM) producing
A finite collection of prototypes $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_c$.

Given any new input **x** we realize its encoding and decoding

Let us recall

Encoding – representation of x in terms of the prototypes

Decoding – development of external representation of the result of processing realized at the level of information granules

$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

Fuzzy Vector Quantization: Encoding

The optimization problem

$$\sum_{i=1}^{c} u_i^m \| \mathbf{x} - \mathbf{v}_i \|^2$$

Minimize w.r.t. u_i subject to

$$u_i(\mathbf{x}) \in [0,1], \quad \sum_{i=1}^{c} u_i(x) = 1$$

$$u_i(\mathbf{x}) = \frac{1}{\sum \left(\frac{\| \mathbf{x} - \mathbf{v}_i \|}{\| \mathbf{x} - \mathbf{v}_j \|} \right)^{\frac{2}{m-1}}}$$

$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

Fuzzy Vector Quantization: Decoding

Reconstruct original mutidimensional input x

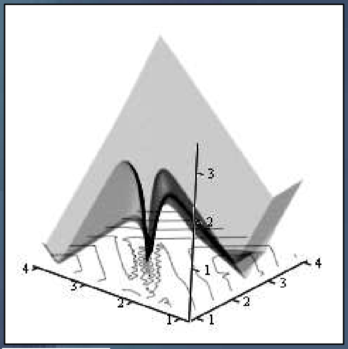
$$Q_2(\hat{\mathbf{x}}) = \sum_{i=1}^{c} u_i^m \, \| \hat{\mathbf{x}} - \mathbf{v}_i \|^2$$

minimize

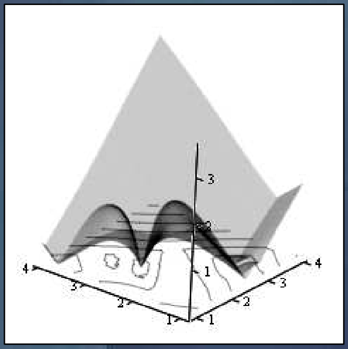
$$\hat{\mathbf{x}} = \frac{\sum_{i=1}^{c} u_i^m \mathbf{v}_i}{\sum_{i=1}^{c} u_i^m}$$

$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

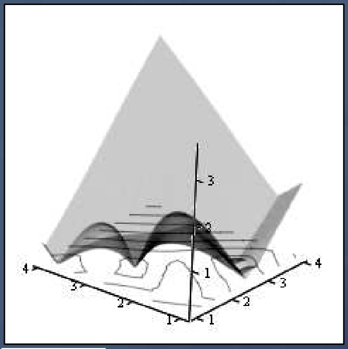
Fuzzy Vector Quantization: Decoding error



m =1.2



m =2.0



m =3.5

$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

Fuzzy encoding and decoding with possibility and necessity measures

Consider a family of fuzzy sets $A_1, A_2, \ldots, A_c$
Input datum X could be either a fuzzy set or a numeric quantity

$$\text{Poss}(A_i, X) = \sup_{x \in \mathbf{X}}[X(x) t A_i(x)]$$

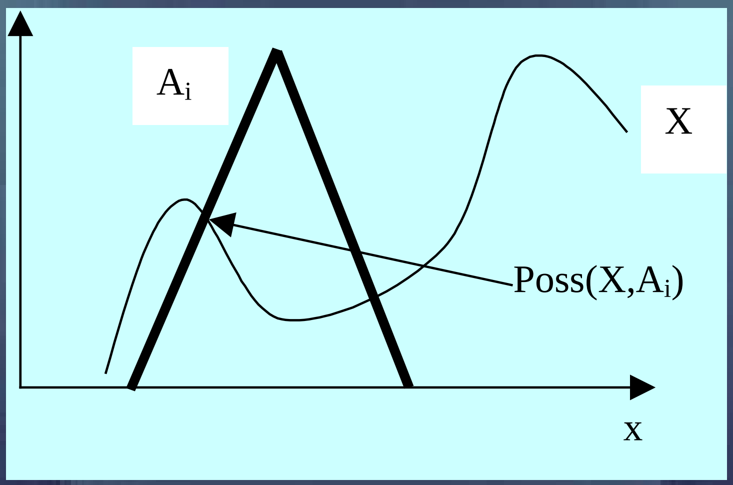
$$\text{Nec}(A_i, X) = \inf_{x \in \mathbf{X}}[(1 - A_i(x)) s X(x)]$$

possibility

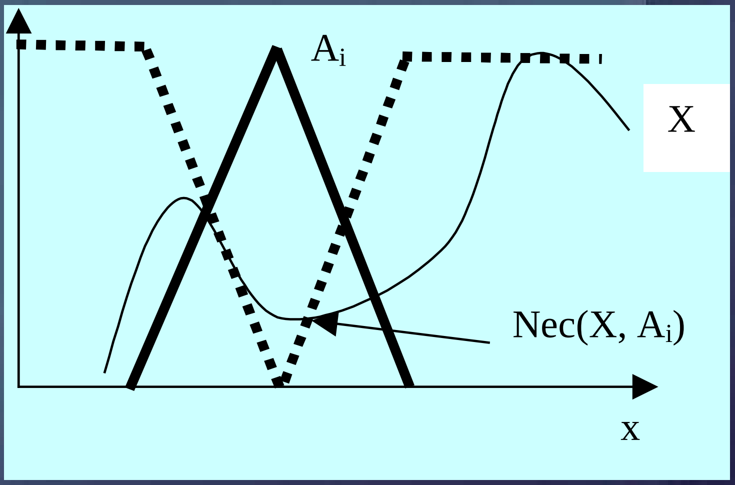
necessity

$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

Possibility and necessity encoding



possibility



necessity

$u_i(\mathbf{x}) \in [0,1], \quad \sum_{i=1}^{c} u_i(x) = 1$

Possibility and necessity encoding: example

X = [0.0 0.2 0.8 1.0 0.9 0.5 0.1 0.0]

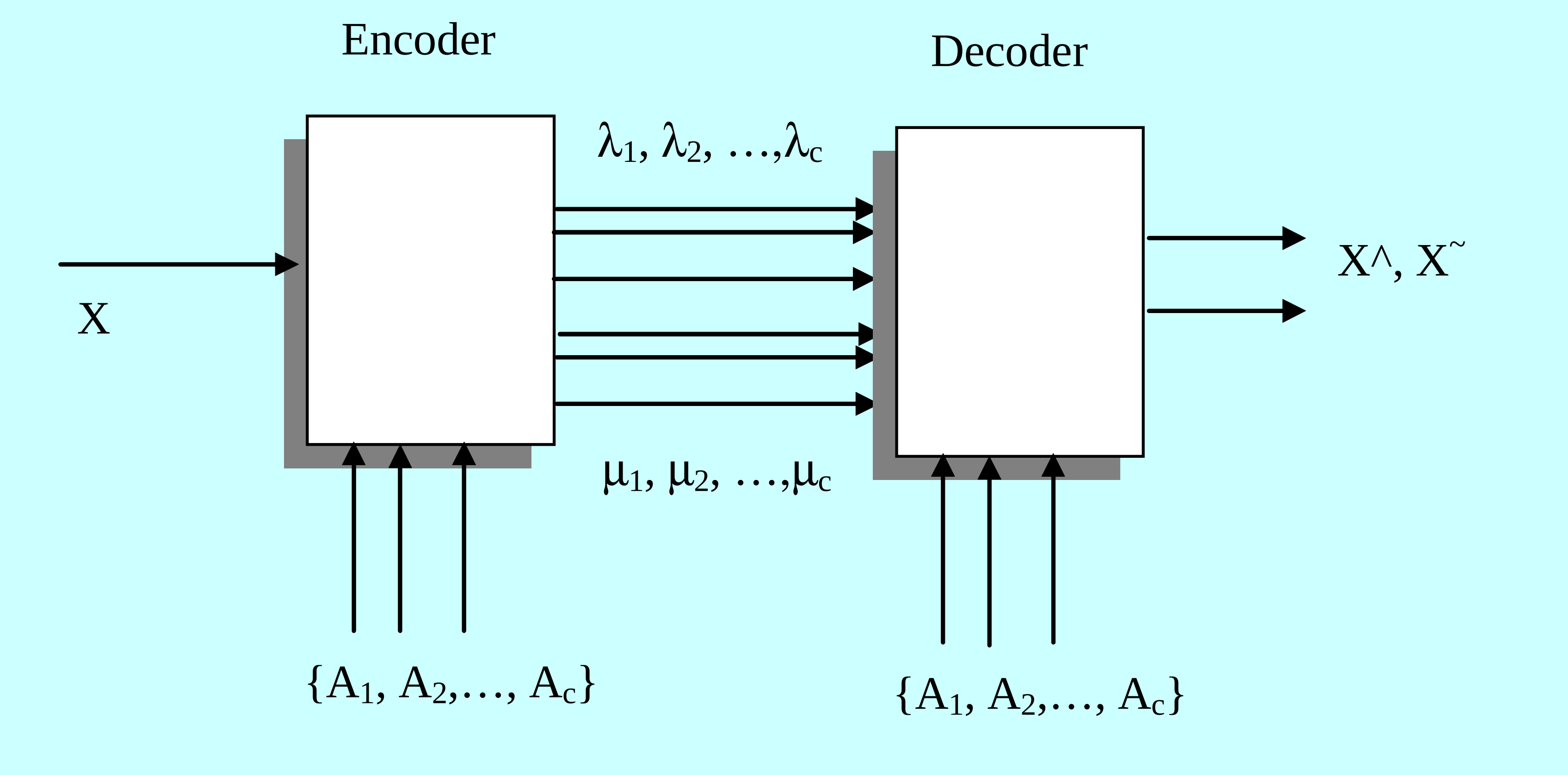
A_i = [0.6 0.5 0.4 0.5 0.6 0.9 1.0 1.0]

Poss(A_i, X) = max (0.0, 0.5, 0.4, 0.5, 0.6, 0.5, 0.1, 0.0) = 0.6

Nec (A_i, X) = min(0.4, 0.5, 0.8, 1.0, 0.9, 0.5, 0.1, 0.0) =0.0

$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

Encoding and decoding: an overview



$u_i(\mathbf{x}) \in [0,1], \ \sum_{i=1}^{c} u_i(x) = 1$

Decoding

Given the nature of encoding (possibility and necessity measures), the decoding is regarded as a certain “inverse” problem in terms of fuzzy relational equations:

Possibility measure – sup-t composition

Necessity measure – inf-s composition

Decoding –possibility measure

Possibility measure – sup-t composition

$$\hat{X}(x) = A(x) \to \lambda \;\; = \begin{cases} 1 & \text{if } A(x) \le \lambda \\ \lambda & \text{otherwise} \end{cases}$$

$$\hat{X}(x) = A(x) \to \lambda \;\; = \sup[a \in [0,1] \mid a t A(x) \le \lambda]$$

$$\hat{X} = \bigcap_{i=1}^{c} \hat{X}_i$$

Decoding –necessity measure

Necessity measure – inf-s composition

$$\tilde{X}(x) = (1 - A \ \ (x))\varepsilon\,\mu \quad = \begin{cases} \mu, & \text{if } 1 - A \ \ (x) < \mu \\ 0\,, & \text{otherwise} \end{cases}$$

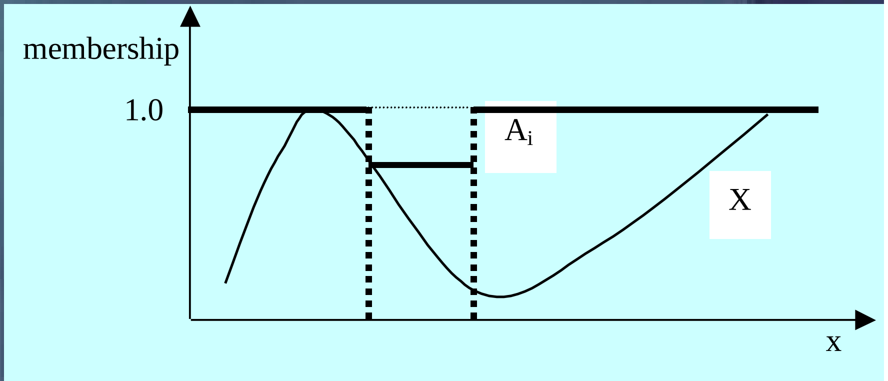
$$\tilde{X}(x) = (1 - A_i\,(x))\varepsilon\mu = \inf\{a \in [0,1] \mid as(1 - A(x)) \geq \mu\}$$

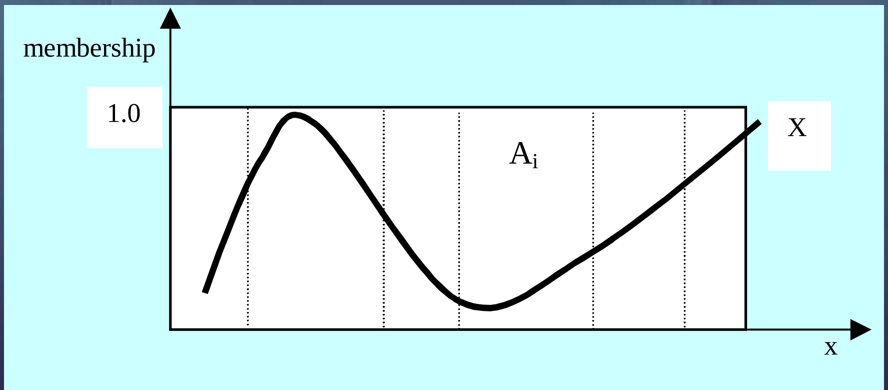
$$\tilde{X} = \bigcup_{i=1}^{c} \tilde{X}_i$$

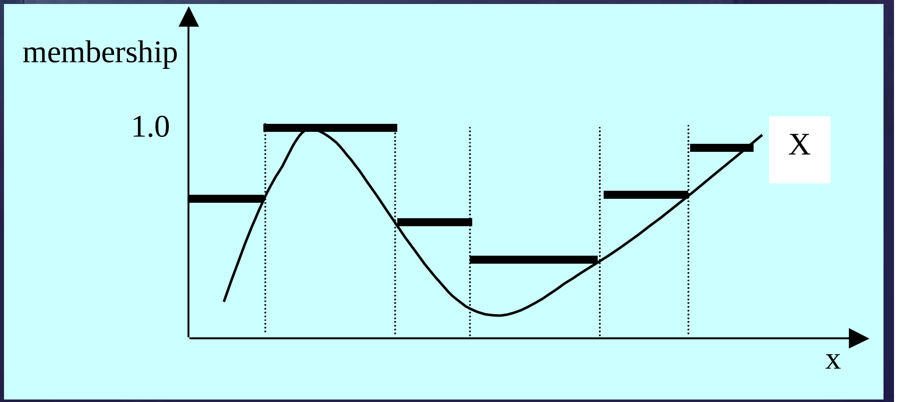
$$\tilde{X} \subseteq X \subseteq \hat{X}$$

Decoding: example

possibility measure



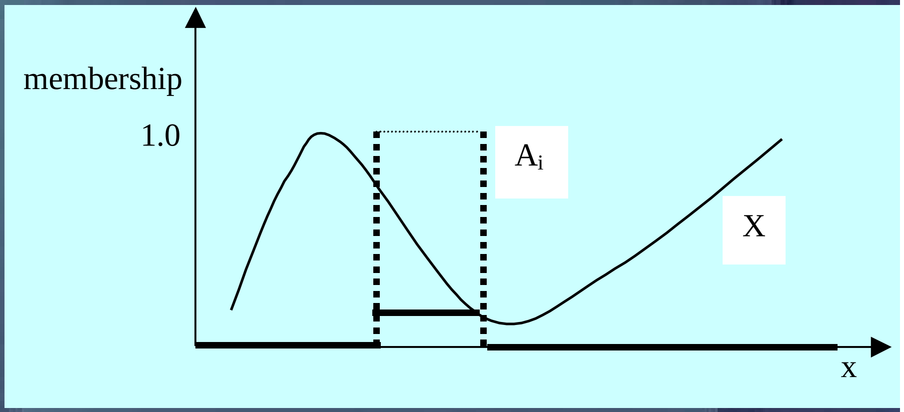


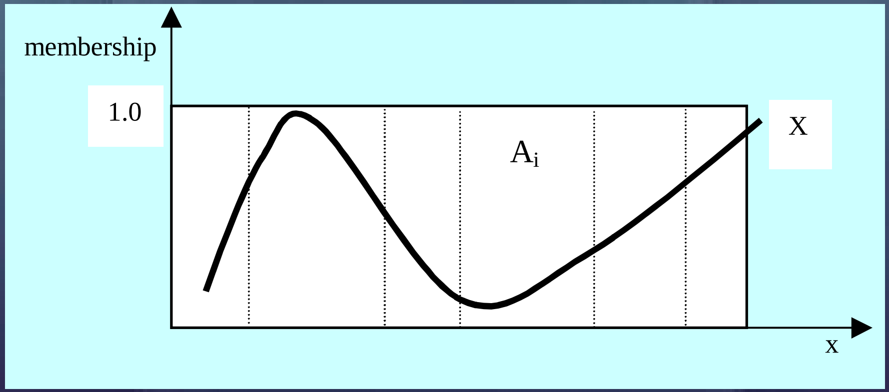


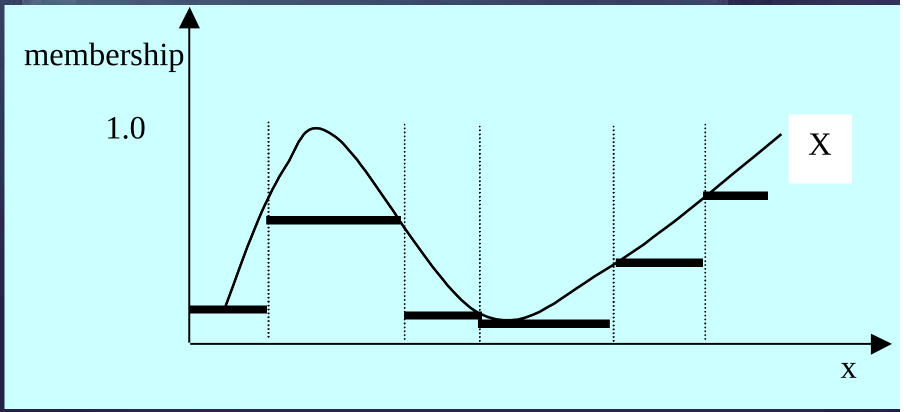
Pedrycz and Gomide, FSE 2007

Decoding: example

necessity measure



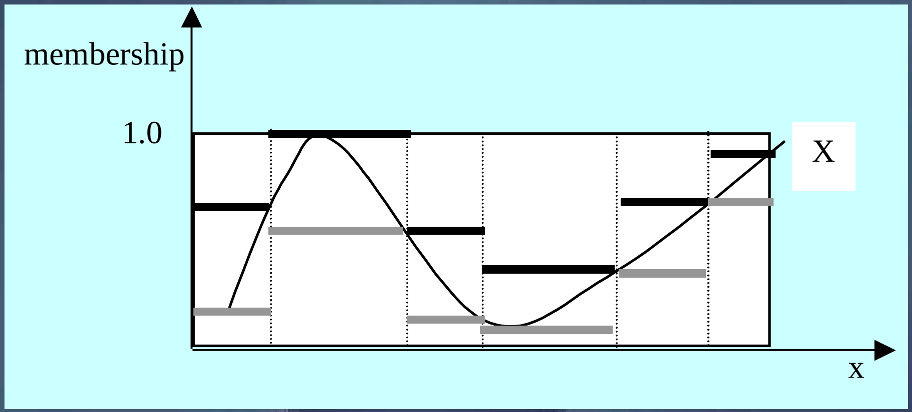


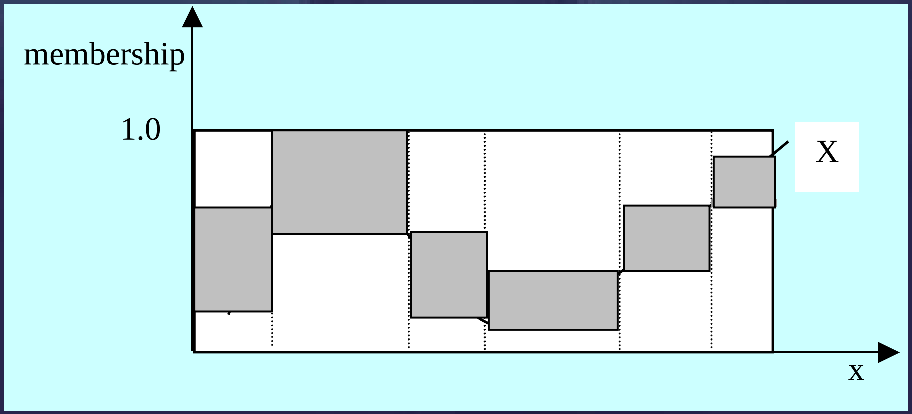


Pedrycz and Gomide, FSE 2007

Decoding: example

bounds of possibility and necessity measure





Taxonomy of data structure with the use of shadowed sets

Core structure

Shadowed data structure

Uncertain data structure

Taxonomy of data structure with the use of shadowed sets

Core data structure patterns that belong to a core of at least one or more shadowed sets

Core data structure = $\{\, \mathbf{x} \mid \exists_i \;\; \mathbf{x} \in \mathrm{core}(A_i) \}$

Taxonomy of data structure with the use of shadowed sets

Shadowed data structure This structure is formed by patterns that do not belong to core of any of the shadowed sets but fall within the shadow of one or more shadowed sets.

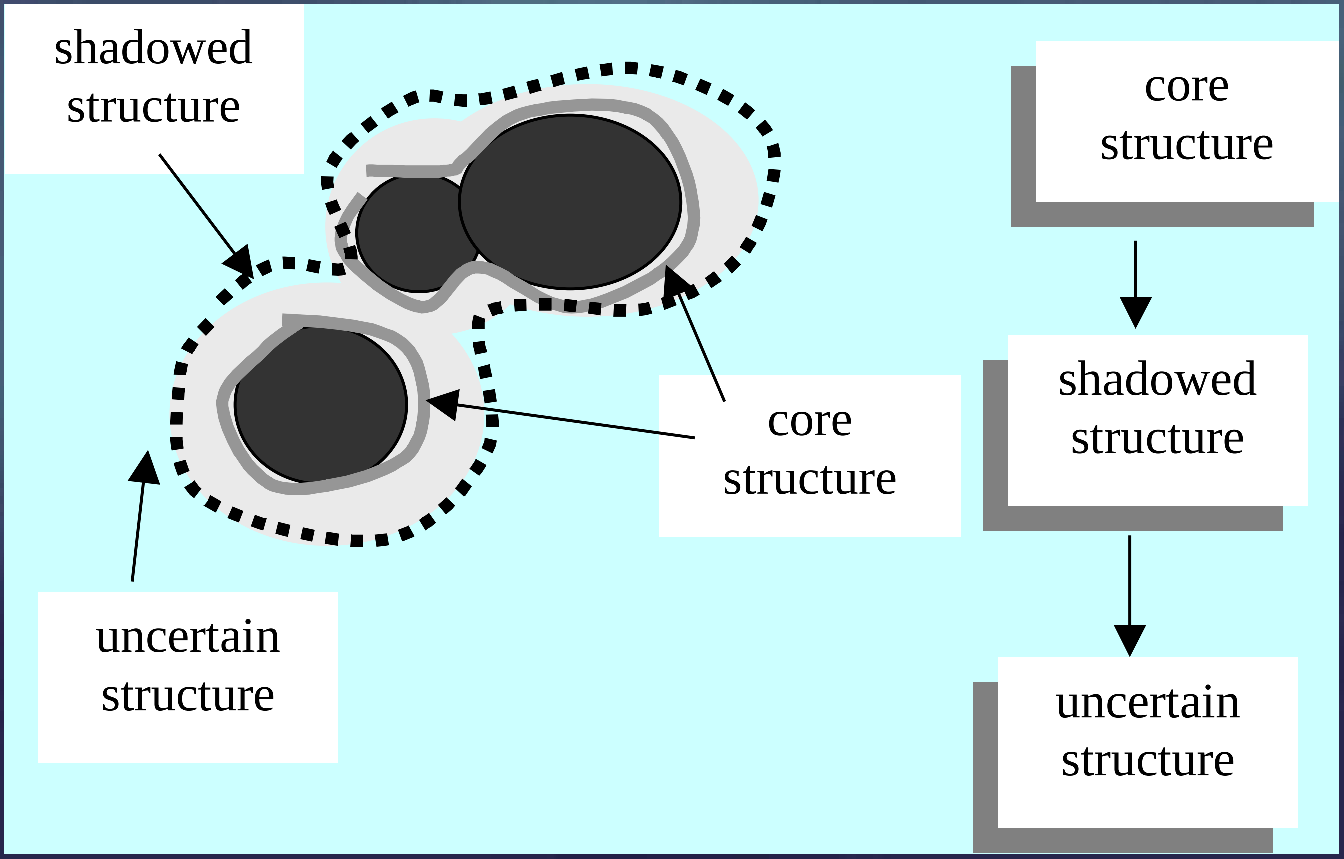
Shadowed data structure = $\{\, \mathbf{x} \mid \exists_i \ \mathbf{x} \in \text{shadow}(A_i) \ \text{ and } \forall_i \mathbf{x} \notin \text{core}(A_i) \,\}$

Taxonomy of data structure with the use of shadowed sets

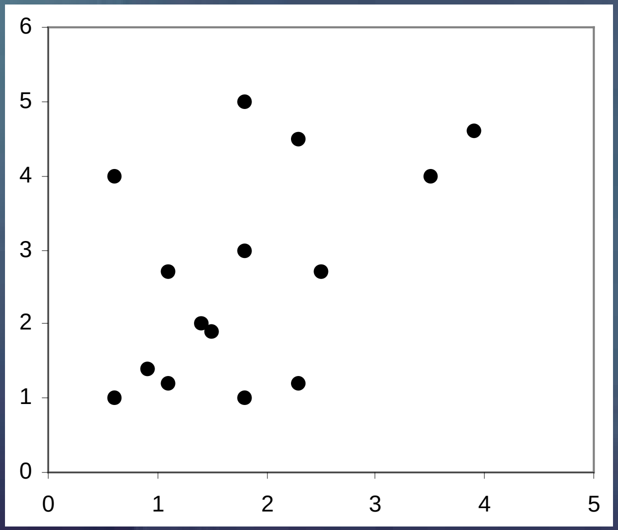
Uncertain data structure The patterns belonging to this structure are those that are left out from all shadows

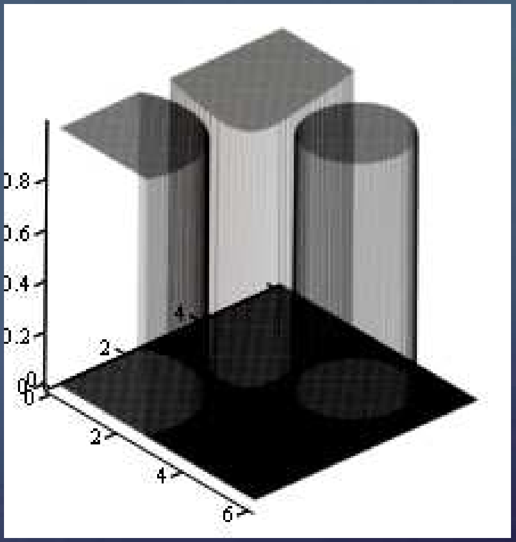
$$\text{Uncertain data structure} = \{\, \mathbf{x} \mid \forall_i \ \mathbf{x} \notin \text{shadow}\,(A_i) \ \text{ and } \forall_i \mathbf{x} \notin \text{core}(A_i)\}$$

Three-valued characterization of data structure with shadowed sets



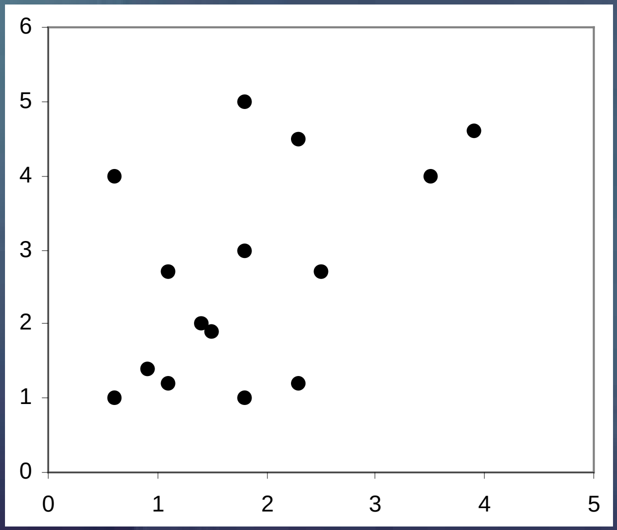
Three-valued characterization of data structure: example

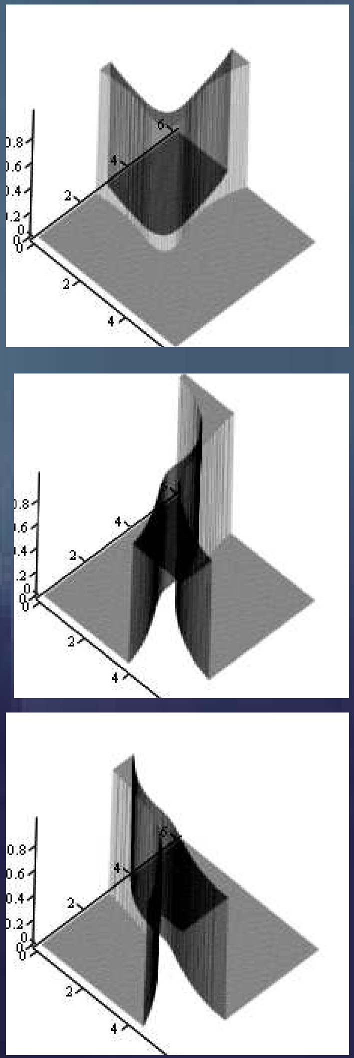




core

Three-valued characterization of data structure: example





shadow