7 Transformations of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

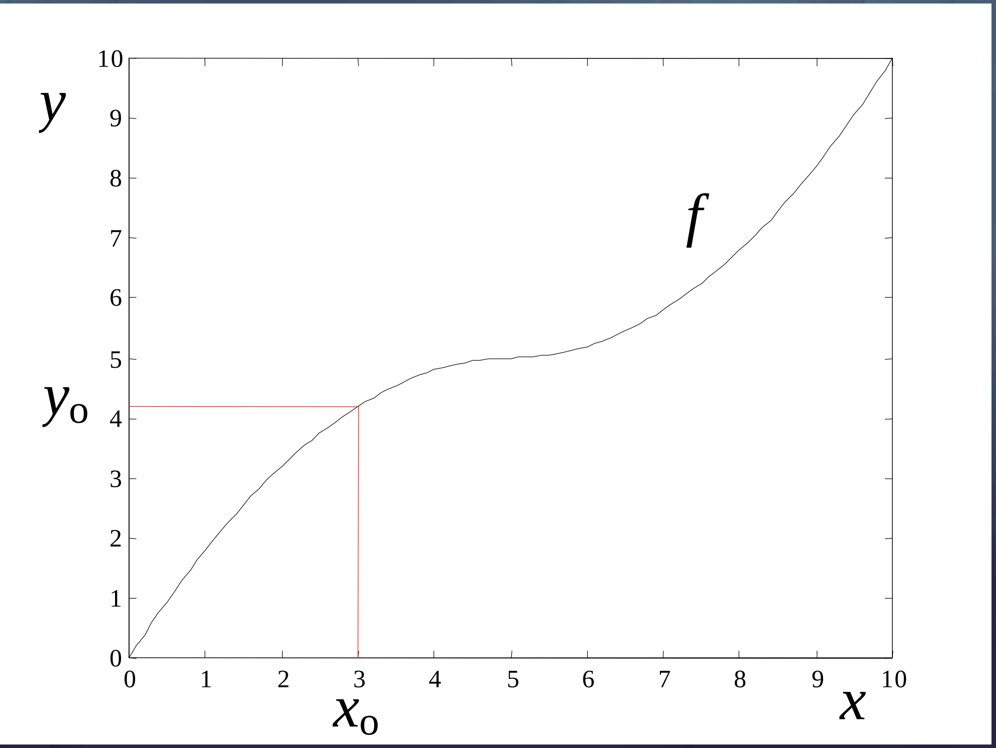
7.1 The extension principle

Extension principle

- Extends point transformations to operations involving
  - sets
  - fuzzy sets
- Given a function f: $\mathbf{X} \rightarrow \mathbf{Y}$ and a set (or fuzzy set) A on $\mathbf{X}$ the extension principle allows to map A into a set (or fuzzy set) on $\mathbf{Y}$ through f

Pointwise transformation

f is a function



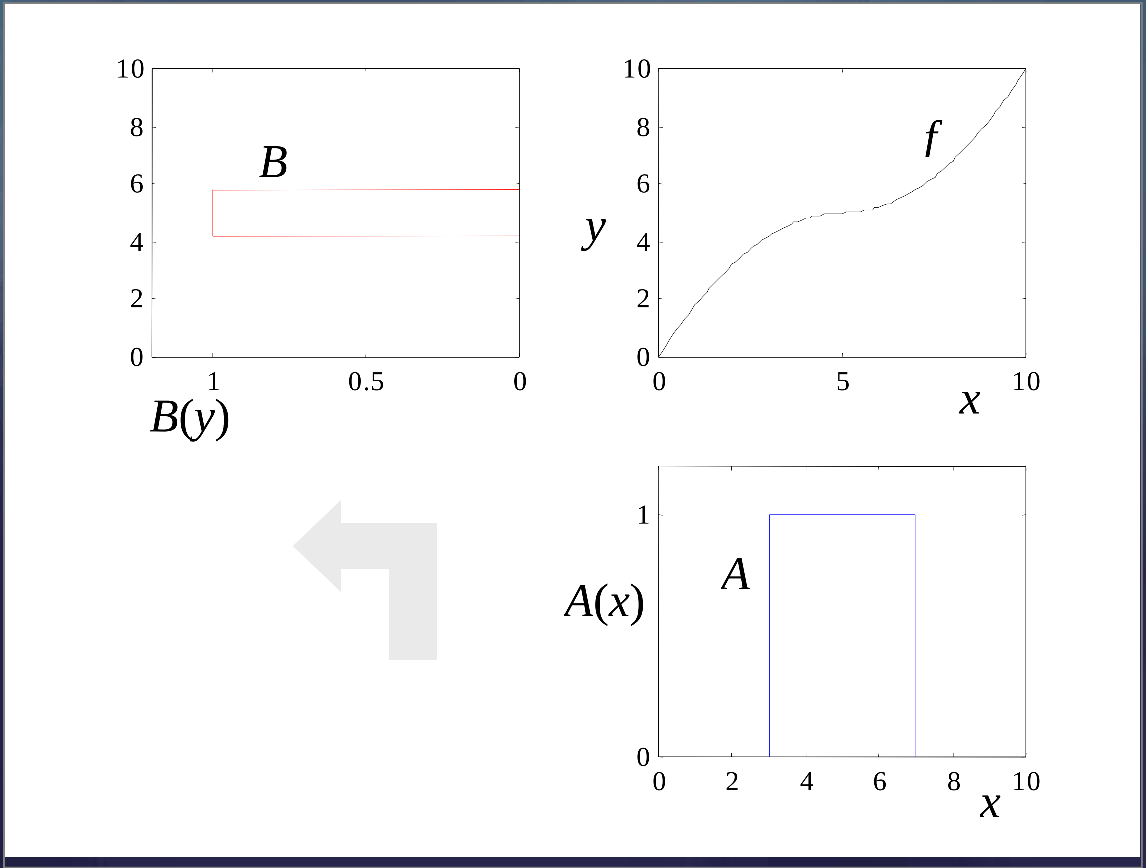
$$f : \mathbf{X} \rightarrow \mathbf{Y}$$

$$y_o = f(x_o)$$

Set transformation

$$f: \mathbf{X} \rightarrow \mathbf{Y}, \quad A \in P(\mathbf{X})$$

$$B = f(A) = \{ y \in \mathbf{Y} \mid y = f(x), \ \forall x \in \mathbf{X} \}$$



$$B \in P(\mathbf{Y})$$

$$B(y) = \sup_{x / y = f(x)} A(x)$$

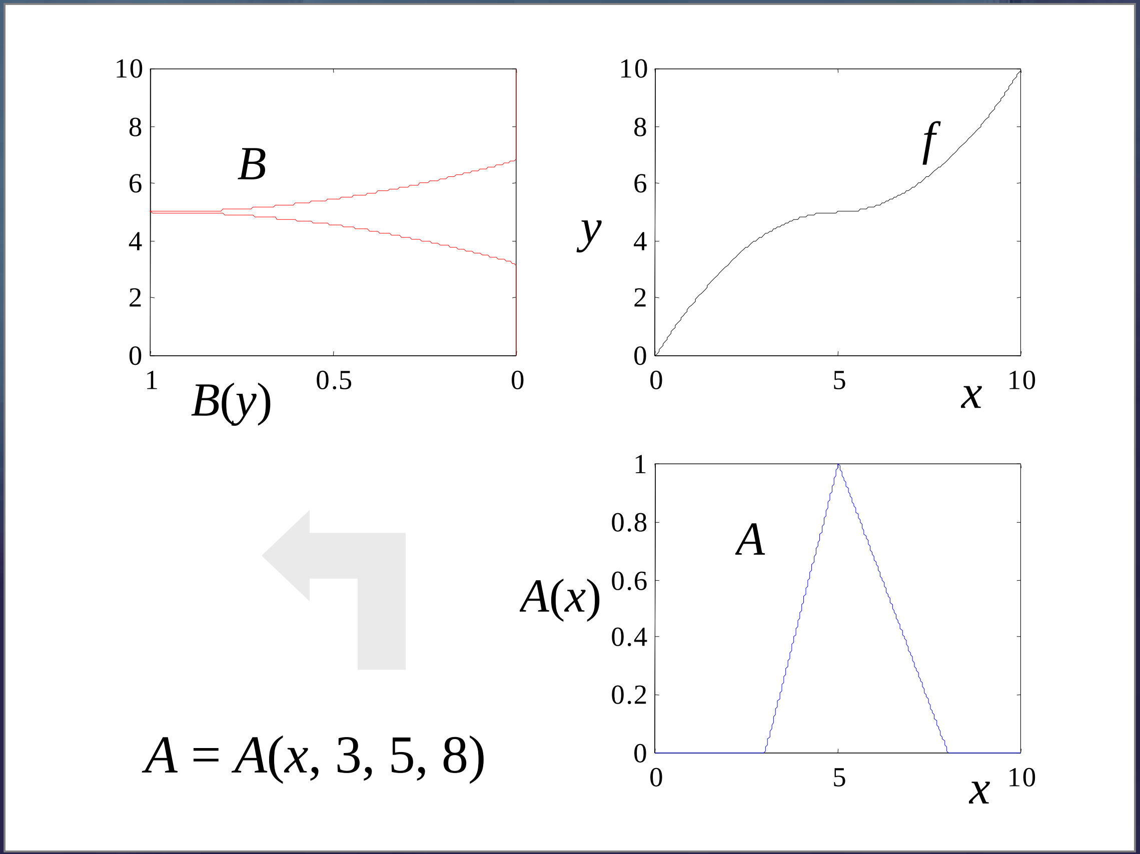
Fuzzy set transformation

$$f: \mathbf{X} \rightarrow \mathbf{Y}, \quad A \in F(\mathbf{X})$$

$$B = f(A),\ B \in F(\mathbf{Y})$$

$$f(x) = \begin{cases} -0.2(x-5)^2 + 5 & \text{if } 0 \le x \le 5 \\ 0.2(x-5)^2 + 5 & \text{if } 5 < x \le 10 \end{cases}$$

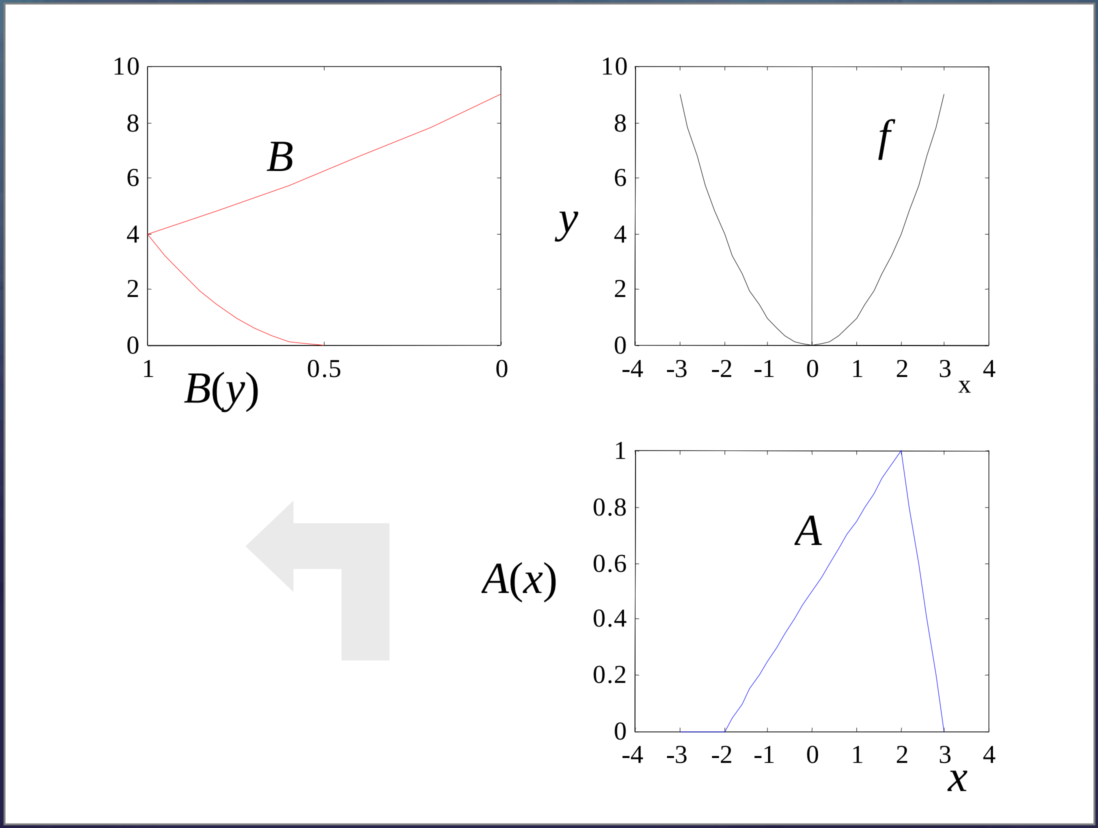
$$B(y) = \sup_{x / y = f(x)} A(x)$$



Example

$y = f(x) = x^2$

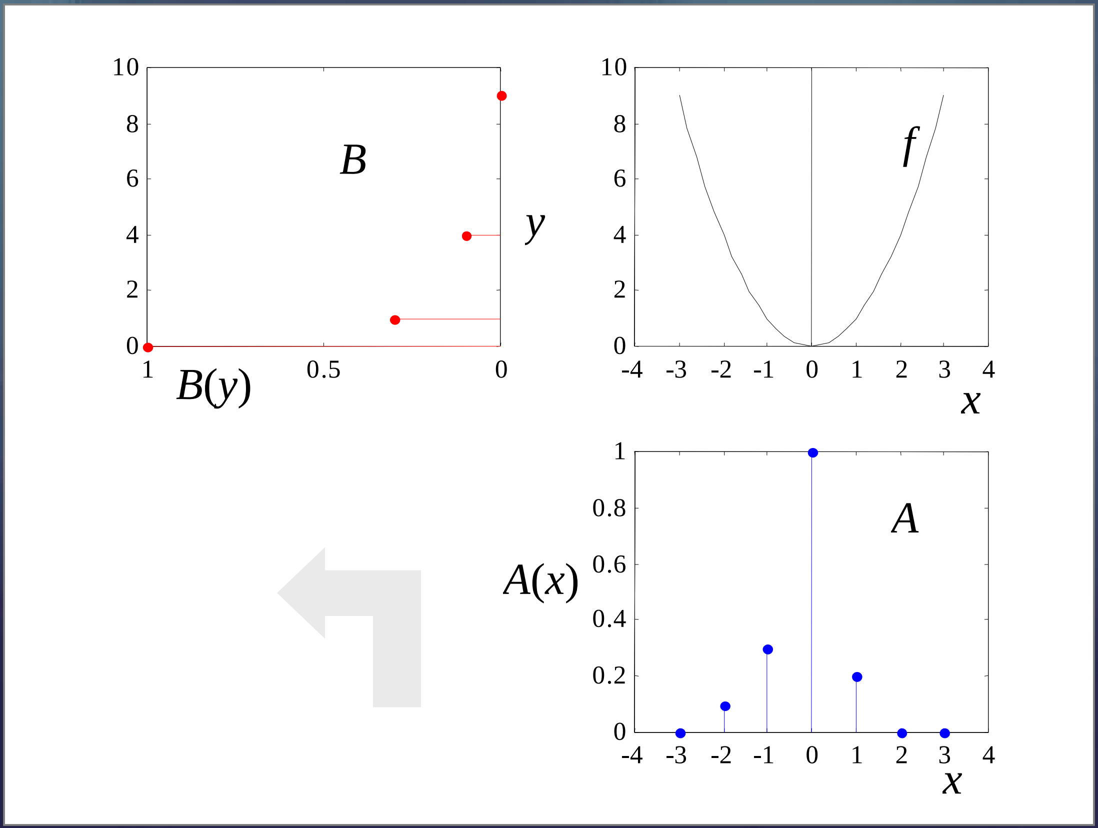
$A = A(x, -2, 2, 3)$



Example

$y = f(x) = x^2$

$\mathbf{X} = \{-3, -2, -1, 0, 1, 2, 3\}$ $\mathbf{Y} = \{0, 1, 4, 9\}$



$B = \{1/0, \max(0.2, 0.3)/1, \max(0, 0.1)/4, 0/9\} = \{1/0, 0.3/1, 0.1/4, 0/9\}$

Pedrycz and Gomide, FSE 2007

Generalization

$$\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n$$

$$A_i \in F(\mathbf{X}_i), \ \ i = 1,\ldots,n$$

$$y = f(\mathbf{x}), \quad \mathbf{x} = [x_1, x_2, \ldots, x_n]$$

$$B(y) = \sup_{\mathbf{x}|y=f(x)} \{\min[A_1(x_1), A_2(x_2), \ldots, A_n(x_n)]\}$$

$$B \in F(\mathbf{Y})$$

Properties

1. $B_i = \varnothing \ \ iff \ \ A_i = \varnothing$

2. $A_1 \subseteq A_2 \Rightarrow B_1 \subseteq B_2$

3. $f(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} f(A_i) = \bigcup_{i=1}^{n} B_i$

4. $f(\bigcap_{i=1}^{n} A_i) \subseteq \bigcap_{i=1}^{n} f(A_i) = \bigcap_{i=1}^{n} B_i$

5. $B_\alpha \supseteq f(A_\alpha)$

6. $B_\alpha^{+} = f(A_\alpha^{+})$

$B^{+}{}_{\alpha} = \{y \in \mathbf{Y} \mid B(y) > \alpha\}$
strong α–cut

7.2 Compositions of fuzzy relations

Sup-t composition

Given the fuzzy relations

$G : \mathbf{X}\times\mathbf{Z} \rightarrow [0,1]$

$W : \mathbf{Z}\times\mathbf{Y} \rightarrow [0,1]$

$R = G \circ W$ sup-t composition

$$R(x, y) = \sup_{z\in\mathbf{Z}}\{\min[G(x, z)\, t\, W(z, y)\}$$

$\forall(x,y) \in \mathbf{X}\times\mathbf{Y}$

$R : \mathbf{X}\times\mathbf{Y} \rightarrow [0,1]$

Example

sup-product composition t = product

$$G(x,z)=\exp[-(x-z)^2]$$

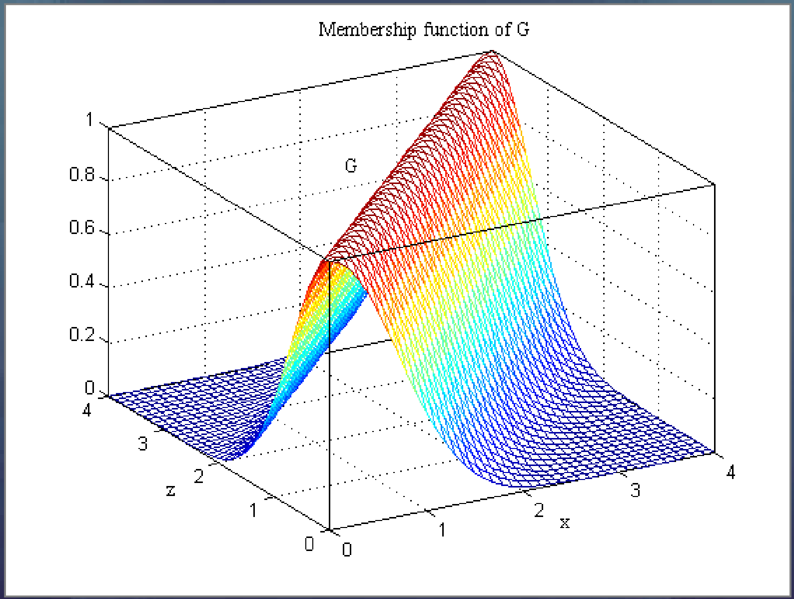
$$W(z,y)=\exp[-(z-y)^2]$$

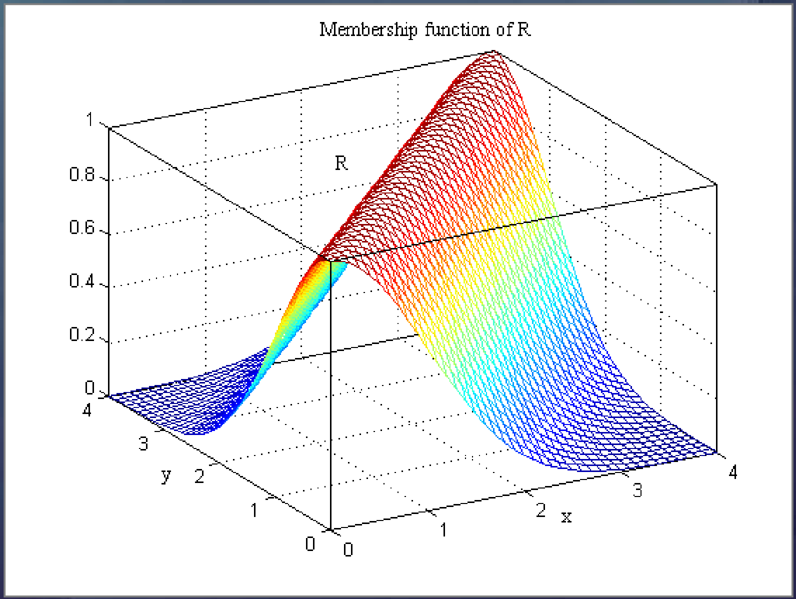
$$R(x,y)=\sup_{z\in\mathbf{Z}}\{e^{-(x-z)^2}e^{-(z-x)^2}\}=\max_{z\in\mathbf{Z}}\{e^{-(x-z)^2}e^{-(z-x)^2}\}$$

$$R(x,y)=\exp[-(x-y)^2/2]$$

$G : \mathbf{X} \times \mathbf{Z}$

$R = G \circ W$





$$G(x, z) = \exp[-(x - z)^2]$$

$$R(x, y) = \exp[-(x - y)^2 / 2]$$

Sup-t composition for matrix relations

procedure SUP-T-COMPOSITION (G,W) **returns** composition of fuzzy relations
static: fuzzy relations: $G = [g_{ik}]$, $W=[w_{kj}]$
0_{nm}: n×m matrix with all entries equal to zero
t: a t-norm

$R = 0_{nm}$

for i = 1:n **do**
for j = 1:m **do**
for k = 1:p **do**
tope ← $g_{ik}\ t\ \mathbf{w}_{\mathbf{kj}}$
r_{ij} ← $\max(r_{ij}, \text{tope})$
return R

Example

$$G = \begin{bmatrix} 1.0 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.8 & 1.0 & 0.2 \\ 0.8 & 0.3 & 0.4 & 0.3 \end{bmatrix} \quad W = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

t = min = $\wedge$

r_{11}=max(1.0$\wedge$0.6, 0.6$\wedge$0.5, 0.5$\wedge$0.7, 0.5$\wedge$0.3) = max (0.6, 0.5, 0.5, 0.3) = 0.6

....

r_{32}=max(0.8$\wedge$0.1, 0.3$\wedge$0.7, 0.4$\wedge$0.8, 0.3$\wedge$0.6) = max (0.1, 0.3, 0.4, 0.3) = 0.4

$$R = G \circ W = \begin{bmatrix} 0.6 & 0.6 \\ 0.7 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}$$

Properties

1. $P \circ (Q \circ R) = (P \circ Q) \circ R$	associativity
2. $P \circ (Q \cup R) = (P \circ Q) \cup (P \circ R)$	distributivity over union
3. $P \circ (Q \cap R) \subseteq (P \circ Q) \cap (P \circ R)$	weak distributivity over intersection
4. If $Q \subseteq S$ then $P \circ Q \subseteq P \circ S$	monotonicity

$\cup$, $\cap$ are standard operations

Interpretations

1. $B(y) = \sup_{x \in \mathbf{X}} [A(x) t R_y(x)]$ possibility

2. $B(y) = \text{truth}[\exists x \mid A(x) \; and \; R_y(x)]$ existential quantifier

3. $B(y) = \sup_{x \in \mathbf{X}} [\mathbf{X}(x) t R(x, y)] = \sup_{x \in \mathbf{X}} [1 t \, R(x, y)] = \sup_{x \in \mathbf{X}} R(x, y)$ projection

Inf-s composition

Given the fuzzy relations

$G: \mathbf{X} \times \mathbf{Z} \rightarrow [0,1]$

$W: \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1]$

$R = G \bullet W$ inf-s composition

$$R(x, y) = \inf_{z \in \mathbf{Z}} \{\min[G(x, z)\, s\, W(z, y)\} \qquad \forall (x,y) \in \mathbf{X} \times \mathbf{Y}$$

$R: \mathbf{X} \times \mathbf{Y} \rightarrow [0,1]$

procedure INF-S-COMPOSITION(G,W) **returns** composition of fuzzy relations
static: fuzzy relations: $G = [g_{ik}]$, $W = [w_{kj}]$
    1_{nm}: $n \times m$ matrix with all entries equal to unity
    s: a s-norm

$R = 1_{nm}$
for $i = 1:n$ **do**
  for $j = 1:m$ **do**
    for $k = 1:p$ **do**
      sope $\leftarrow$ $g_{ik}\ s\ w_{kj}$
      r_{ij} $\leftarrow$ min(r_{ij}, sope)
return R

Example

$$G = \begin{bmatrix} 1.0 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.8 & 1.0 & 0.2 \\ 0.8 & 0.3 & 0.4 & 0.3 \end{bmatrix} \quad W = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

s = probabilistic sum

r_{11}= min (1.0+0.6-0.6, 0.6+0.5-0.3, 0.5+0.7-0.35, 0.5+0.3-0.15)
= min (1.0, 0.8, 0.85, 0.65) = 0.65

………………………

r_{32}= min (0.8+0.1-0.08, 0.3+0.7-0.21, 0.4+0.8-0.32, 0.3+0.6-01.8)
= min (0.82, 0.79, 0.88, 0.72) = 0.72

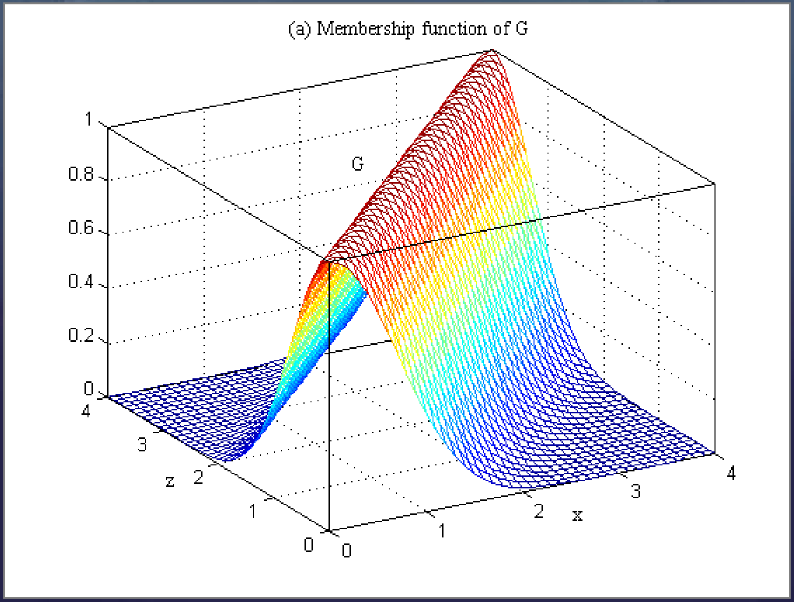
$$R = G \bullet W = \begin{bmatrix} 0.65 & 0.80 \\ 0.44 & 064 \\ 0.51 & 0.72 \end{bmatrix}$$

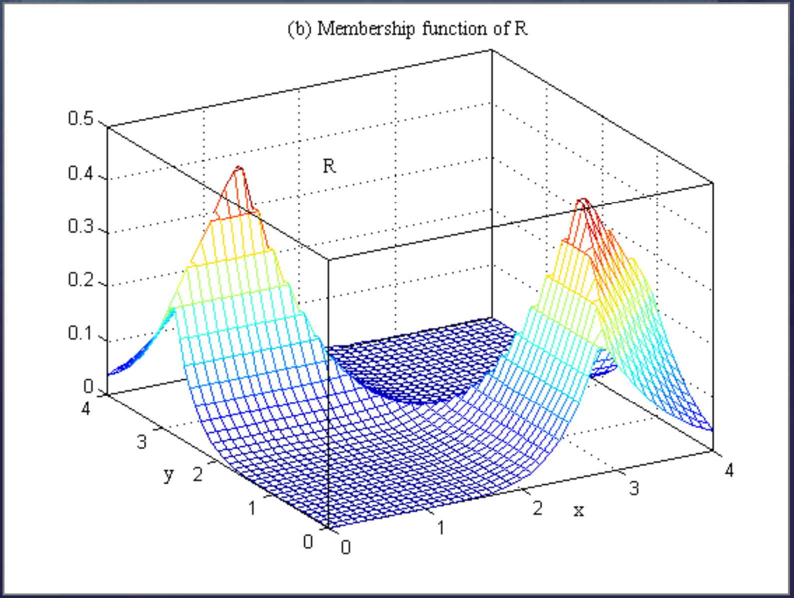
Example

$$G(x,z)=\exp[-(x-z)^2], \qquad W(z,y)=\exp[-(z-y)^2]$$

$G:\mathbf{X}\times\mathbf{Z}$

$R = G \bullet W$





Properties

1. $P \bullet (Q \bullet R) = (P \bullet Q) \bullet R$	associativity
2. $P \bullet (Q \cup R) \supseteq (P \bullet Q) \cup (P \bullet R)$	weak distributivity over union
3. $P \bullet (Q \cap R) = (P \bullet Q) \cap (P \bullet R)$	distributivity over intersection
4. If $Q \subseteq S$ then $P \bullet Q \supseteq P \bullet S$	monotonicity

$\cup$, $\cap$ are standard operations

Interpretations

$$1.\ B(y) = \inf_{x \in \mathbf{X}} [A(x) s R_y(x)] = \inf_{x \in \mathbf{X}} [R_y(x) s A(x)] = \inf_{x \in \mathbf{X}} [R_y(x) s \overline{\overline{A}}(x)]$$

necessity

$$2.\ B(y) = \text{truth}[\forall\, x \mid A(x)\, or\, R_y(x)]$$

universal quantifier

Inf-φ composition

Given the fuzzy relations

$$G : \mathbf{X} \times \mathbf{Z} \rightarrow [0,1]$$

$$W : \mathbf{Z} \times \mathbf{Y} \rightarrow [0,1]$$

$$R = G \ \varphi \ W$$ inf-φ composition

$$a \varphi b = \{c \in [0,1] \mid atc \leq b\}, \quad \forall a, b \in [0,1]$$

$$\varphi : [0,1] \rightarrow [0,1$$

$$R(x, y) = \inf_{z \in \mathbf{Z}} \{G(x, z) \ \varphi \ W(z, y)\}$$

$$\forall (x,y) \in \mathbf{X} \times \mathbf{Y}$$

Example

$$G = \begin{bmatrix} 1.0 & 0.6 & 0.5 & 0.5 \\ 0.6 & 0.8 & 1.0 & 0.2 \\ 0.8 & 0.3 & 0.4 & 0.3 \end{bmatrix} \quad W = \begin{bmatrix} 0.6 & 0.1 \\ 0.5 & 0.7 \\ 0.7 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

If t is the bounded difference: $a\ t\ b = \max(0, a + b - 1)$

then $a \varphi b = \min(1, 1 - a + b)$ Lukasiewicz implication

$$R = G \varphi W = \begin{bmatrix} 0.6 & 0.1 \\ 0.7 & 06 \\ 0.8 & 0.3 \end{bmatrix}$$

Properties

1. $P\varphi(Q\varphi R) = (P \circ Q)\varphi R$	associative
2. $P\varphi(Q \cup R) \supseteq (P\varphi Q) \cup (P\varphi R)$	weak distributivity over union
3. $P\varphi(Q \cap R) = (P\varphi Q) \cap (P\varphi R)$	distributivity over intersection
4. If $Q \subseteq S$ then $P\varphi Q \subseteq P\varphi S$	monotonicity

$\cup$, $\cap$ are standard operations

Interpretation

$$B(y) = \inf_{x \in \mathbf{X}} [A(x) \varphi R_y(x)] = \inf_{x \in \mathbf{X}} [A(x) \Rightarrow R_y(x)]$$

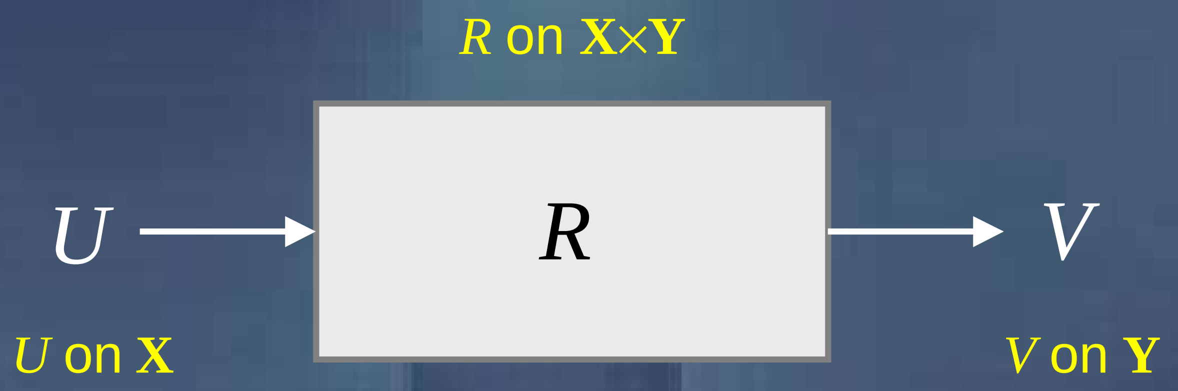
$$= \inf_{x \in \mathbf{X}} [A(x) \Rightarrow R_y(x)] = \inf_{x \in \mathbf{X}} [A(x) \subset R_y(x)]$$

$$\Downarrow$$

$$B(y) = \forall x [A(x) \Rightarrow R_y(x)]$$

7.3 Fuzzy relational equations

Single–input, single–output fuzzy system



Fundamental problems

- given U and V, determine R — estimation
- given V and R, determine U — inverse

Solution to the estimation problem
Sup-t composition

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$

$\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$

U: $\mathbf{X} \rightarrow [0,1]$ $\quad U = [u_1, u_2,\ldots, u_i,\ldots, u_n] = [u_i]\ (1\times n)$

V: $\mathbf{Y} \rightarrow [0,1]$ $\quad V = [v_1, v_2,\ldots, v_j,\ldots, v_m] = [v_j]\ (1\times m)$

$R: \mathbf{X}\times\mathbf{Y} \rightarrow [0,1]$

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}]\ \ (n\times m)$$

$S_e = \{R \in F(\mathbf{X}) \times F(\mathbf{Y}) \mid V = U \circ R\}$ solution set

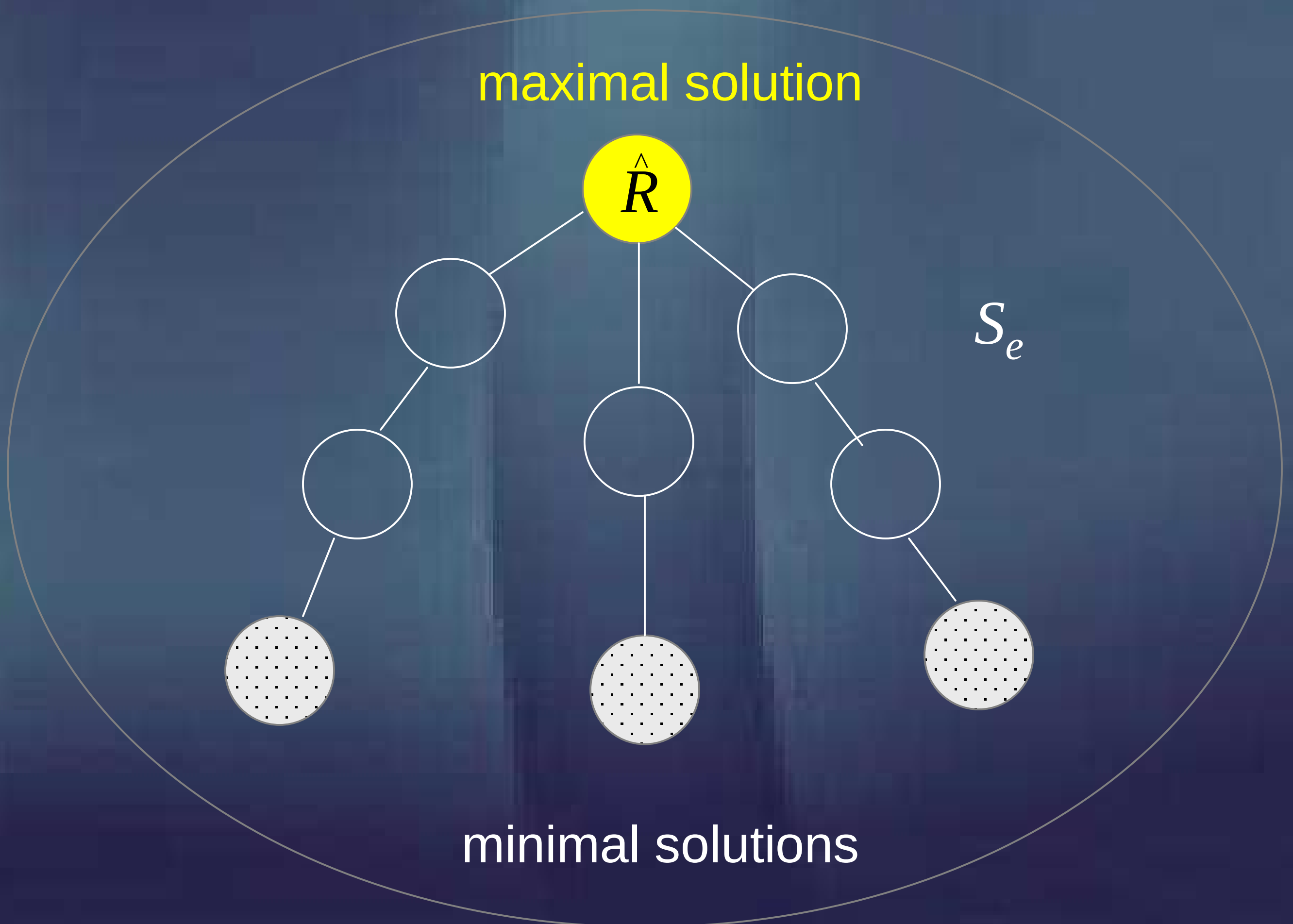
$a\ \varphi\ b = \sup\ \{c \in [0,1] \mid a\ t\ c \leq b$ φ operator

Proposition

if $S_e \neq \varnothing$, then the unique maximal solution $\hat{R}$ of the sup-t relational equation $V = U \circ R$ is

$$\hat{R} = U^T \varphi\, V$$

$\hat{R}$ is maximal (in the sense that, if $R \in S_e$, then $R \subseteq \hat{R}$)


maximal solution
$\hat{R}$
S_e
minimal solutions

procedure ESTIMATE-SOLUTION (*U*,*V*) **returns** fuzzy relation
static: fuzzy unary relations $U = [u_i]$, $V = [v_j]$
t: a t-norm
define φ operator

for $i = 1{:}n$ **do**
for $j = 1{:}m$ **do**
$\hat{r}_{ij} \leftarrow u_i \, \varphi \, v_j$

return $\hat{R}$

Example

U = [0.8, 0.5, 0.3] V = [0.4, 0.2, 0.0, 0.7]

$$t = \min \Rightarrow a\,\varphi\, b = \begin{cases} 1 & \text{if} \quad a \le b \\ b & \text{if} \quad a > b \end{cases}$$

$$\hat{R} = \begin{bmatrix} 0.8 \\ 0.5 \\ 0.3 \end{bmatrix} \varphi \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} 0.8\varphi 0.4 & 0.8\varphi 0.2 & 0.8\varphi 0.0 & 0.8\varphi 0.7 \\ 0.5\varphi 0.4 & 0.5\varphi 0.2 & 0.5\varphi 0.0 & 0.5\varphi 0.7 \\ 0.3\varphi 0.4 & 0.3\varphi 0.2 & 0.3\varphi 0.0 & 0.3\varphi 0.7 \end{bmatrix}$$

$$\Downarrow$$

$$\hat{R} = \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \\ 0.4 & 0.2 & 0.0 & 1.0 \\ 1.0 & 0.2 & 0.0 & 1.0 \end{bmatrix}$$

maximal solution

$$R_1 = \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \\ 0.0 & 0.0 & 0.0 & 0.5 \\ 0.3 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0.0 & 0.2 & 0.0 & 0.7 \\ 0.4 & 0.0 & 0.0 & 0.2 \\ 0.6 & 0.2 & 0.0 & 1.0 \end{bmatrix}$$

$R_1 \in S_e$ and $R_2 \in S_e$

$R_1 \subset \hat{R}$ and $R_2 \subset \hat{R}$

Fuzzy relational system

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$

U_k: $\mathbf{X} \rightarrow [0,1]$ $\quad U_k = [u_{1k}, u_{2k},\ldots, u_{ik},\ldots, u_{nk}] = [u_{ik}]\ (1\times n)$

V_k: $\mathbf{Y} \rightarrow [0,1]$ $\quad V_k = [v_{1k}, v_{2k},\ldots, v_{jk},\ldots, v_{mk}] = [v_{jk}]\ (1\times m)$

$k = 1,\ldots,N$

$R : \mathbf{X}\times\mathbf{Y} \rightarrow [0,1]$

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}]\ \ (n\times m)$$

$$V_k = U_k \circ R, \quad k = 1, \ldots, N$$

$$S_e^k = \{R \in F(\mathbf{X}) \times F(\mathbf{Y}) \mid V_k = U_k \circ R\} \neq \varnothing$$

$$S_e^N = \bigcap_{k=1}^{N} S_e^k \neq \varnothing$$

$$\Downarrow$$

$$\hat{R} = \bigcap_{k=1}^{N} \hat{R}_k$$

$$\hat{R}_k = U_k^T \varphi V_k$$

maximal solution

Relation-relation fuzzy equations

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$, $\mathbf{Z}=\{z_1, z_2,\ldots,z_p\}$

U: $\mathbf{Z}\times\mathbf{X} \rightarrow [0,1]$ $U = [u_{ki}]$ $(p\times n)$

V: $\mathbf{Z}\times\mathbf{Y} \rightarrow [0,1]$ $V = [v_{kj}]$ $(p\times m)$

$$V = U \circ R$$

$$R: \mathbf{X}\times\mathbf{Y} \rightarrow [0,1] \qquad R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}] \ (n\times m)$$

Let

$U^k = [u_{k1}, u_{k2},\ldots, u_{ki},\ldots, u_{kn}]$ $(1\times n)$ k-th row of U

$V^k = [v_{k1}, v_{k2},\ldots, v_{ki},\ldots, v_{km}]$ $(1\times n)$ k-th row of V

$R^j = [r_{1j}, r_{2j},\ldots, r_{ij},\ldots, r_{nj}]^T$ $(n\times 1)$ j-th column of R

then

$$V = U \circ R =$$

$$\begin{bmatrix} V^1 \\ V^2 \\ \vdots \\ V^p \end{bmatrix} = \begin{bmatrix} U^1 \\ U^2 \\ \vdots \\ U^p \end{bmatrix} \circ \begin{bmatrix} R^1 & R^2 & \cdots & R^m \end{bmatrix} = \begin{bmatrix} U^1 \circ R^1 & U^1 \circ R^2 & \cdots & U^1 \circ R^m \\ U^2 \circ R^1 & U^2 \circ R^2 & \cdots & U^2 \circ R^m \\ \vdots & \vdots & \cdots & \vdots \\ U^p \circ R^1 & U^p \circ R^2 & \cdots & U^p \circ R^m \end{bmatrix}$$

$$V^1 = \left[U^1 \circ R^1 \quad U^1 \circ R^2 \quad \cdots \quad U^1 \circ R^m \right] = U^1 \circ R$$

$$V^2 = \left[U^2 \circ R^1 \quad U^2 \circ R^2 \quad \cdots \quad U^2 \circ R^m \right] = U^2 \circ R$$

$$\vdots$$

$$V^p = \left[U^p \circ R^1 \quad U^p \circ R^2 \quad \cdots \quad U^p \circ R^m \right] = U^p \circ R$$

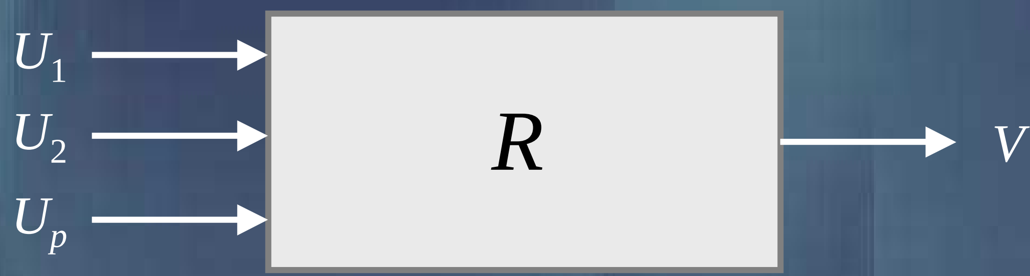
Therefore, using the previous result we get

$$\hat{R} = \bigcap_{k=1}^{p} \hat{R}_k$$

$$\hat{R}_k = U^{kT} \varphi V^k$$

$$U^{kT} = (U^k)^T$$

Multi–input, single–output fuzzy equations



$U_i \in F(\mathbf{X}_i)$, $i = 1,\ldots, p$

$V \in F(\mathbf{Y})$

$R \in F(\mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_p \times \mathbf{Y})$

$\rightarrow$

$$V = U_1 \circ U_2 \circ \cdots \circ U_p \circ R$$

If $U = U_1 t U_2 t \ldots t U_p$ then

$$V = U \circ R$$

$\downarrow$

$$\hat{R} = U^T \varphi\, V$$

Solution to the estimation problem
Inf-s composition

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$

$U: \mathbf{X} \rightarrow [0,1]$ $U = [u_i]\ (1\times n)$

$V: \mathbf{Y} \rightarrow [0,1]$ $V = [v_j]\ (1\times m)$

$R: \mathbf{X}\times\mathbf{Y} \rightarrow [0,1]$ $R = [r_{jj}]\ (n\times m)$

$$V = U \bullet R$$

$S_e^s = \{R \in F(\mathbf{X}) \times F(\mathbf{Y}) \mid V = U \bullet R\}$ solution set

$a \,\beta\, b = \inf \{c \in [0,1] \mid a \, s \, c \geq b$ β operator

Proposition

if $S_e^s \neq \varnothing$, then the unique minimal solution $\hat{R}$ of the sup-t relational equation $V = U \bullet R$ is

$$\hat{R} = U^T \beta\, V$$

$\hat{R}$ is minimal (in the sense that, if $R \in S_e^s$, then $\hat{R} \subseteq R$)

Solution to the inverse problem
Sup-t composition

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$

$U: \mathbf{X} \to [0,1]$ $U = [u_i]\ (1\times n)$

$V: \mathbf{Y} \to [0,1]$ $V = [v_j]\ (1\times m)$

$R: \mathbf{X}\times\mathbf{Y} \to [0,1]$ $R = [r_{jj}]\ (n\times m)$

$$V = U \circ R$$

$S_i = \{U \in F(\mathbf{X}) \mid V = U \circ R\}$ solution set

$v_j\ \theta\ s_{ji} = \min\ (v_j\ \varphi\ s_{ji},\ j = 1,\ldots,m),\ i = 1,\ldots,n$ θ operator

Proposition

if $S_i \neq \varnothing$, then the unique maximal solution $\hat{U}$ of the sup-t relational equation $V = U \circ R$ is

$$\hat{U} = V\ \theta\ R^T$$

$\hat{U}$ is maximal (in the sense that, if $U \in S_i$, then $U \subseteq \hat{U}$)

procedure INVERSE-SOLUTION (R,V) **returns** fuzzy unary relation
static: fuzzy relations: $R=[r_{ij}]$, $V=[v_j]$
M: large number
t: a t-norm
define: φ operator

for $i = 1{:}n$ **do**
$u \leftarrow M$
for $j = 1{:}m$ **do**
$u \leftarrow \min(u, v_j \,\varphi\, r_{ij})$
$\hat{u}_i \leftarrow u$
return $\hat{U}$

Example

V = [0.4, 0.2, 0.0, 0.7]

$$\hat{R} = \begin{bmatrix} 0.4 & 0.2 & 0.0 & 0.7 \\ 0.4 & 0.2 & 0.0 & 1.0 \\ 1.0 & 0.2 & 0.0 & 1.0 \end{bmatrix}$$

$$t = \min \Rightarrow a\,\varphi\,b = \begin{cases} 1 & \text{if} \quad a \le b \\ b & \text{if} \quad a > b \end{cases}$$

$$\hat{U} = [0.4 \quad 0.2 \quad 0.0 \quad 0.7]\theta \begin{bmatrix} 0.4 & 0.4 & 1.0 \\ 0.2 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.7 & 1.0 & 1.0 \end{bmatrix}$$

$$= \min\left\{ [0.4 \quad 0.2 \quad 0.0 \quad 0.7]\theta \begin{bmatrix} 0.4 & 0.4 & 1.0 \\ 0.2 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.0 \\ 0.7 & 1.0 & 1.0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \min(0.4\varphi 0.4, 0.2\varphi 0.2, 0.0\varphi 0.0, 0.7\varphi 0.7 \\ \min(0.4\varphi 0.4, 0.2\varphi 0.2, 0.0\varphi 0.0, 1.0\varphi 0.7 \\ \min(1.0\varphi 0.4, 0.2\varphi 0.2, 0.0\varphi 0.0 1.0\varphi 0.7 \end{bmatrix}^T = [1.0 \quad 0.7 \quad 0.4]$$

Relation-relation fuzzy equations

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$, $\mathbf{Z}=\{z_1, z_2,\ldots,z_p\}$

U: $\mathbf{Z}\times\mathbf{X} \rightarrow [0,1]$ $U = [u_{ki}]$ $(p\times n)$

V: $\mathbf{Z}\times\mathbf{Y} \rightarrow [0,1]$ $V = [v_{kj}]$ $(p\times m)$

$$V = U \circ R$$

$$R : \mathbf{X}\times\mathbf{Y} \rightarrow [0,1] \qquad R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & r_{ij} & \vdots \\ r_{n1} & \cdots & r_{nm} \end{bmatrix} = [r_{ij}] \ (n\times m)$$

As before, let

$U^k = [u_{k1}, u_{k2},\ldots, u_{ki},\ldots, u_{kn}]$ $(1\times n)$ k-th row of U

$V^k = [v_{k1}, v_{k2},\ldots, v_{ki},\ldots, v_{km}]$ $(1\times n)$ k-th row of V

$R^j = [r_{1j}, r_{2j},\ldots, r_{ij},\ldots, r_{nj}]^T$ $(n\times 1)$ j-th column of R

thus

$$V = U \circ R =$$

$$\begin{bmatrix} V^1 \\ V^2 \\ \vdots \\ V^p \end{bmatrix} = \begin{bmatrix} U^1 \\ U^2 \\ \vdots \\ U^p \end{bmatrix} \circ \begin{bmatrix} R^1 & R^2 & \cdots & R^m \end{bmatrix} = \begin{bmatrix} U^1 \circ R^1 & U^1 \circ R^2 & \cdots & U^1 \circ R^m \\ U^2 \circ R^1 & U^2 \circ R^2 & \cdots & U^2 \circ R^m \\ \vdots & \vdots & \cdots & \vdots \\ U^p \circ R^1 & U^p \circ R^2 & \cdots & U^p \circ R^m \end{bmatrix}$$

$$V^1 = \left[U^1 \circ R^1 \quad U^1 \circ R^2 \quad \cdots \quad U^1 \circ R^m\right] = U^1 \circ R$$

$$V^2 = \left[U^2 \circ R^1 \quad U^2 \circ R^2 \quad \cdots \quad U^2 \circ R^m\right] = U^2 \circ R$$

$$\vdots$$

$$V^p = \left[U^p \circ R^1 \quad U^p \circ R^2 \quad \cdots \quad U^p \circ R^m\right] = U^p \circ R$$

Using the previous result we get

$$\hat{U}_i = V^i \theta R^T, \; i = 1, \ldots, p$$

Multi–input, single–output fuzzy equations

$U_i \in F(\mathbf{X}_i)$, $i = 1,\ldots, p$

$V \in F(\mathbf{Y})$

$R \in F(\mathbf{X}_1 \times \mathbf{X}_2 \times \ldots . \times \mathbf{X}_p \times \mathbf{Y})$

$$V = U_1 \circ U_2 \circ \cdots \circ U_p \circ R$$

$$V(y) = \sup_{x \in \mathbf{X}} [U_1(x_1)\, t\, U_2(x_2)\, t \cdots t\, U_p(x_p)\, t\, R(x_1, x_2, \ldots, x_n, y)]$$

$$\hat{U}_i = V \,\theta\, R_i^T$$

$$R_i = U_1 \circ \cdots \circ U_{i-1} \circ U_{i+1} \circ \cdots \circ U_p \circ R$$

Solvability conditions for maximal solutions

- $hgt(U) \geq hgt(V)$ estimation problem
- $\max_i r_{ij} \geq v_j$ necessary condition for inverse problem
- concise and practically relevant solvability is difficult (in general)
- if system is not solvable, then look for approximate solution

7.4 Associative memories

Sup-t fuzzy associative memories

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$

$U_k\colon \mathbf{X} \rightarrow [0,1]$ $U_k = [u_{1k}, u_{2k},\ldots, u_{ik},\ldots, u_{nk}] = [u_{ik}]\ (1\times n)$

$V_k\colon \mathbf{Y} \rightarrow [0,1]$ $V_k = [v_{1k}, v_{2k},\ldots, v_{jk},\ldots, v_{mk}] = [v_{jk}]\ (1\times m)$

$k = 1,\ldots,N$

U_k and V_k are patterns to be encoded into memory R

Sup-t fuzzy associative memories

- Encoding

$$R = \bigcap_{k=1}^{N} R_k, \qquad R_k = U_k^T \varphi V_k$$

- Decoding

$$V_k = U_k \circ R$$

Semioverlapping fuzzy sets

- $U_1, U_2, \ldots, U_N$ form a partition
- adjacent and overlap at ½
- $hgt(U_k \cap U_{k-1}) = 0.5$ and $\sum_k U_k(\mathbf{x}) = 1 \;\; \forall \mathrm{x} \in \mathbf{X}$
- $\mathbf{x} = [x_1, x_2, \ldots, x_n]$

Proposition

if fuzzy patterns U_k are semioverlapped, then the pairwise encoding of U_k and V_k , $k = 1,\ldots, N$ using

$$R = \bigcap_{k=1}^{N} R_k, \quad R_k = U_k^T \varphi V_k$$

produces perfect recall realized as

$$V_k = U_k \circ R$$

Inf-s fuzzy associative memories

$\mathbf{X}=\{x_1, x_2,\ldots,x_n\}$, $\mathbf{Y}=\{y_1, y_2,\ldots,y_m\}$

U_k: $\mathbf{X} \rightarrow [0,1]$ $U_k = [u_{1k}, u_{2k},\ldots, u_{ik},\ldots, u_{nk}] = [u_{ik}]\ (1\times n)$

V_k: $\mathbf{Y} \rightarrow [0,1]$ $V_k = [v_{1k}, v_{2k},\ldots, v_{jk},\ldots, v_{mk}] = [v_{jk}]\ (1\times m)$

$k = 1,\ldots,N$

U_k and V_k are patterns to be encoded into memory R

Inf-s fuzzy associative memories

- Encoding

$$R = \bigcup_{k=1}^{N} R_k, \qquad R_k = U_k^T \beta V_k$$

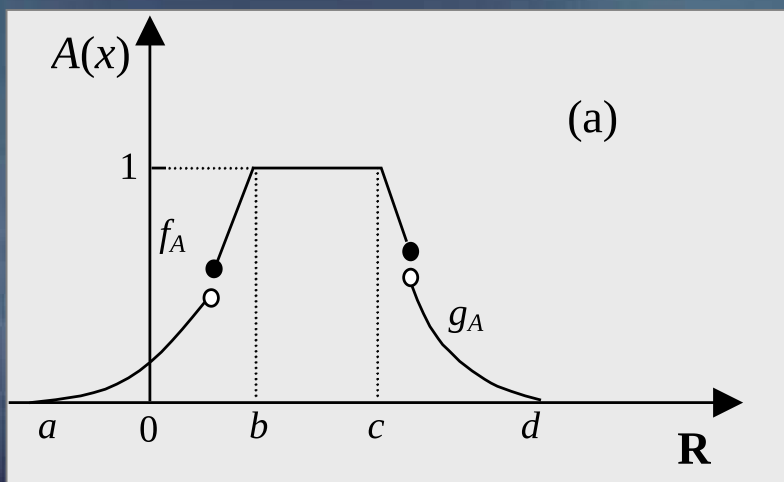
- Decoding

$$V_k = U_k \bullet R$$

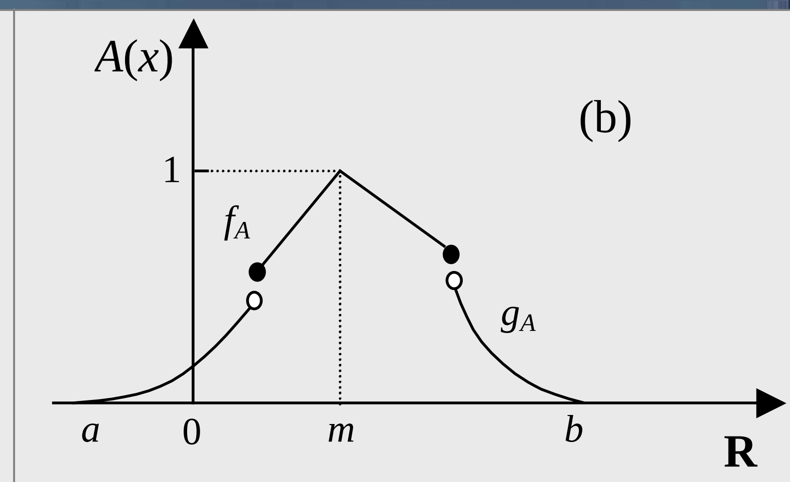
7.5 Fuzzy numbers and fuzzy arithmetic

Algebraic operations on fuzzy numbers

Fuzzy interval



Fuzzy number

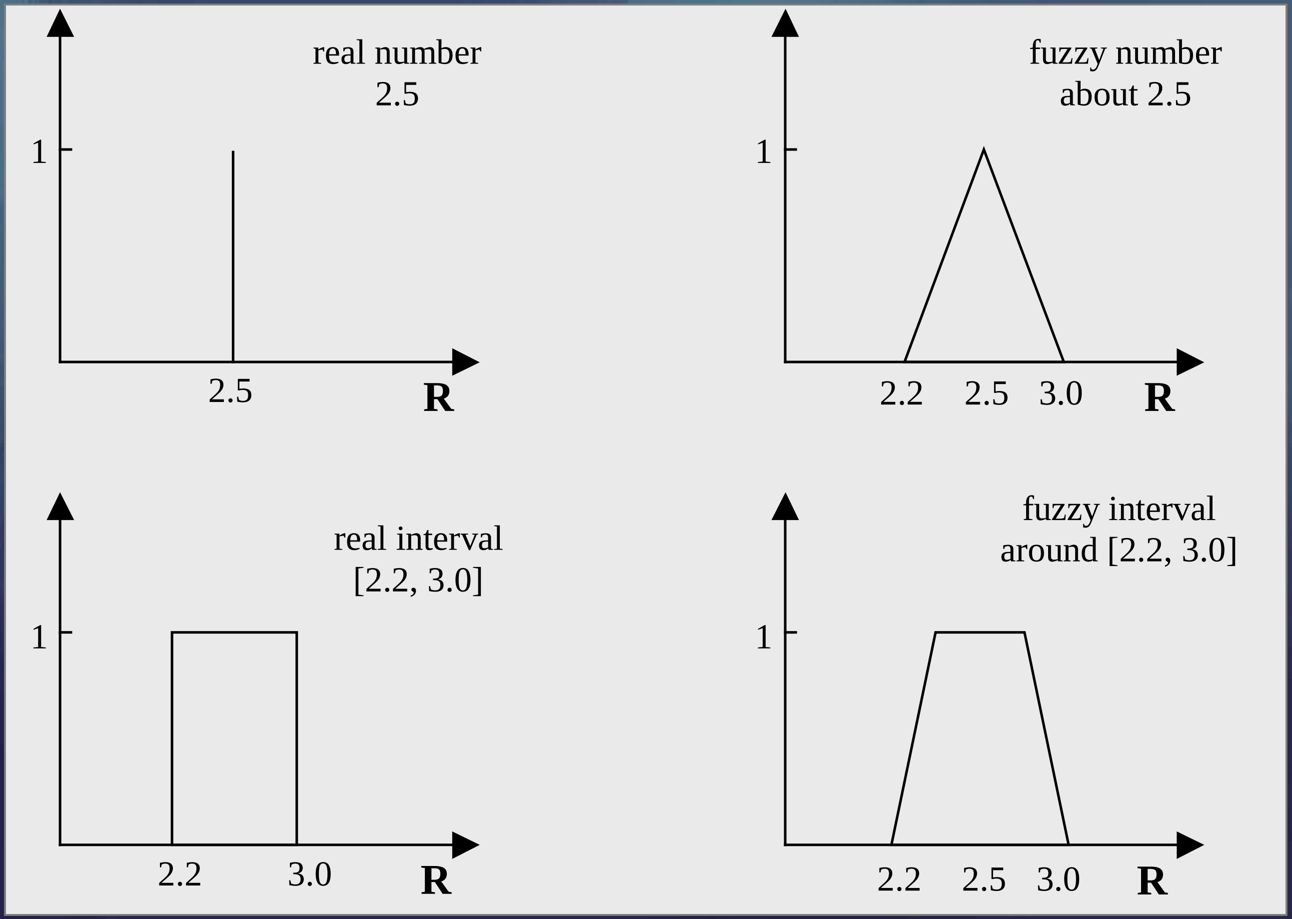


$$A(x) = \begin{cases} f_A(x) & \text{if } x \in [a, b) \\ 1 & \text{if } x \in [b, c] \\ g_A(x) & \text{if } x \in (c, d] \\ 0 & \text{otherwise} \end{cases}$$

f_A right semicontinuous

g_A left semicontinuous

Examples


real number
2.5
1
2.5
R
fuzzy number
about 2.5
1
2.2
2.5
3.0
R
real interval
[2.2, 3.0]
1
2.2
3.0
R
fuzzy interval
around [2.2, 3.0]
1
2.2
2.5
3.0
R

Computing with fuzzy numbers

- *Consider a 2 h travel at a speed of about 110 km/h. What was the distance you traveled?*

- *In a given manufacturing process, there are five operations completed in series. Each manufacturing task has durations of about T_1, T_2,…, T_n time units. What is the completion time of the process?*

- Two fundamental methods to perform algebraic operations
  - based on interval arithmetic and α-cuts
  - extension principle

Interval arithmetic and α-cuts

$$[a,b]+[c,d]=[a+b,c+d]$$

$$[a,b]-[c,d]=[a-d,b-c]$$

$$[a,b].[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)]$$

$$[a,b]/[c,d]=[\min(a/c,a/d,b/c,b/d),\max(a/c,a/d,b/c,b/d)]$$

If $*$ is any of the four basic algebraic operations

and A and B are fuzzy sets on **R** and $\alpha \in [0,1]$, then

$(A*B)_\alpha = A_\alpha * B_\alpha$

$$A * B = \bigcup_{\alpha \in [0,1]} (A * B)_\alpha$$

$$(A * B)(x) = \sup_{\alpha \in [0,1]} [\alpha (A * B)(x)]$$

Example

$A(x,a,m,b)$, $B(x,c,n,d)$ triangular fuzzy numbers

$A_\alpha = [(m - a)\alpha + a, (m - b)\alpha + b]$, $B_\alpha = [(n - c)\alpha + c, (n - d)\alpha + d]$

$A = A(x,1,2,3)$, $B = B(x,2,3,5)$

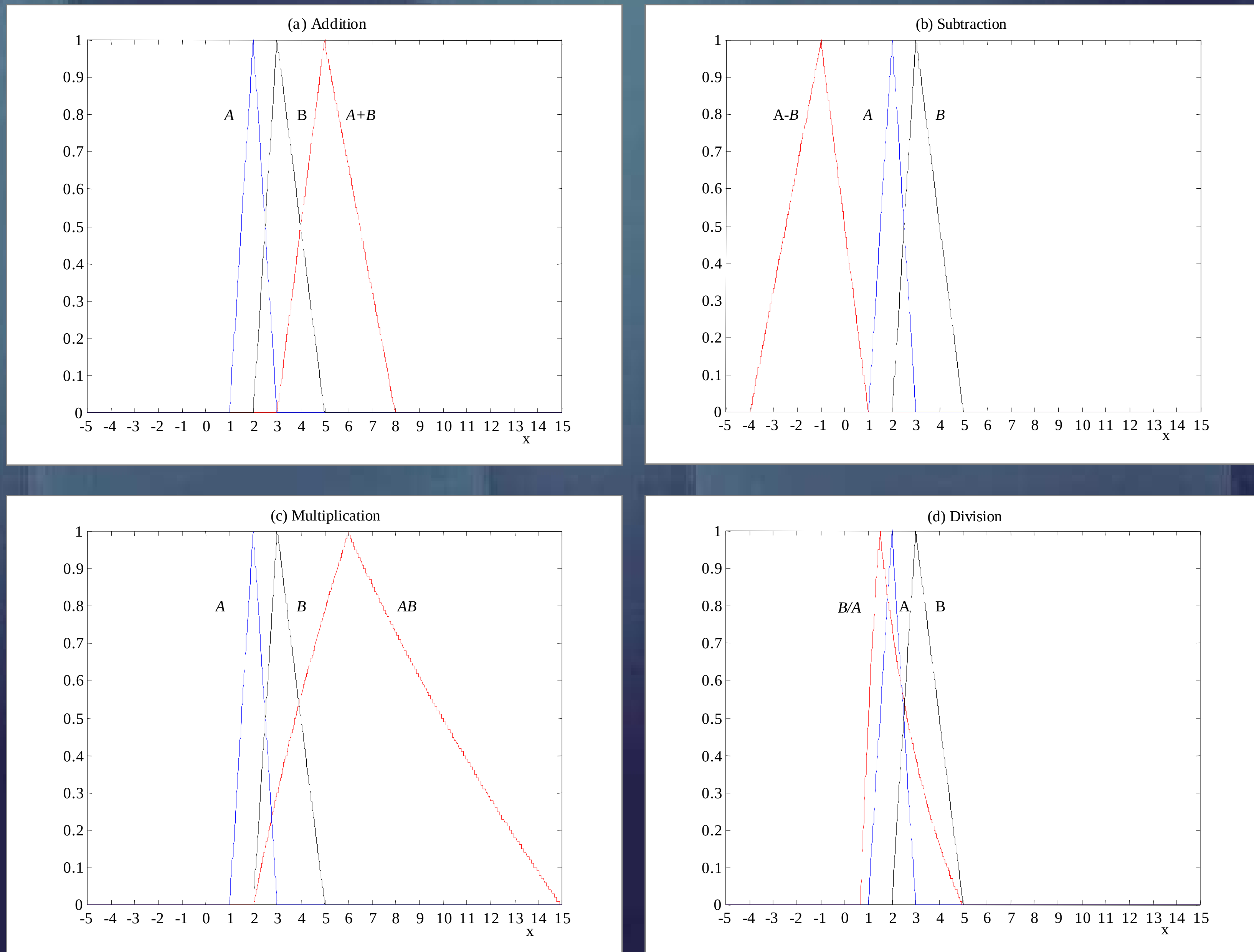
$A_\alpha = [\alpha + 1, - \alpha + 3]$, $B_\alpha = [\alpha + 2, - 2\alpha + 5]$

$(A+B)_\alpha = [2\alpha + 3, - 3\alpha + 3]$

$(A - B)_\alpha = [3\alpha - 4, - 2\alpha + 1]$

$(AB)_\alpha = [(\alpha + 1)(\alpha + 2), (- \alpha + 3)(- 2\alpha + 5)]$

$(A/B)_\alpha = [(\alpha + 1)/(-2\alpha + 5), (- \alpha + 3)/(\alpha + 2)]$


(a) Addition
A
B
A+B
(b) Subtraction
A-B
A
B
(c) Multiplication
A
B
AB
(d) Division
B/A
A
B
x

Fuzzy arithmetic and the extension principle

Extension principle and standard operations on real numbers

$$(A * B)(z) = \sup_{z=x*y} \min[A(x), B(y)]\,, \quad \forall z \in \mathbf{R}$$

$$* \in \{+, -, \cdot\,, /\,\}$$

In general, if t is a t-norm and $*: \mathbf{R}^2 \rightarrow \mathbf{R}$ then

$$(A * B)(z) = \sup_{z=x*y} [A(x)\, t\, B(y)]\,, \qquad \forall z \in \mathbf{R}$$

$$t_1 \le t_2 \Rightarrow a\, t_1 b \le a t_2 b, \;\; \forall a, b \in [0,1]$$

$$\sup_{z=x*y}[A(x)\, t_d\, B(y)] \le \sup_{z=x*y}[A(x)\, t\, B(y)] \le \sup_{z=x*y}[A(x)\, t_m\, B(y)], \quad \forall z \in \mathbf{R}$$

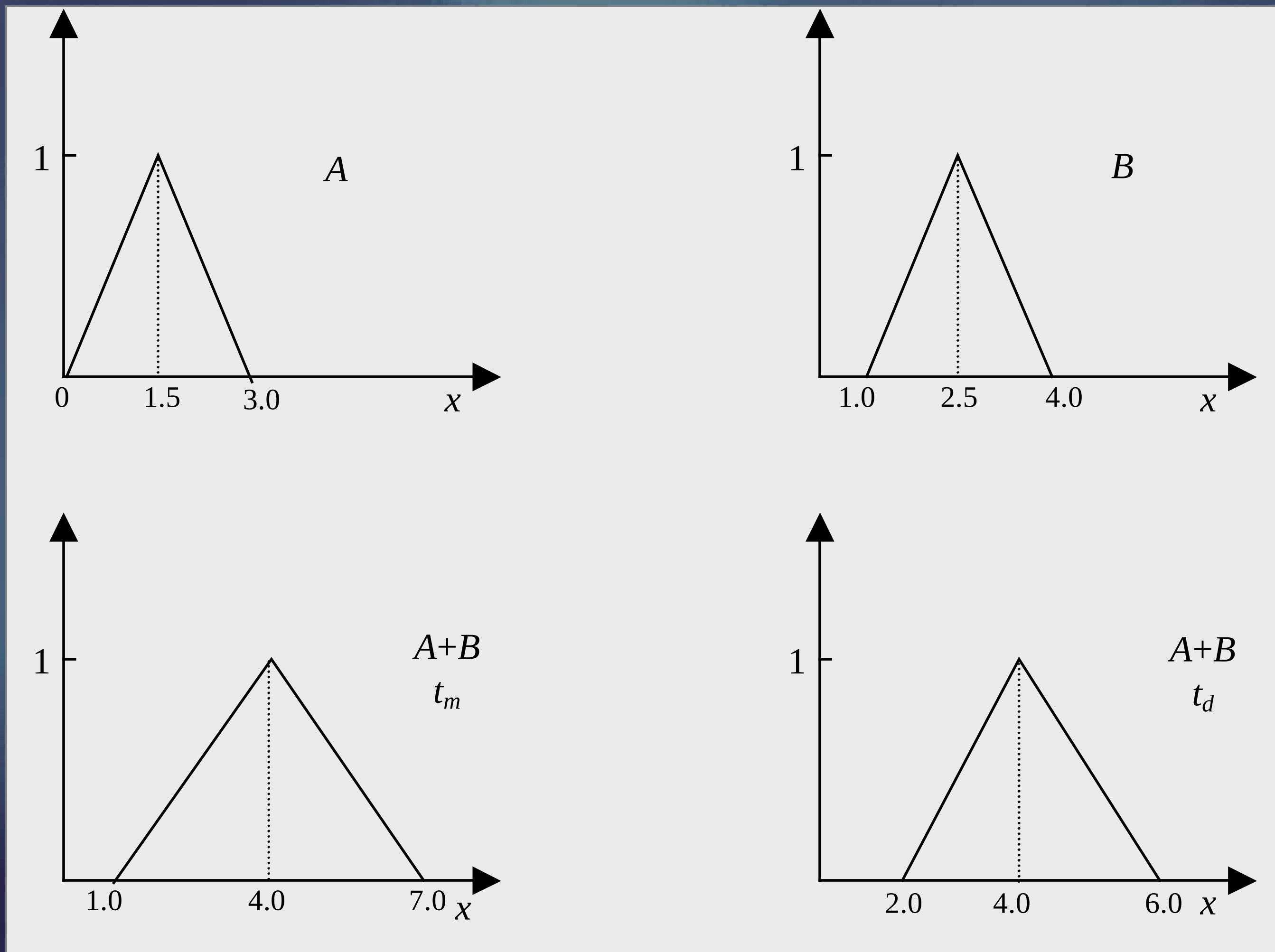
$$^{t_d}(A*B)(z) \le {}^{t}(A*B)(z) \le {}^{t_d}(A*B)(z), \quad \forall z \in \mathbf{R}$$

Example

$A(x,a,m,b)$, $B(x,c,n,d)$ triangular fuzzy numbers

$^{tm}(A+B) = (A+B)$ using *minimum* t-norm

$^{td}(A+B) = (A+B)$ using *drastic product* t-norm



Different choices of t-norms, different results

Proposition

For any fuzzy numbers A and B and a continuous monotone binary operation $*$ on $\mathbf{R}$, the following equality holds for all α-cuts with $\alpha \in [0,1]$:

$$(A*B)_\alpha = A_\alpha * B_\alpha$$

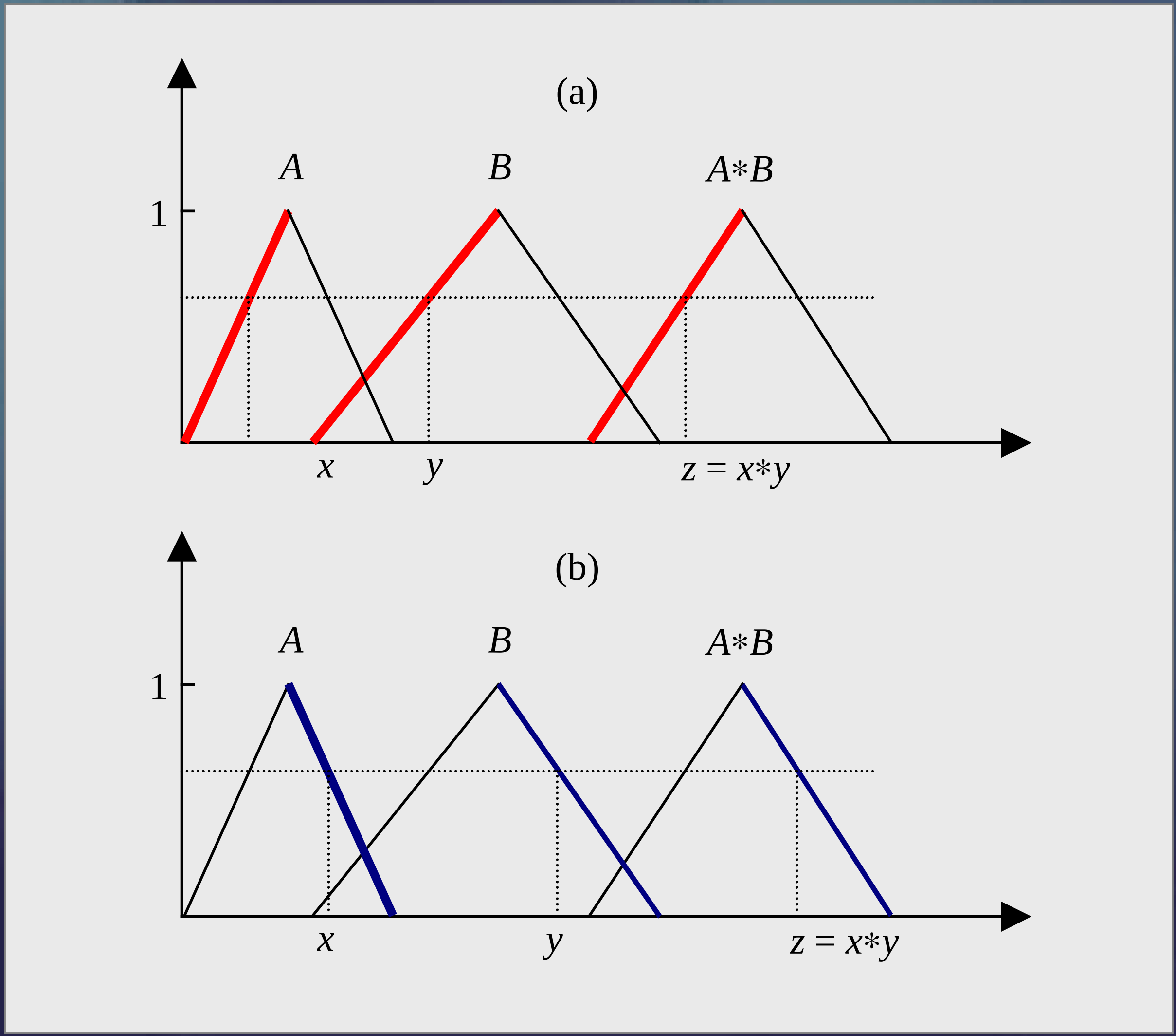
(Nguyen and Walker, 1999)

Important consequences of the proposition:

1. A_α and B_α closed and bounded $\forall\alpha \Rightarrow (A*B)_\alpha$ closed and bounded

2. A and B normal $\Rightarrow (A*B)$ normal

3. Computation of $(A*B)$ can be done combining the increasing and decreasing parts of the membership functions of A and B.



Computation of $(A*B)$ combining the increasing and decreasing parts of the membership functions

Computing with triangular fuzzy numbers

- $A(x,a,m,b)$ and $B(x,c,n,d)$ $\rightarrow$ triangular fuzzy numbers
- membership functions

$$A(x)=\begin{cases}\dfrac{x-a}{m-a} & \text{if } x\in[a,m)\\ \dfrac{b-x}{b-m} & \text{if } x\in[m,b]\\ 0 & \text{otherwise}\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-c}{n-c} & \text{if } x\in[c,n)\\ \dfrac{d-x}{d-n} & \text{if } x\in[n,d]\\ 0 & \text{otherwise}\end{cases}$$

Addition

$$C(z) = \sup_{z=x+y} \min[A(x), B(y)], \quad \forall z \in \mathbf{R}1.$$

1. $C(z) = 1$ for $z = m + n$

2. $z < m + n$

$x < m$ and $y < n$

$A(x) = B(y) = \alpha$

$$\frac{x-a}{m-a} = \alpha \quad \text{and} \quad \frac{y-c}{n-c} = \alpha \qquad x \in [a, m), \; y \in [c, n)$$

$x = (m-a)\alpha + a \qquad y = (n-c)\alpha + c$ and from $z = x + y$

$$\alpha = \frac{z-(a+c)}{(m+n)-(a+c)}$$

3. $z > m+n$

$x > m$ and $y > n$

$A(x) = B(y) = \alpha$

$$\frac{b-x}{b-m} = \alpha \quad \text{and} \quad \frac{d-y}{d-n} = \alpha \qquad x \in [m,b],\ y \in [n,d]$$

$$x = (m-b)\alpha + b \qquad y = (n-d)\alpha + d \quad \text{and from } z = x+y$$

$$\alpha = \frac{(b+d)}{(b+d)-(m+n)}$$

$$4.\ C(x) = \begin{cases} \dfrac{z-(a+c)}{(M+n)-(a+c)} & \text{if} \quad z < m+n \\ 1 & \text{if} \quad z = m+n \\ \dfrac{(b+d)-z}{(b+d)-(m+n)} & \text{if} \quad z > m+n \end{cases}$$

$C = A + B$

$\Downarrow$

$C(x) = C(x, a+c, m+n, b+d)$

Multiplication

- Looking at the increasing part of the membership function

$$x = (m - a)\alpha + a$$

$$y = (n - c)\alpha + c$$

$$z = xy = [(m - a)\alpha + a][(n - c)\alpha + c]$$

$$z = (m - a)(n - c)\alpha^2 + (m - a)\alpha c + a(n - c)\alpha + ac = f_1(\alpha)$$

if $ac \leq z \leq mn$ then the membership function of $D = A.B$ is

$$D(z) = f_1^{-1}(z)$$

• Looking at the decreasing part of the membership function

$$x = (m - b)\alpha + b$$

$$y = (n - d)\alpha + d$$

$$z = xy = [(m - b)\alpha + b][(n - d)\alpha + d]$$

$$z = (m - b)(n - d)\alpha^2 + (m - b)\alpha d + b(n - d)\alpha + bd = f_2(\alpha)$$

if $mn \le z \le bd$ then the membership function of $D = A.B$ is

$$D(z) = f_2^{-1}(z)$$