

# 5 Operations and Aggregations of Fuzzy Sets

*Fuzzy Systems Engineering  
Toward Human-Centric Computing*

# Contents

**5.1 Standard operations on sets and fuzzy sets**

**5.2 Generic requirements for operations on fuzzy sets**

**5.3 Triangular norms**

**5.4 Triangular conorms**

**5.5 Triangular norms as a general category of logical operations**

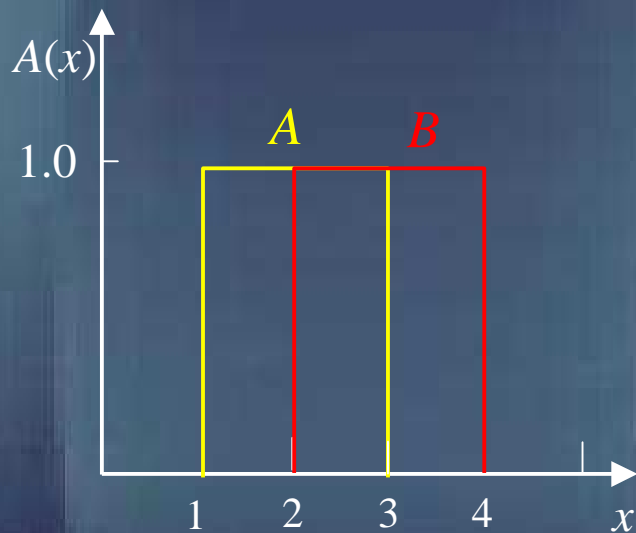
**5.6 Aggregation operations**

**5.7 Fuzzy measure and integral**

**5.8 Negations**

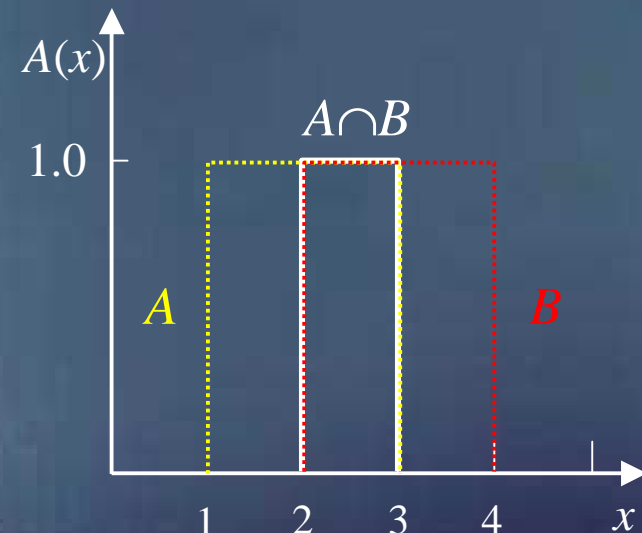
# 5.1 Standard operations on sets and fuzzy sets

# Intersection of sets



$$A = \{x \in \mathbf{R} \mid 1 \leq x \leq 3\}$$

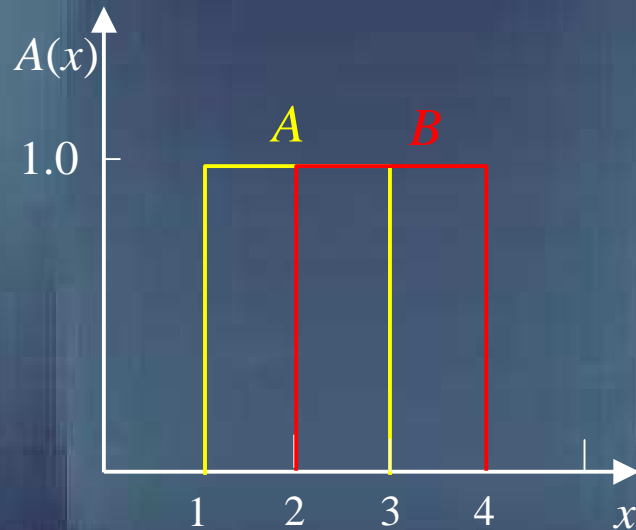
$$B = \{x \in \mathbf{R} \mid 2 \leq x \leq 4\}$$



$$A \cap B: \{x \in \mathbf{R} \mid 2 \leq x \leq 3\}$$

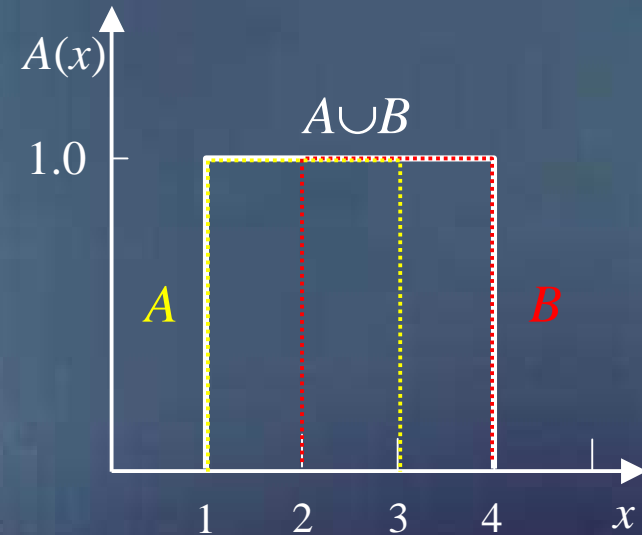
$$(A \cap B)(x) = \min [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

# Union of sets



$$A = \{x \in \mathbf{R} \mid 1 \leq x \leq 3\}$$

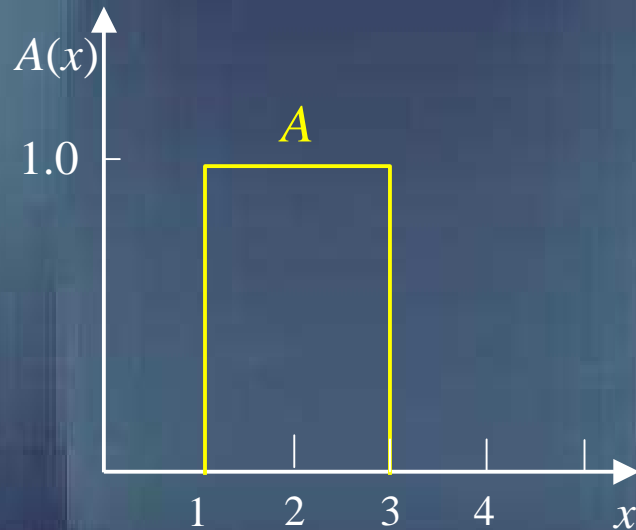
$$B = \{x \in \mathbf{R} \mid 2 \leq x \leq 4\}$$



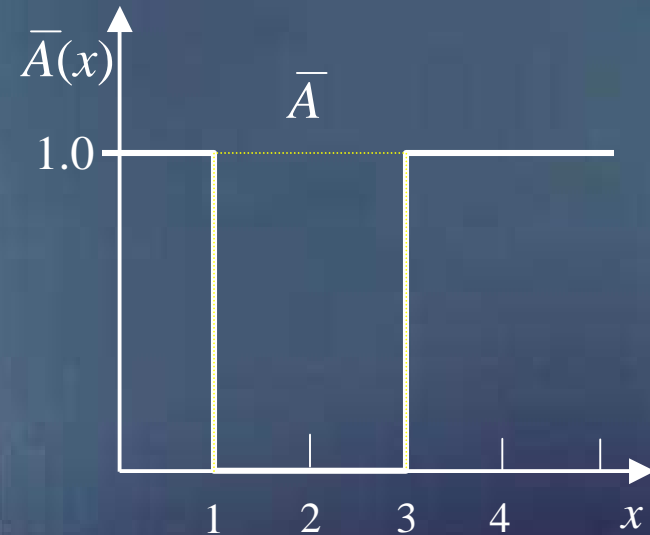
$$A \cup B: \{x \in \mathbf{R} \mid 1 \leq x \leq 4\}$$

$$(A \cup B)(x) = \max [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

# Complement of sets



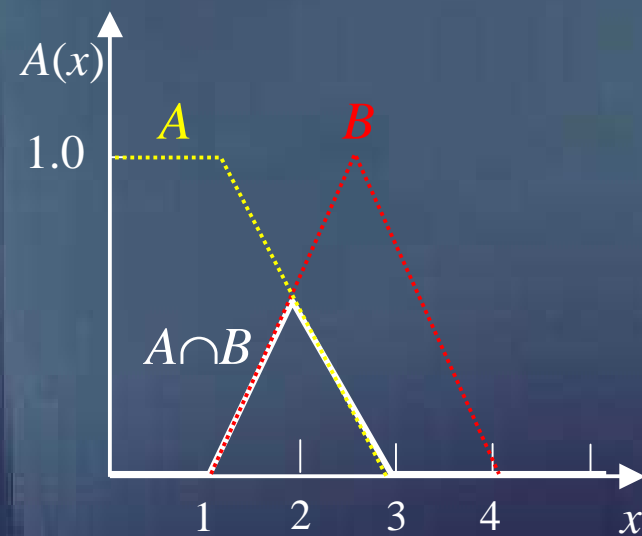
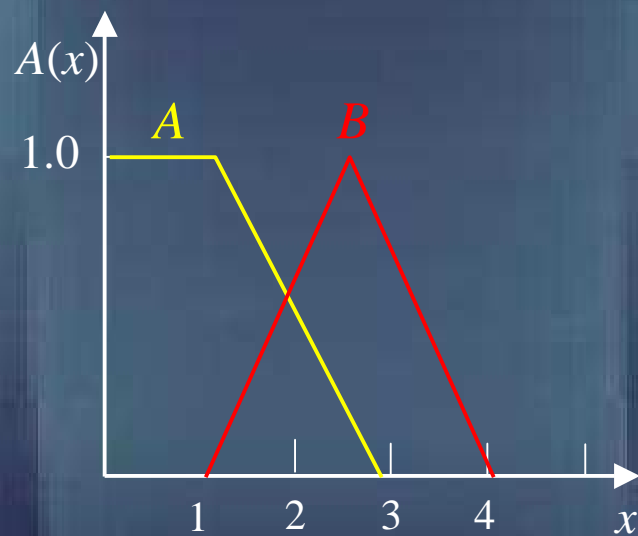
$$A = \{x \in \mathbf{R} \mid 1 \leq x \leq 3\}$$



$$\bar{A}(x) = \{x \in \mathbf{R} \mid x < 1, x > 3\}$$

$$\bar{A}(x) = 1 - A(x) \quad \forall x \in \mathbf{X}$$

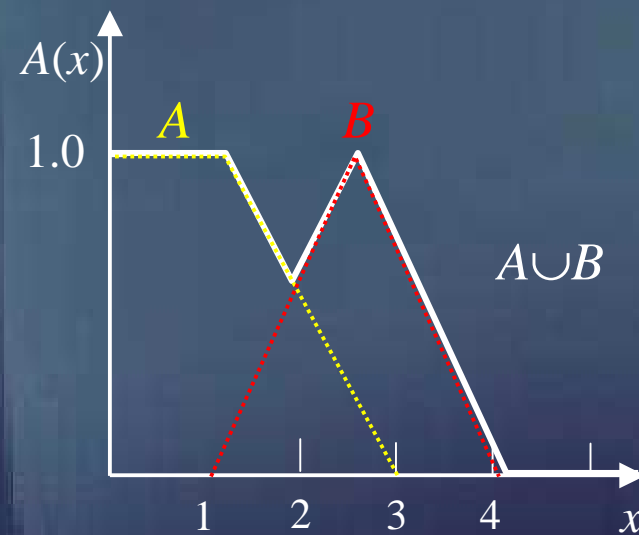
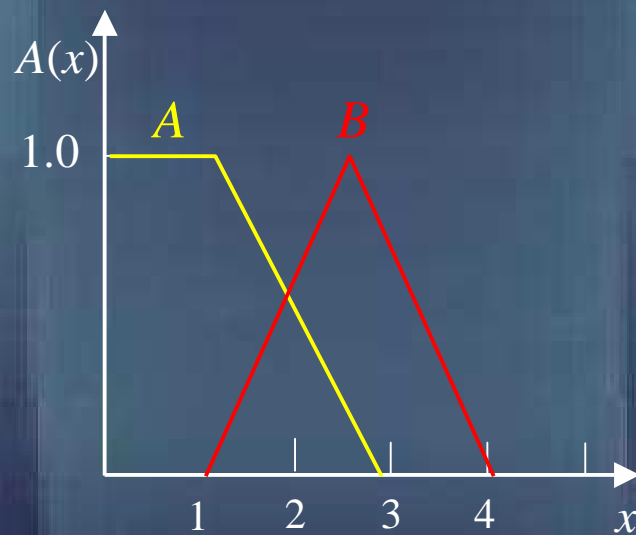
# Intersection on fuzzy sets



$$(A \cap B)(x) = \min [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

Standard intersection

# Union on fuzzy sets

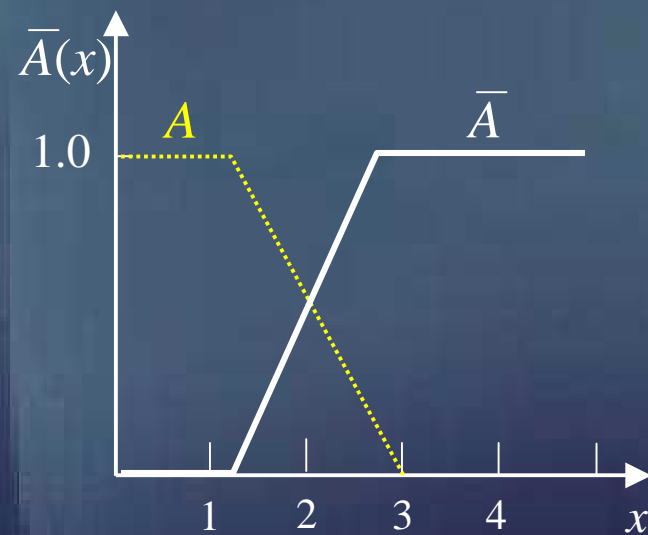
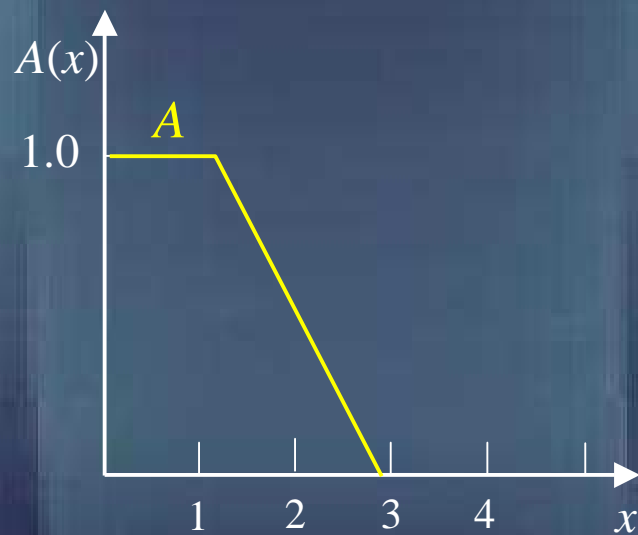


$$(A \cup B)(x) = \max [A(x), B(x)] \quad \forall x \in \mathbf{X}$$

Standard union



# Complement of fuzzy sets



$$\bar{A}(x) = 1 - A(x) \quad \forall x \in \mathbf{X}$$

Standard complement

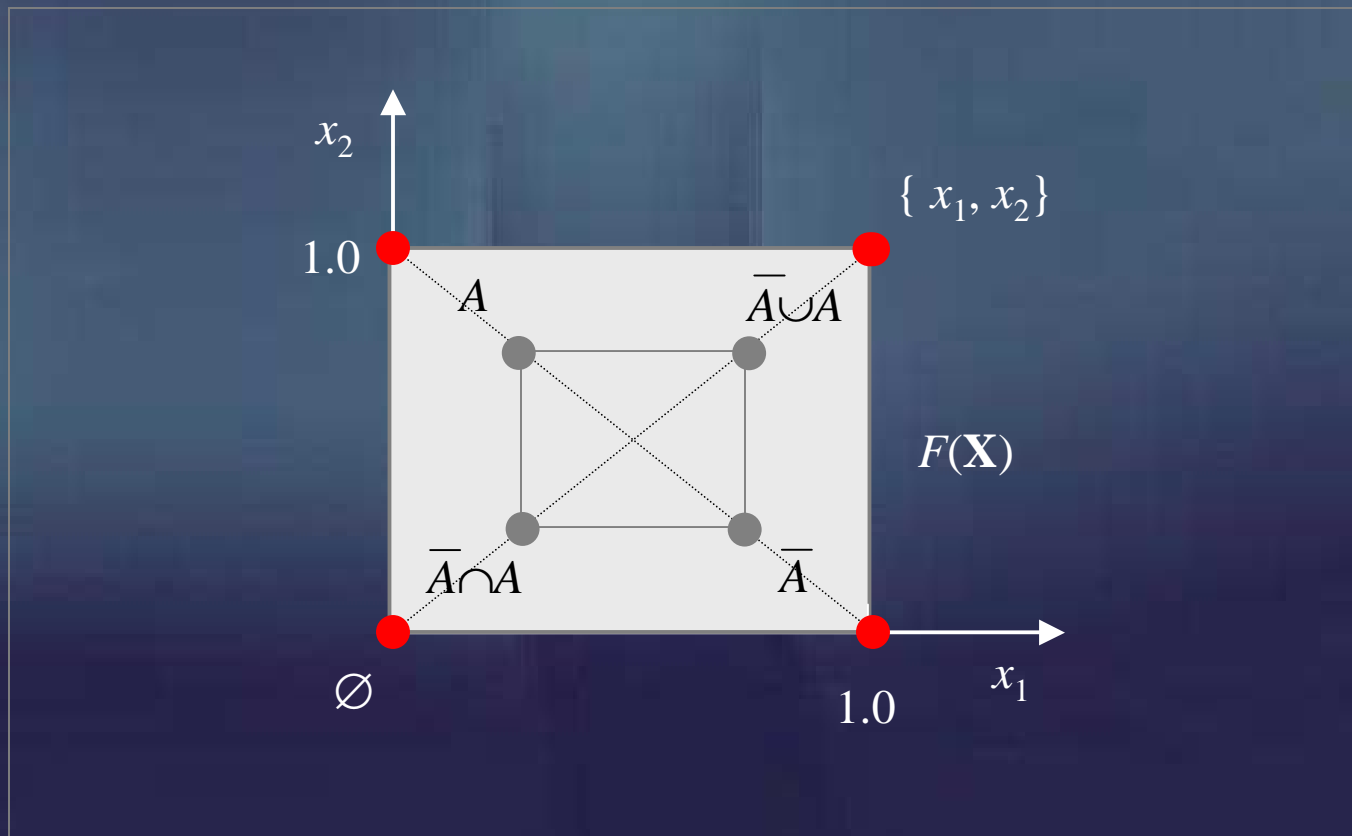
# Basic properties of sets and fuzzy sets

1 Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
2 Associativity	$A \cup (B \cap C) = (A \cup B) \cap C$
	$A \cap (B \cup C) = (A \cap B) \cup C$
3 Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4 Idempotency	$A \cup A = A$
	$A \cap A = A$
5 Boundary Conditions	$A \cup \phi = A$ and $A \cup \mathbf{X} = \mathbf{X}$
	$A \cap \phi = \phi$ and $A \cap \mathbf{X} = A$
6 Involution	$\overline{\overline{A}} = A$
7 Transitivity	if $A \subset B$ and $B \subset C$ then $A \subset C$

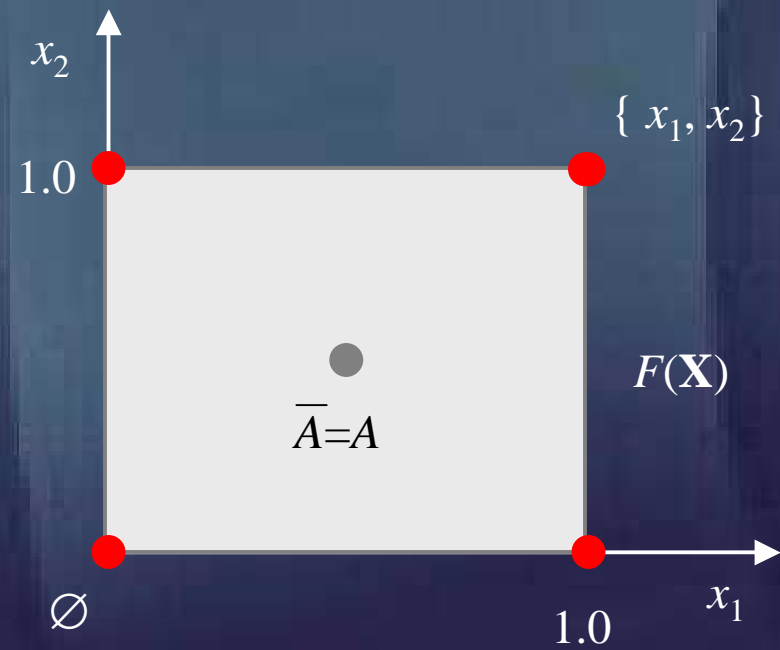
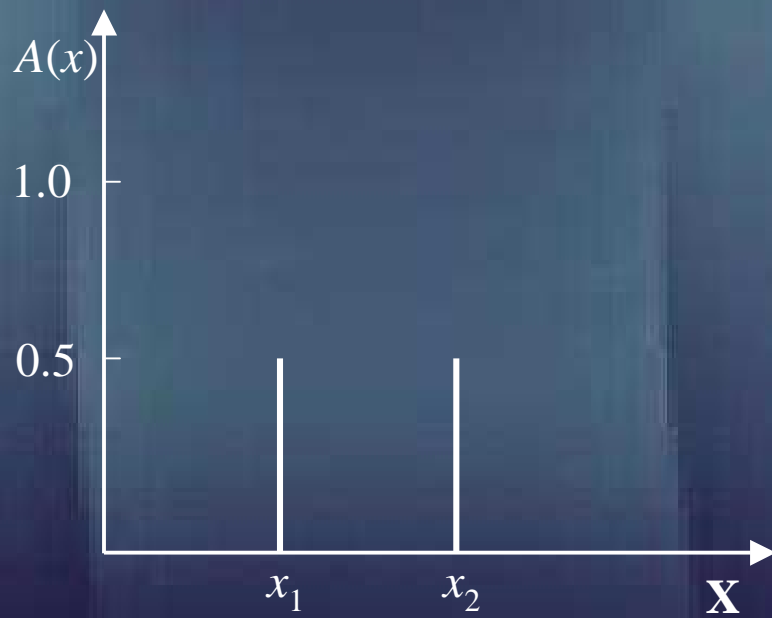
# Noncontradiction and excluded middle for standard operations

	Sets	Fuzzy sets
8-Noncontradiction	$\bar{A} \cap A = \phi$	$\bar{A} \cap A \neq \phi$
9-Excluded middle	$\bar{A} \cup A = \mathbf{X}$	$\bar{A} \cup A \neq \mathbf{X}$

# Geometric view of standard operations



# Example



## **5.2 Generic requirements for operations on fuzzy sets**

# Set operation is a binary operator

$$[0,1] \times [0,1] \rightarrow [0,1]$$

- **Requirements:**

- commutativity
- associativity
- identity

- **Identity:**

- its form depends on the operation

## 5.3 Triangular norms



# Definition

$$t : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity:

$$a \ t \ b = b \ t \ a$$

- Associativity:

$$a \ t \ (b \ t \ c) = (a \ t \ b) \ t \ c$$

- Monotonicity:

$$\text{if } b \leq c \text{ then } a \ t \ b \leq a \ t \ c$$

- Boundary conditions:

$$a \ t \ 1 = a$$

$$a \ t \ 0 = 0$$

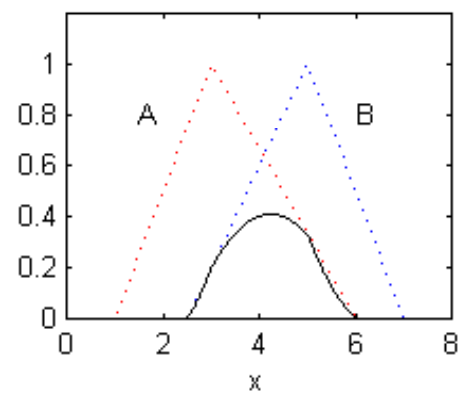
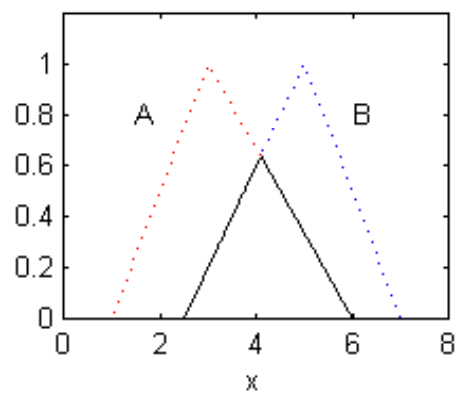
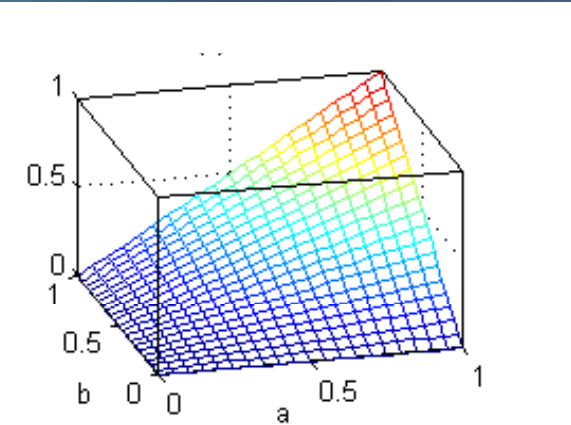
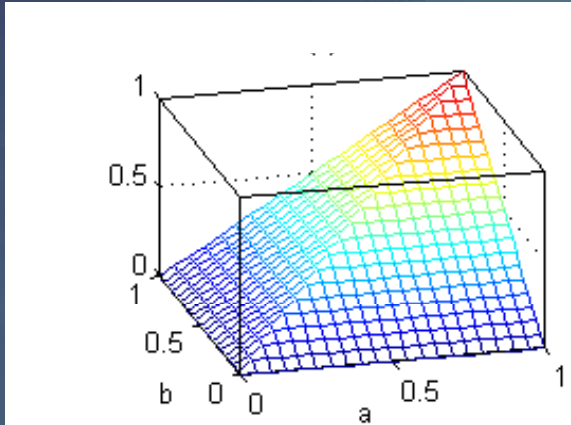
# Examples

$$a \text{ t}_m b = \min(a, b)$$

$$a \text{ t}_p b = ab$$

minimum

product

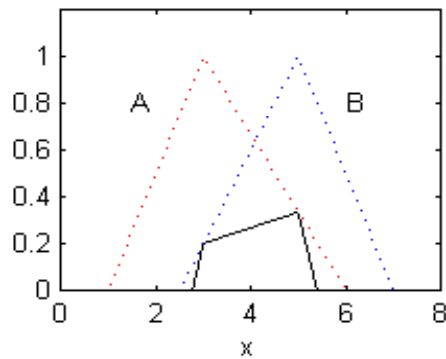
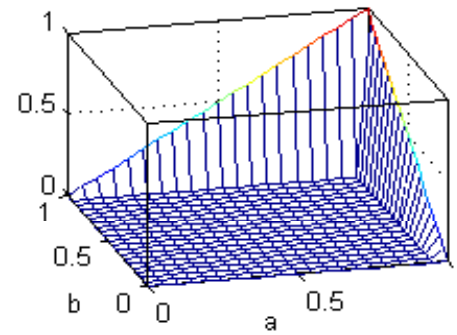
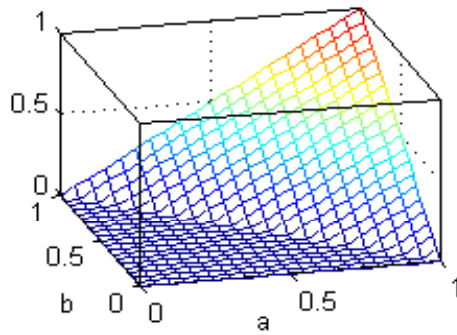


(a)

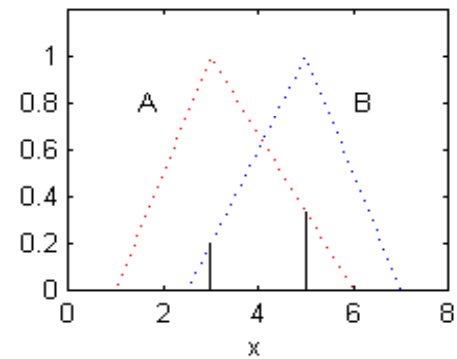
(b)

$$a \text{ t}_1 b = \max(a+b-1, 0)$$

$$a \text{ t}_d b = \begin{cases} a & \text{if } b=1 \\ b & \text{if } a=1 \\ 0 & \text{otherwise} \end{cases}$$



(c)



(d)

Lukasiewicz

drastic  
product

$$a t_d b \leq a t b \leq \min(a, b)$$

lower and upper bounds

$$a t a \leq a$$

Archimedean (if  $t$  is continuous)

$$a^n = 0$$

nilpotent (e.g. Lukasiewicz)

$$a t a = a^2, \dots, a^{n-1} t a = a^n$$

# Constructors of t-norms

- Monotonic function transformation
- Additive and multiplicative generators
- Ordinal sums

# Monotonic function transformation

- If  $h: [0,1] \rightarrow [0,1]$  is a strictly increasing bijection

then  $t_h(a,b) = h^{-1}(t(h(a),h(b)))$  is a t-norm

- $h$  is a scaling transformation

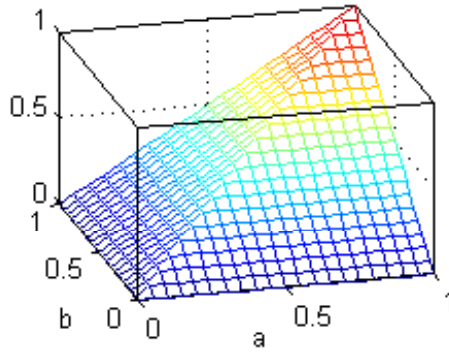
Obs:  $h$  bijective means both,  $h$  injective (one-to-one) and  $h$  surjective (onto)

# Examples

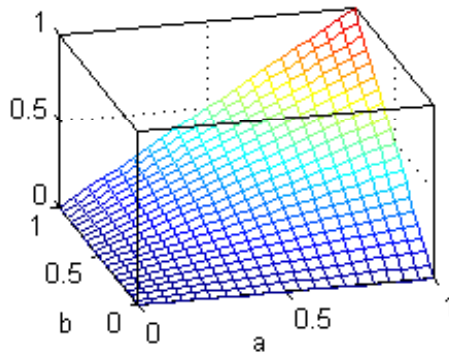
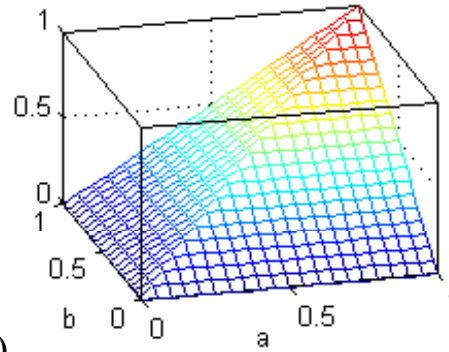
$$h = x^2, \quad x \in [0, 1]$$

$t$

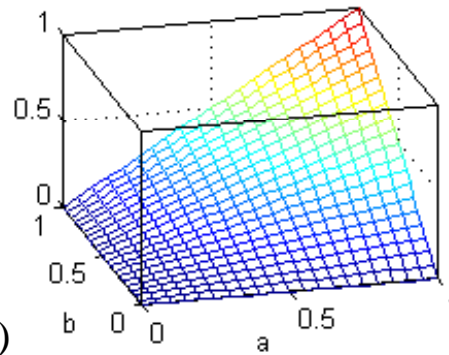
$t_h$



(a)



(b)



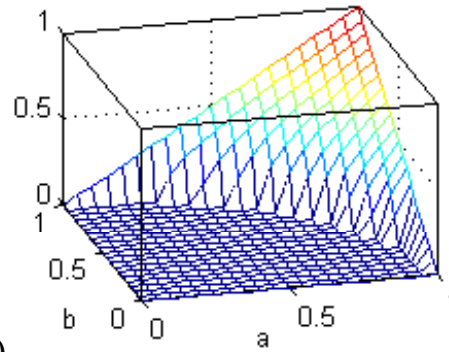
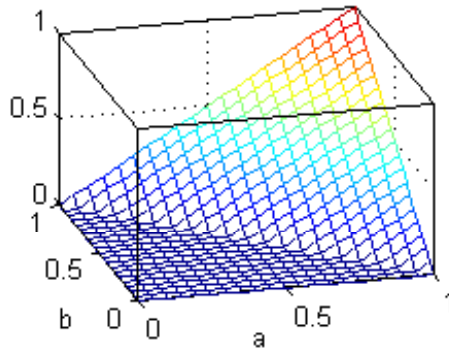
minimum

product

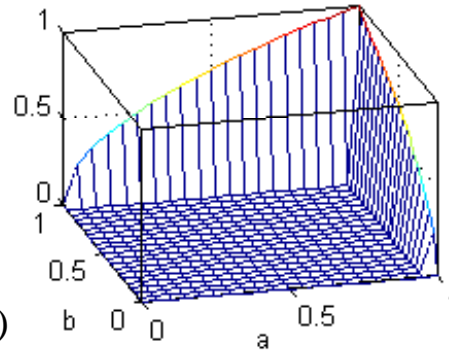
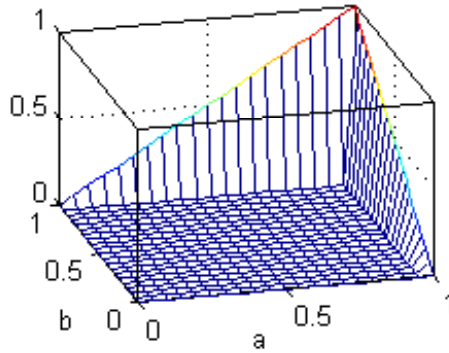
$$h = x^2, \quad x \in [0,1]$$

$t$

$t_h$



(c)



(d)

Lukasiewicz

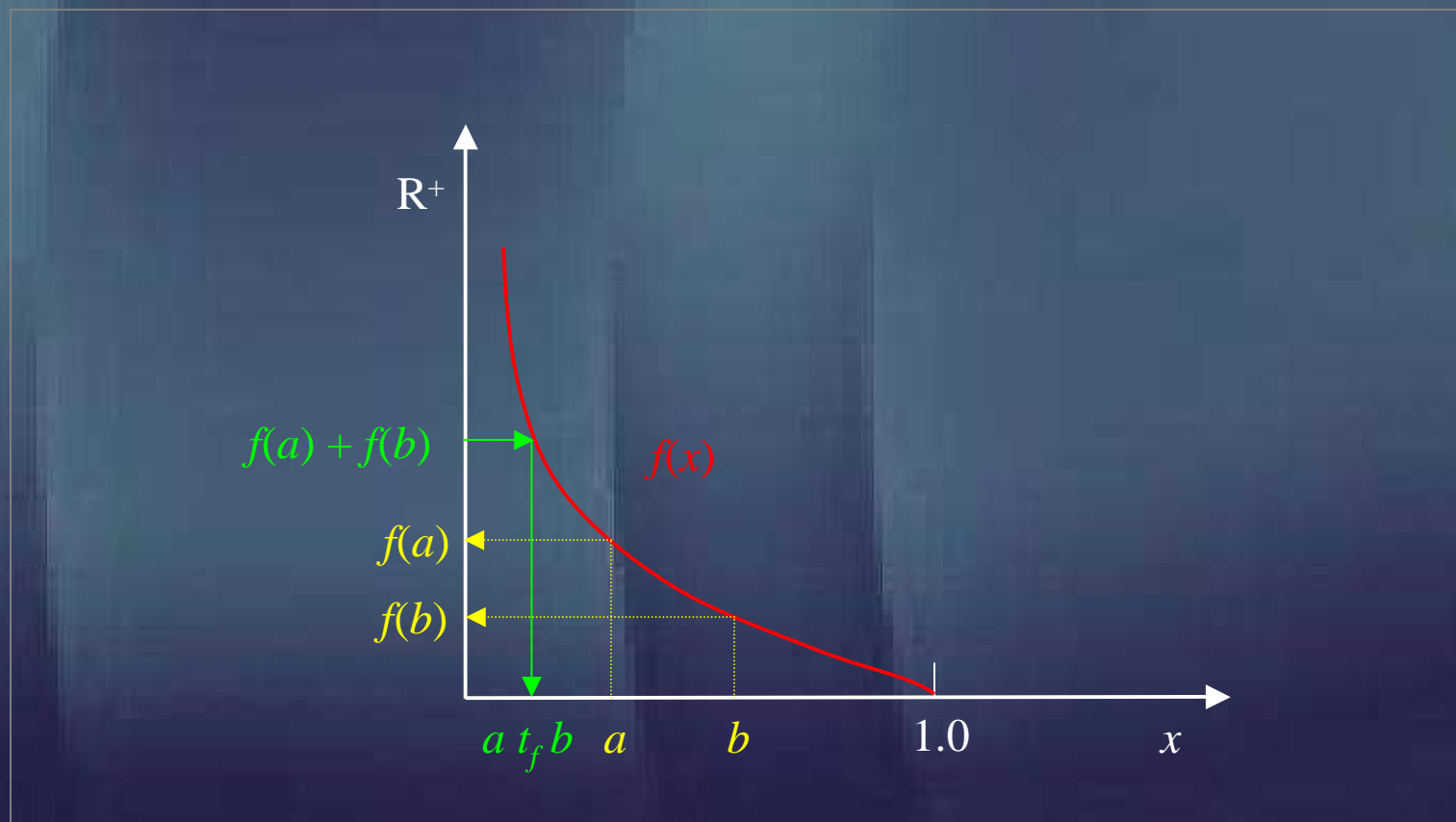
drastic product



# Additive generators

- $f: [0,1] \rightarrow [0, \infty), f(1) = 0$ 
  - continuous
  - strictly decreasing
- $a t_f b = f^{-1}(f(a) + f(b)) \iff$  is a Archimedean t-norm

# Additive generators of t-norms



$$\mathbb{R}^+ \equiv [0, \infty),$$

# Example

$$f = -\log(x)$$

$$\begin{aligned} f(a) + f(b) &= -\log(a) - \log(b) \\ &= -(\log(a) + \log(b)) \\ &= -\log(ab) \end{aligned}$$

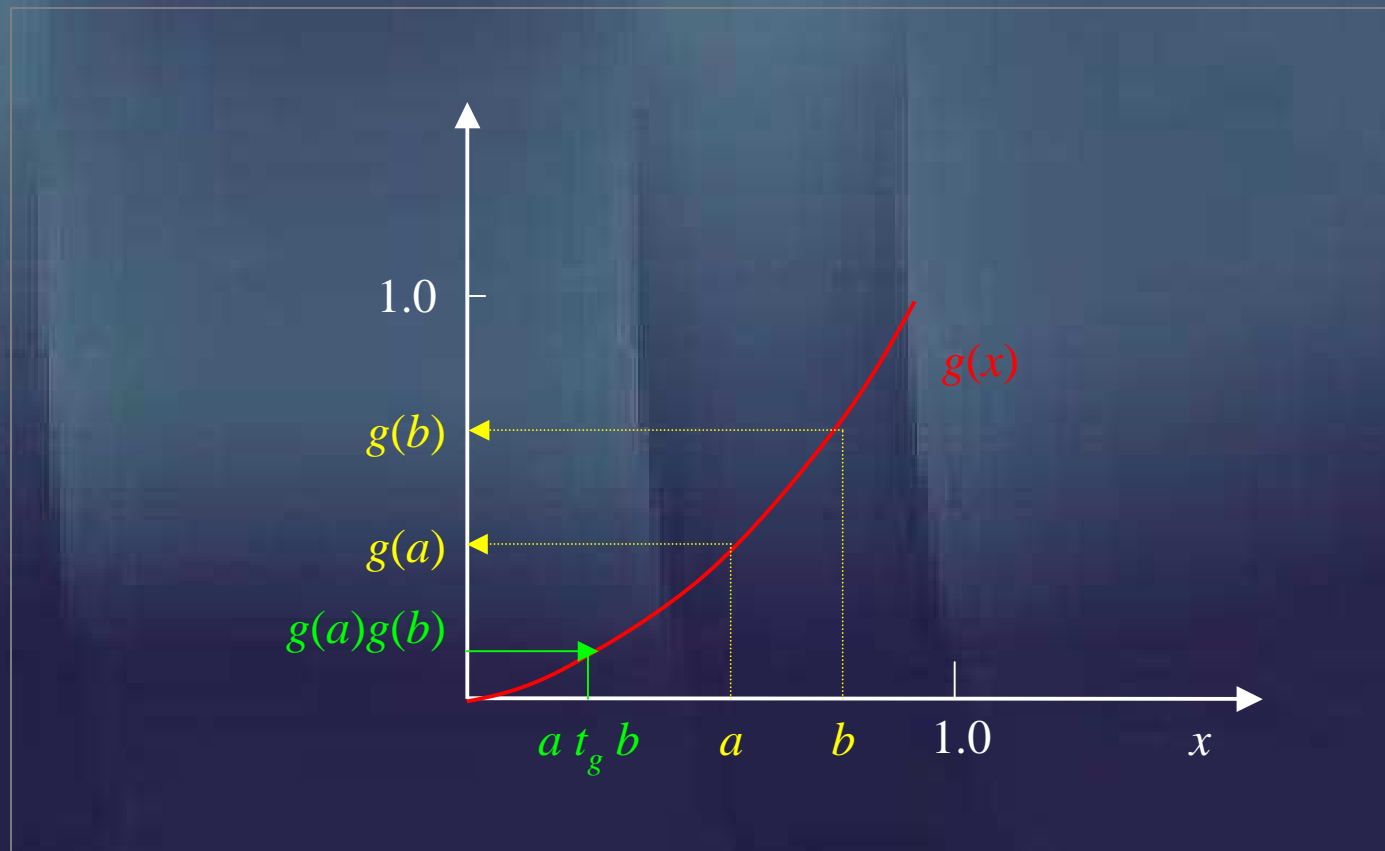
$$f^{-1}(f(a) + f(b)) = e^{\log(ab)} = ab$$

- $a t_f b = ab$  (Archimedean t-norm)

# Multiplicative generators

- $g: [0,1] \rightarrow [0, 1], g(1) = 1$ 
  - continuous
  - strictly increasing
- $a t_g b = g^{-1}(g(a)g(b)) \iff$  is a Archimedean t-norm

# Multiplicative generators of t-norms



# Example

- $g = x^2$

$$a \ t_f \ b = ab \quad (\text{Archimedean t-norm})$$

- $g = e^{-f(x)}$

multiplicative and additive generators  $\rightarrow$  same t-norm

# Ordinal sums

$$t_o: [0,1] \rightarrow [0, 1]$$

denoted  $t_o = (\langle \alpha_k, \beta_k, t_k \rangle, k \in K)$

$$t_o(a, b, I, \tau) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) t_k \left( \frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } a, b \in [\alpha_k, \beta_k] \\ \min(a, b) & \text{otherwise} \end{cases}$$

$$I = \{ [\alpha_k, \beta_k], k \in K \}$$

- nonempty, countable family
- pairwise disjoint subintervals of  $[0,1]$

$$\tau = \{ t_k, k \in K \}$$

- family of t-norms

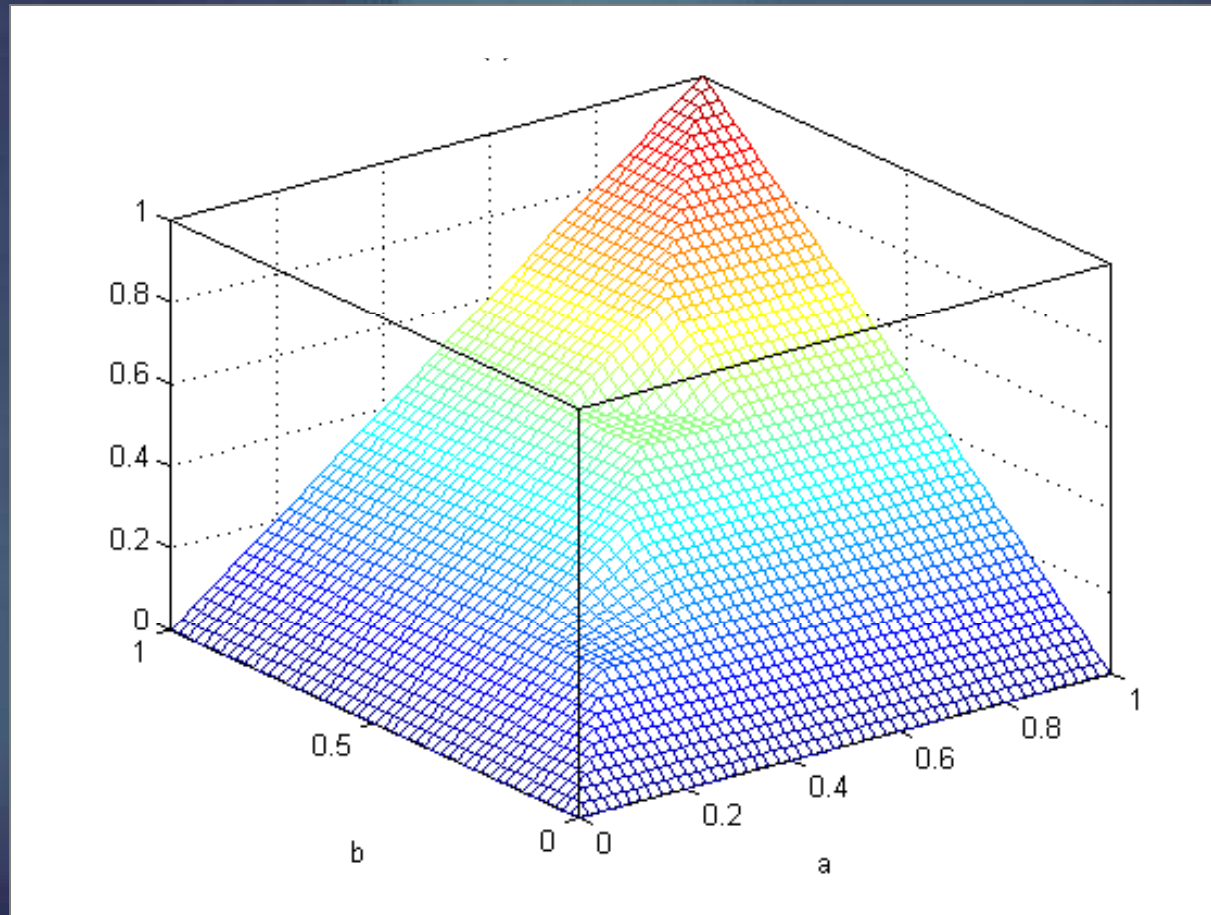
# Example

$$t_o(a,b,I,\tau) = \begin{cases} 0.2 + 5(a - 0.2)(b - 0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + \max(a + b - 1.2, 0) & \text{if } a,b \in [0.5,0.7] \\ \min(a,b) & \text{otherwise} \end{cases}$$

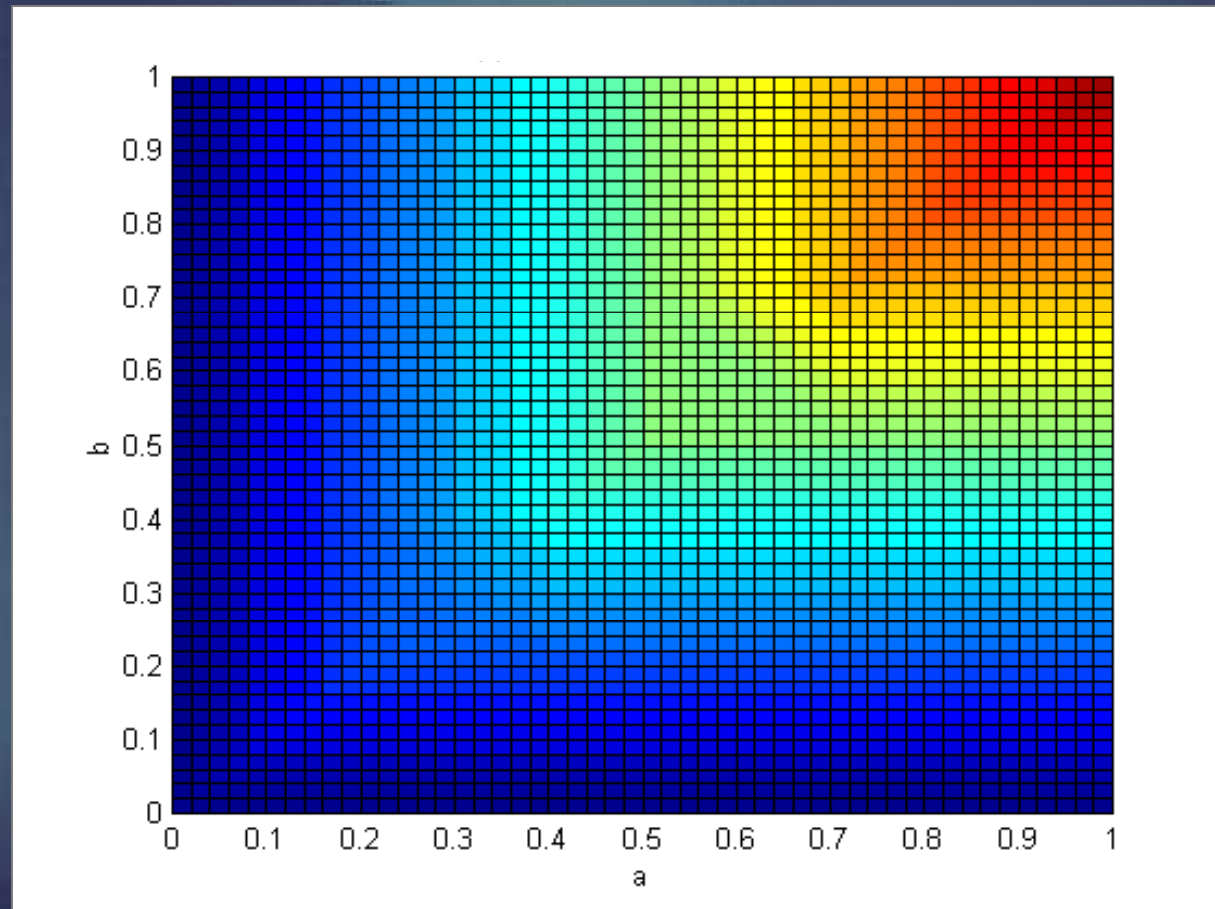
$$I = \{[0.2, 0.4], [0.5, 0.7]\} \quad K = \{1,2\}$$

$$\tau = \{t_p, t_1\}, t_1 = t_p, t_2 = t_1$$





$$t_0(a, b, I, \tau) = \begin{cases} 0.2 + 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + \max(a + b - 1.2, 0) & \text{if } a, b \in [0.5, 0.7] \\ \min(a, b) & \text{otherwise} \end{cases}$$



$$t_0(a, b, I, \tau) = \begin{cases} 0.2 + 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + \max(a + b - 1.2, 0) & \text{if } a, b \in [0.5, 0.7] \\ \min(a, b) & \text{otherwise} \end{cases}$$

## 5.4 Triangular conorms

# Definition

$$s : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity:

$$a \ s \ b = b \ s \ a$$

- Associativity:

$$a \ s \ (b \ s \ c) = (a \ s \ b) \ s \ c$$

- Monotonicity:

$$\text{if } b \leq c \text{ then } a \ s \ b \leq a \ s \ c$$

- Boundary conditions:

$$a \ s \ 1 = 1$$

$$a \ s \ 0 = a$$

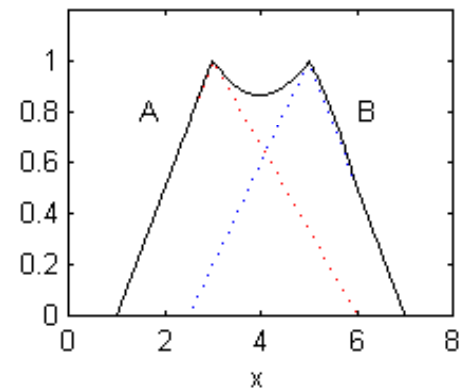
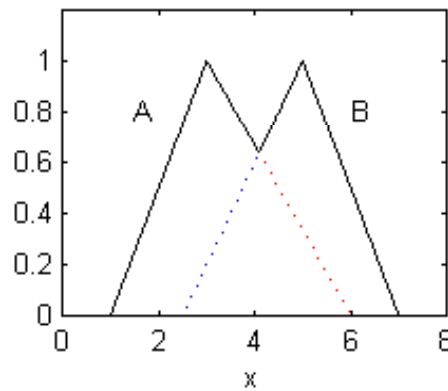
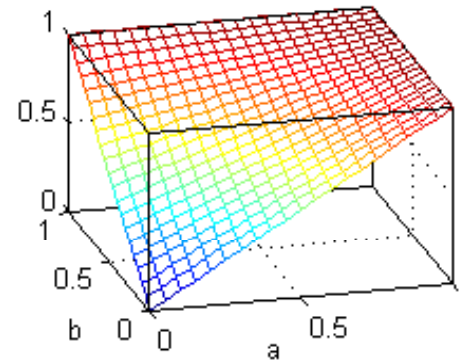
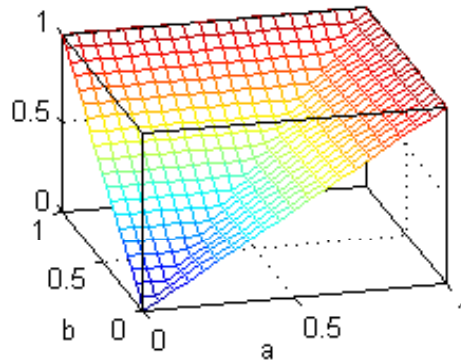
# Examples

$$a s_m b = \max(a, b)$$

$$a s_p b = a + b - ab$$

maximum

probabilistic  
sum



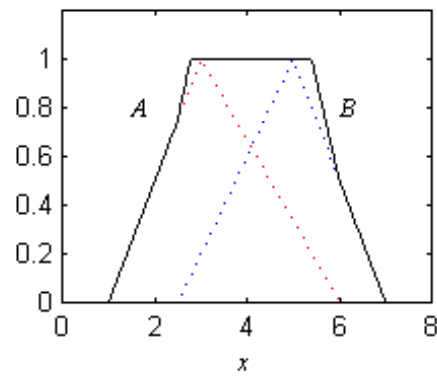
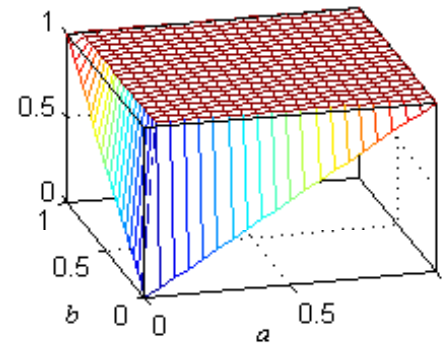
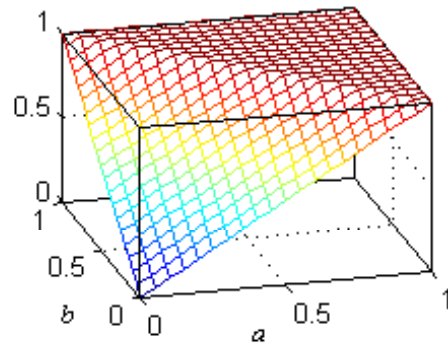
(a)

(b)

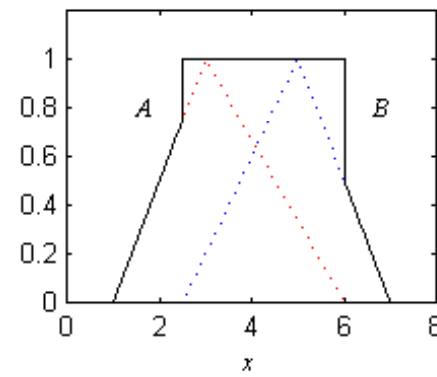
Lukasiewicz

$$a \text{ s}_1 b = \max(a+b, 1)$$

$$a \text{ s}_d b = \begin{cases} a & \text{if } b=0 \\ b & \text{if } a=0 \\ 1 & \text{otherwise} \end{cases}$$



(c)



(d)

drastic  
sum

$$\max(a, b) \leq a s b \leq a s_d b$$

lower and upper bounds

$$a s a \geq a$$

Archimedean (if  $t$  is continuous)

$$a^n = 1$$

nilpotent (e.g. Lukasiewicz)

$$a s a = a^2, \dots, a^{n-1} s a = a^n$$

# Dual norms and De Morgan laws

$$a \, s \, b = 1 - (1 - a) \, t \, (1 - b)$$

$$a \, t \, b = 1 - (1 - a) \, s \, (1 - b)$$

$$(1 - a) \, t \, (1 - b) = 1 - a \, s \, b$$

$$(1 - a) \, s \, (1 - b) = 1 - a \, t \, b$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Dual triangular norms

De Morgan



# Constructors of s-norms

- Monotonic function transformation
- Additive and multiplicative generators
- Ordinal sums

# Monotonic function transformation

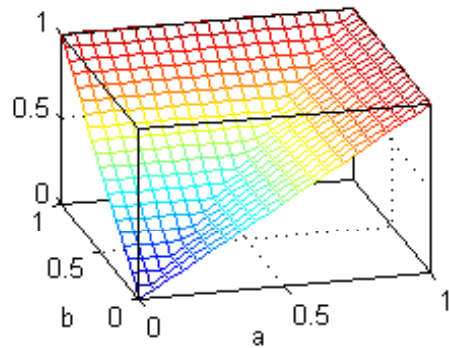
- If  $h: [0,1] \rightarrow [0,1]$  is a strictly increasing bijection  
then  $s_h(a,b) = h^{-1}(s(h(a),s(b)))$  is a t-norm
- $h$  is a scaling transformation

# Examples

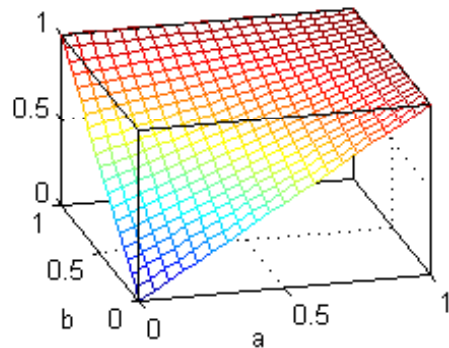
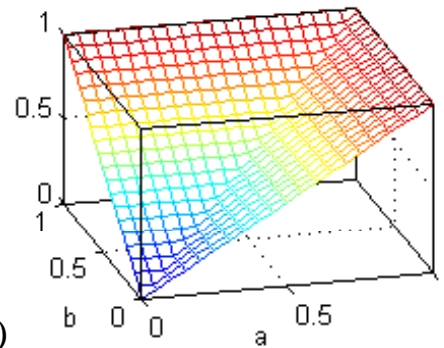
$$h = x^2, x \in [0, 1]$$

$S$

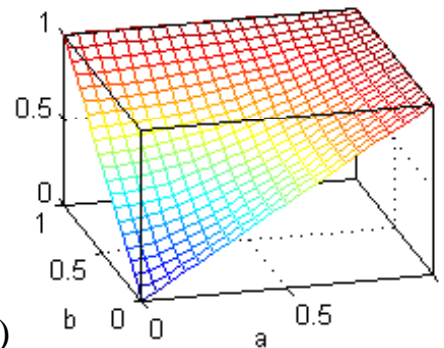
$S_h$



(a)



(b)



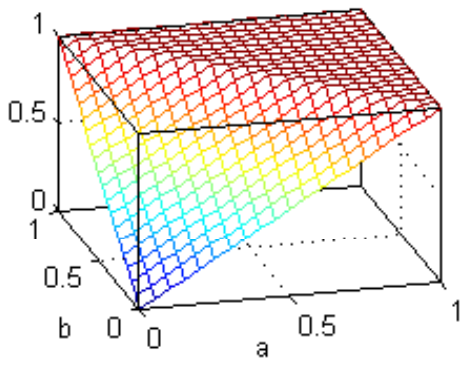
maximum

probabilistic  
sum

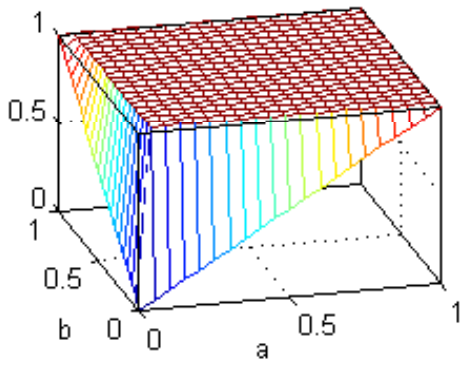
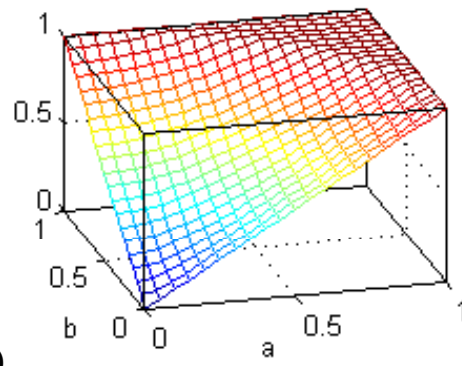
$$h = x^2, x \in [0,1]$$

$S$

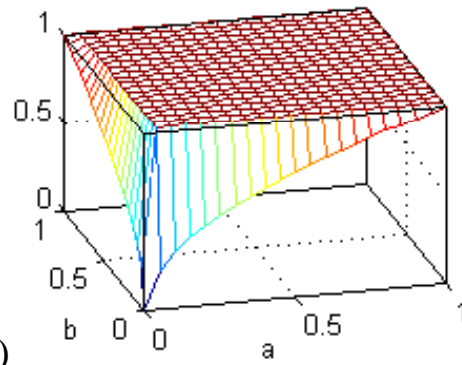
$S_h$



(c)



(d)



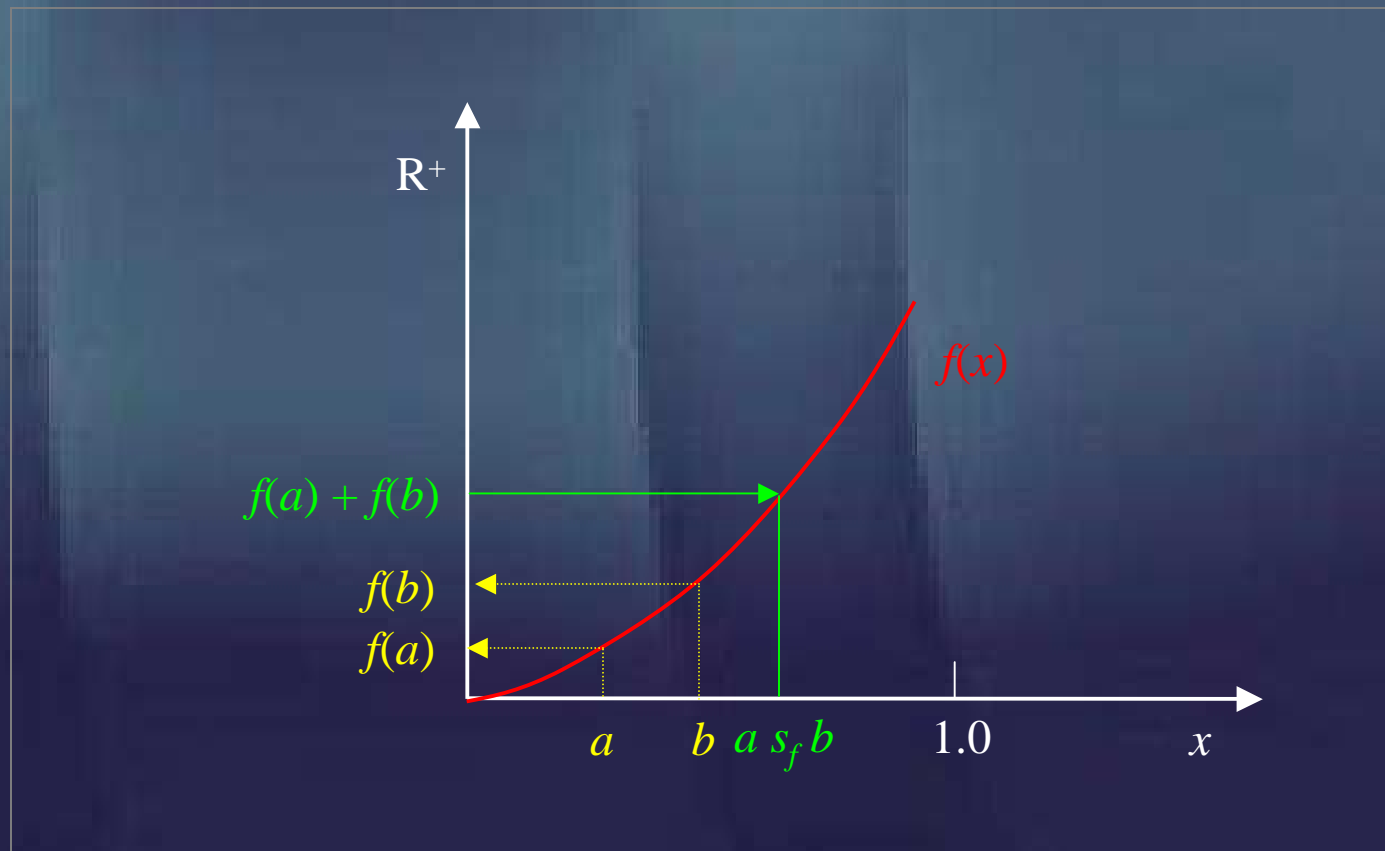
Lukasiewicz

drastic sum

# Additive generators

- $f: [0,1] \rightarrow [0, \infty), f(0) = 0$ 
  - continuous
  - strictly increasing
- $a s_f b = f^{-1}(f(a) + f(b)) \Leftrightarrow$  is a Archimedean t-conorm

# Additive generators of t-conorms



# Example

$$f = -\log(1 - x)$$

$$\begin{aligned} f(a) + f(b) &= -\log(1 - a) - \log(1 - b) \\ &= -(\log(1 - a) + \log(1 - b)) \\ &= -\log(1 - a)(1 - b) \end{aligned}$$

$$f^{-1}(f(a) + f(b)) = 1 - e^{\log(1 - a)(1 - b)} = a + b - ab$$

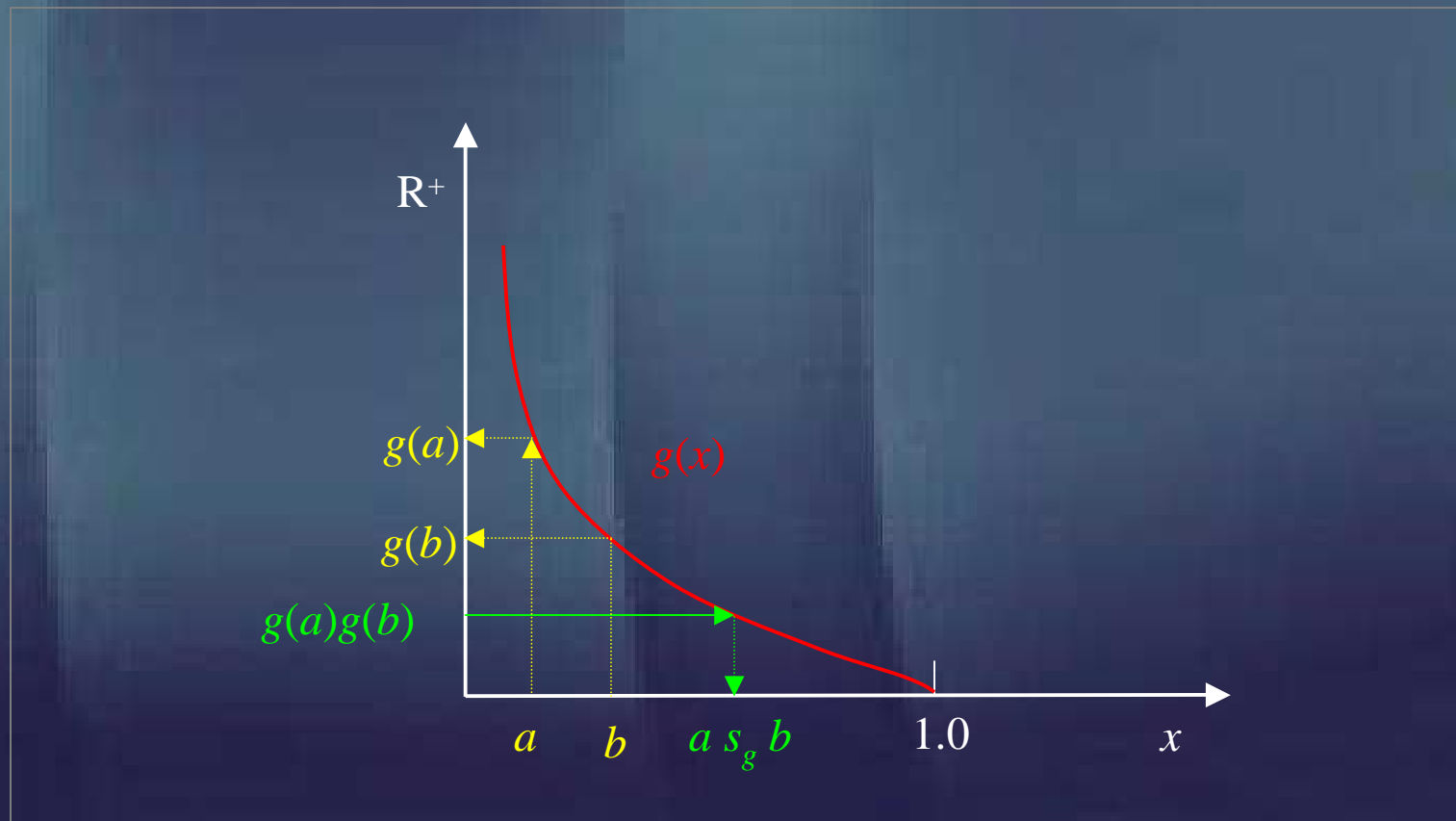
- $a s_f b = a + b - ab$  (Archimedean t-conorm)

# Multiplicative generators

- $g: [0,1] \rightarrow [0, 1], g(0) = 1$ 
  - continuous
  - strictly decreasing
- $a s_g b = g^{-1}(g(a)g(b)) \Leftrightarrow$  is a Archimedean t-norm



# Multiplicative generators of t-conorms



$$\mathbb{R}^+ \equiv [0, \infty),$$

# Example

- $g = 1 - x$

$$a s_g b = a + b - ab \quad (\text{Archimedean t-norm})$$

- $g = e^{-f(x)}$

multiplicative and additive generators  $\rightarrow$  same t-conorm

# Ordinal sums

$$s_o: [0,1] \rightarrow [0, 1]$$

denoted  $s_o = (\langle \alpha_k, \beta_k, s_k \rangle, k \in K)$

$$s_o(a, b, I, \sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) s_k \left( \frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } a, b \in [\alpha_k, \beta_k] \\ \max(a, b) & \text{otherwise} \end{cases}$$

$$I = \{ [\alpha_k, \beta_k], k \in K \}$$

- nonempty, countable family
- pairwise disjoint subintervals of  $[0,1]$

$$\sigma = \{ s_k, k \in K \}$$

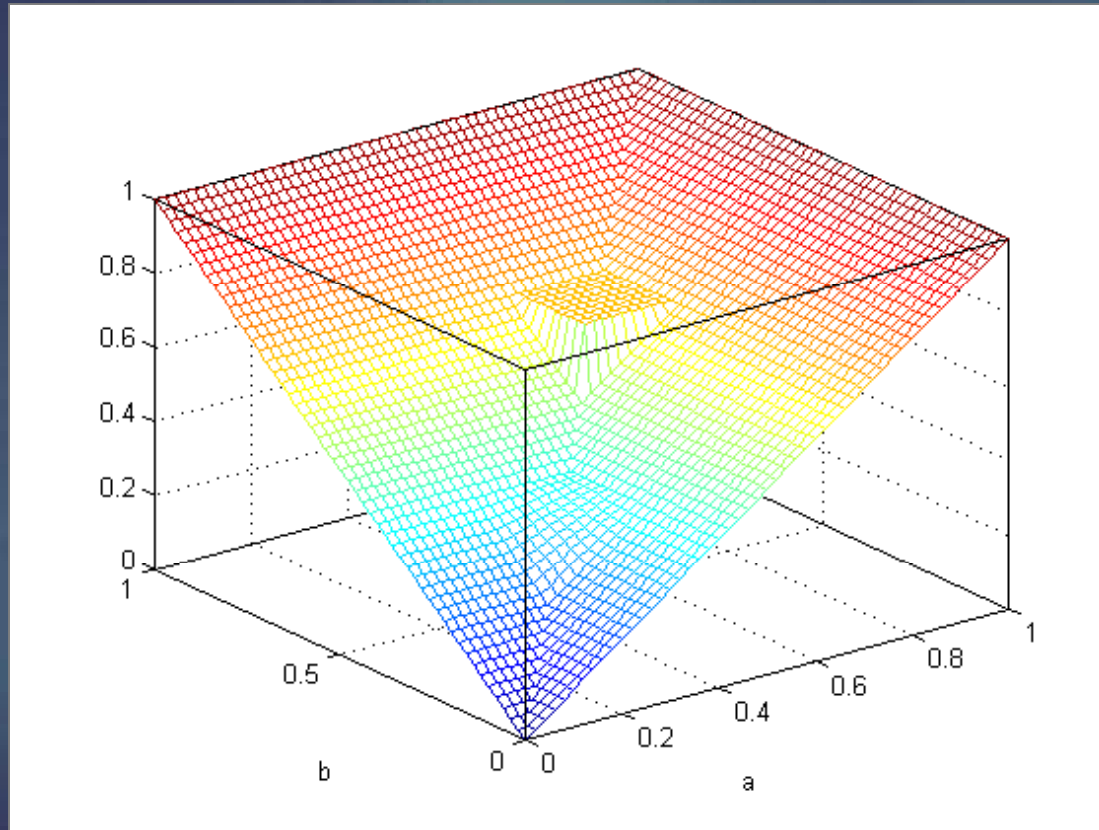
- family of t-conorms

# Example

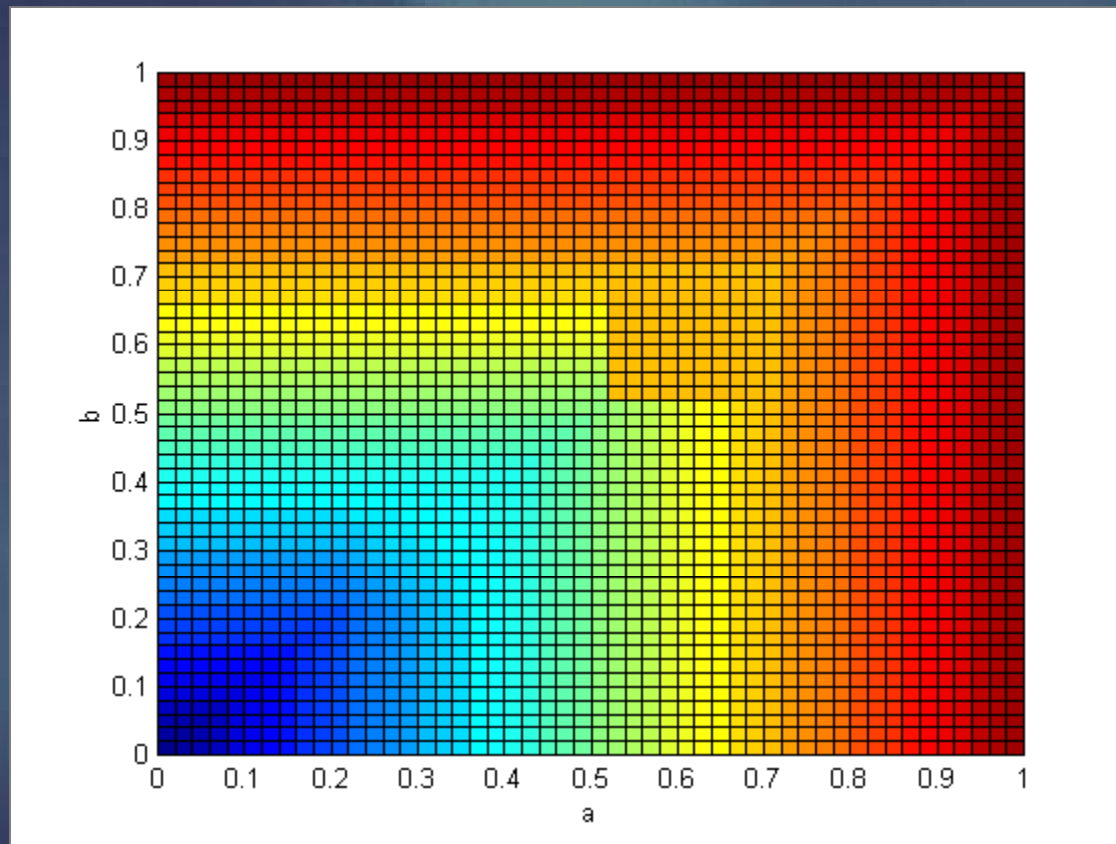
$$s_o(a,b,I,\sigma) = \begin{cases} 0.2 + (a - 0.2) + (b - 0.2) - 5(a-0.2)(b-0.2) & \text{if } a,b \in [0.2,0.4] \\ 0.5 + 0.2\min(5(a - 0.2) + 5(b - 0.2), 1) & \text{if } a,b \in [0.5,0.7] \\ \max(a,b) & \text{otherwise} \end{cases}$$

$$I = \{[0.2, 0.4], [0.5, 0.7]\} \quad K = \{1,2\}$$

$$\sigma = \{s_p, s_1\}, s_1 = s_p, s_2 = s_1$$



$$s_0(a, b, I, \sigma) = \begin{cases} 0.2 + (a - 0.2) + (b - 0.2) - 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + 0.2 \min(5(a - 0.2) + 5(b - 0.2), 0), 1 & \text{if } a, b \in [0.5, 0.7] \\ \max(a, b) & \text{otherwise} \end{cases}$$



$$s_0(a, b, I, \sigma) = \begin{cases} 0.2 + (a - 0.2) + (b - 0.2) - 5(a - 0.2)(b - 0.2) & \text{if } a, b \in [0.2, 0.4] \\ 0.5 + 0.2 \min(5(a - 0.2) + 5(b - 0.2), 0), 1 & \text{if } a, b \in [0.5, 0.7] \\ \max(a, b) & \text{otherwise} \end{cases}$$

## **5.5 Triangular norms as general category of logical operators**

# Motivation

- **Fuzzy propositions involves linguistic statements:**
  - temperature is low *and* humidity is high
  - velocity is high *or* noise level is low
- **Logical operations:**
  - and ( $\wedge$ )
  - or ( $\vee$ )



# Truth value assignment

$L = \{P, Q, \dots\}$      $P, Q, \dots$     atomic statements

truth:  $L \rightarrow [0, 1]$      $p, q, \dots \in [0, 1]$

truth ( $P$  and  $Q$ ) = truth ( $P \wedge Q$ )  $\rightarrow p \wedge q = p \text{ t } q$

truth ( $P$  or  $Q$ ) = truth ( $P \vee Q$ )  $\rightarrow p \vee q = p \text{ s } q$

# Examples

p	q	$\min(p, q)$	$\max(p, q)$	$pq$	$p + q - pq$
1	1	1	1	1	1
1	0	0	1	0	1
0	1	0	1	0	1
0	0	0	0	0	0
0.2	0.5	0.2	0.5	0.1	0.6
0.5	0.8	0.5	0.8	0.4	0.9
0.8	0.7	0.7	0.8	0.56	0.94

# Implication induced by a t-norm

$$a \varphi b \equiv a \Rightarrow b$$

$$a \varphi b = \sup \{ c \in [0, 1] \mid a t c \leq b \} \quad \forall a, b \in [0, 1] \quad \text{residuation}$$

$\varphi$  operator or Boolean values of its arguments

$a$	$b$	$a \Rightarrow b$	$a \varphi b$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

# 5.6 Aggregation operations

# Definition

$$g : [0,1]^n \rightarrow [0,1]$$

## 1. Monotonicity:

$$g(x_1, x_2, \dots, x_n) \geq g(y_1, y_2, \dots, y_n) \text{ if } x_i \geq y_i, i = 1, \dots, n$$

## 2. Boundary conditions:

$$g(0, 0, \dots, 0) = 0$$

$$g(1, 1, \dots, 1) = 1$$

## 1. Neutral element ( $e$ ):

$$g(x_1, x_2, \dots, x_{i-1}, e, x_{i+1}, \dots, x_n) = g(x_1, x_2, \dots, x_{i-1}, e, x_{i+1}, \dots, x_n) \quad n \geq 2$$

## 2. Annihilator ( $l$ ):

$$g(x_1, x_2, \dots, x_{i-1}, l, x_{i+1}, \dots, x_n) = l$$

**Observation: Annihilator  $\equiv$  absorbing element**

# Averaging operations

$$g(x_1, x_2, \dots, x_n) = \sqrt[p]{\frac{1}{n} \sum_{i=1}^n (x_i)^p}, \quad p \in \mathbf{R}, \quad p \neq 0$$

$$p = 1 \quad g(x_1, x_2, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i$$

arithmetic mean

$$p \rightarrow 0 \quad g(x_1, x_2, \dots, x_n) = \sqrt[n]{\prod_{i=1}^n x_i}$$

geometric mean

$$p = -1 \quad g(x_1, x_2, \dots, x_n) = \frac{n}{\sum_{i=1}^n 1/x_i}$$

harmonic mean

$$p \rightarrow -\infty \quad g(x_1, x_2, \dots, x_n) = \min(x_1, x_2, \dots, x_n)$$

$$p \rightarrow \infty \quad g(x_1, x_2, \dots, x_n) = \max(x_1, x_2, \dots, x_n)$$

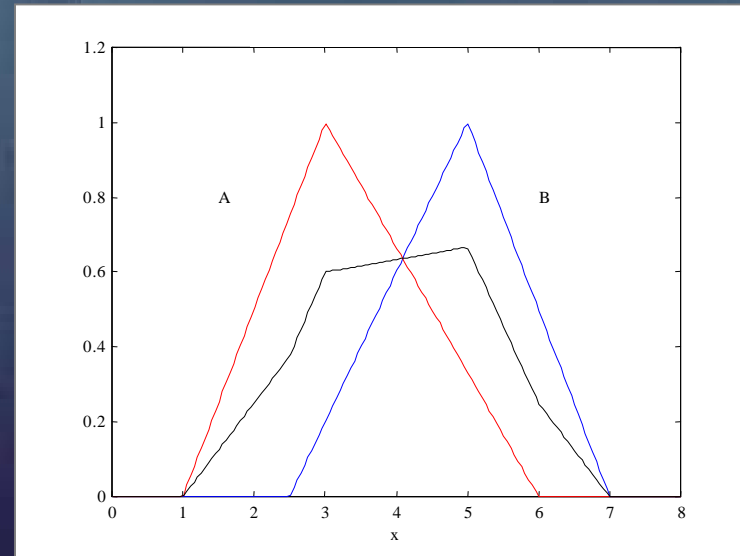
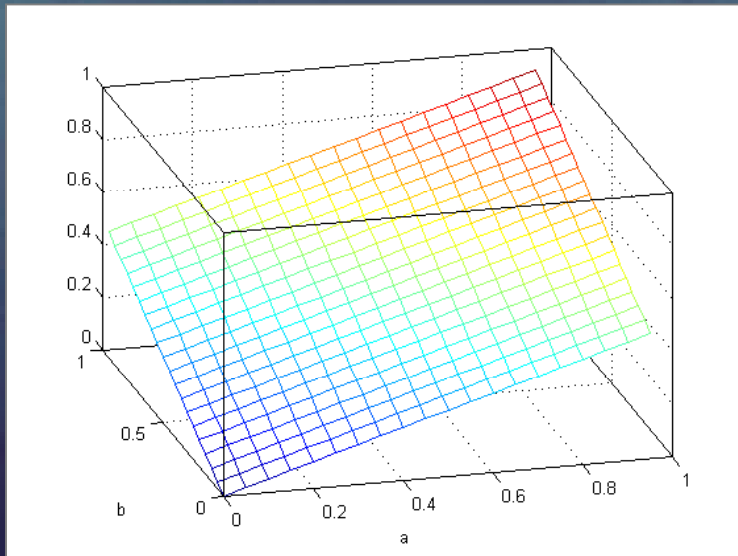
## Bounds

$$\min(x_1, x_2, \dots, x_n) \leq g(x_1, x_2, \dots, x_n) \leq \max(x_1, x_2, \dots, x_n)$$

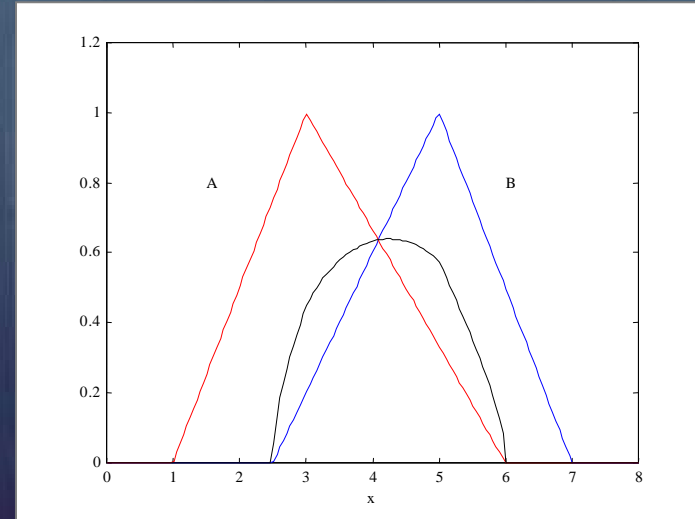
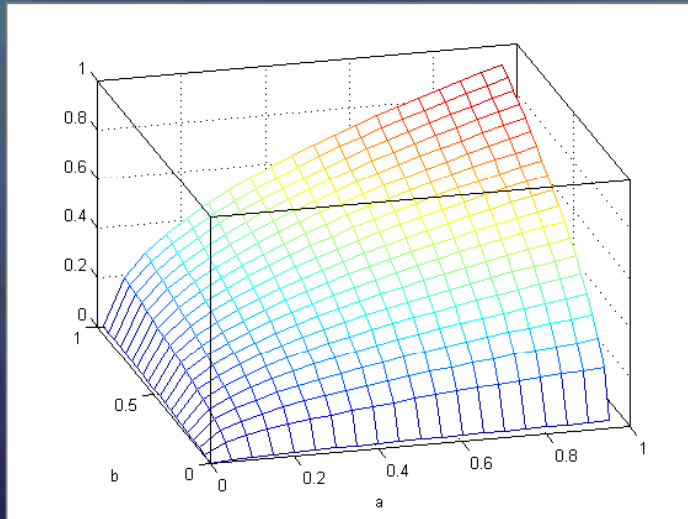


# Examples

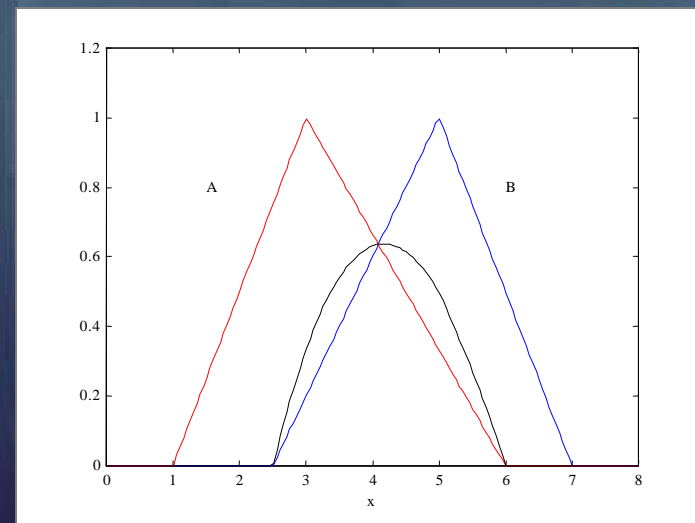
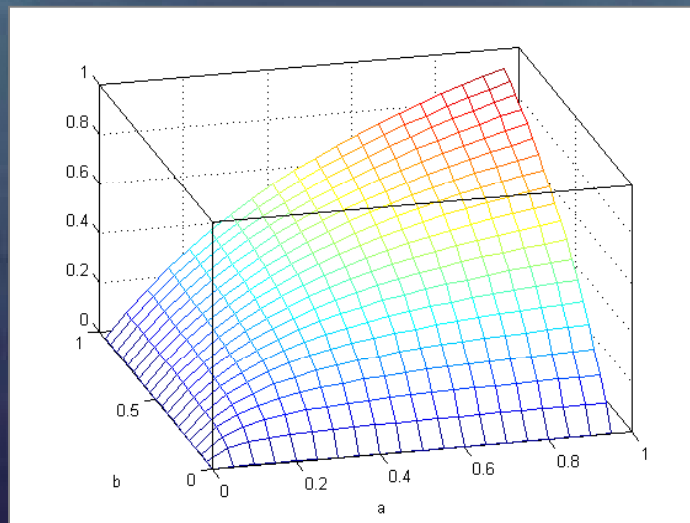
## Arithmetic mean



# Geometric mean



# Harmonic mean



# Ordered Weighted Averaging (OWA)

$$\text{OWA}(A, \mathbf{w}) = \sum_{i=1}^n w_i A(x_i)$$

$$\sum w_i = 1, \quad w_i \in [0, 1]$$

$$A(x_1) \leq A(x_2) \leq \dots \leq A(x_n)$$

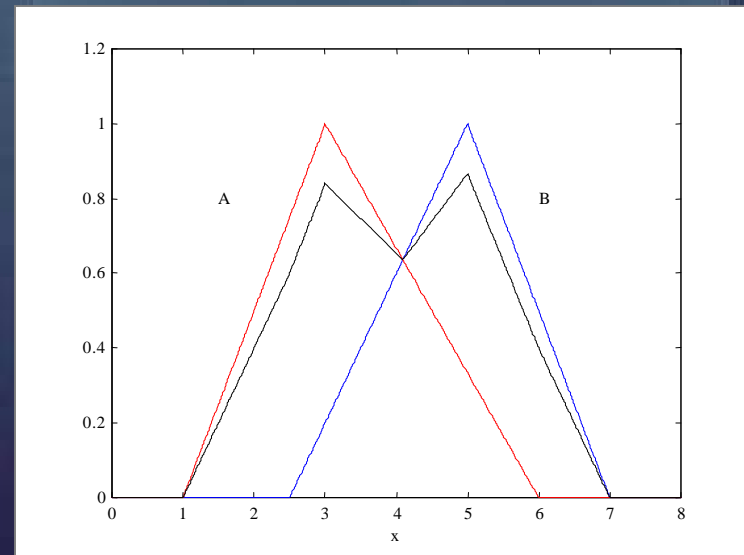
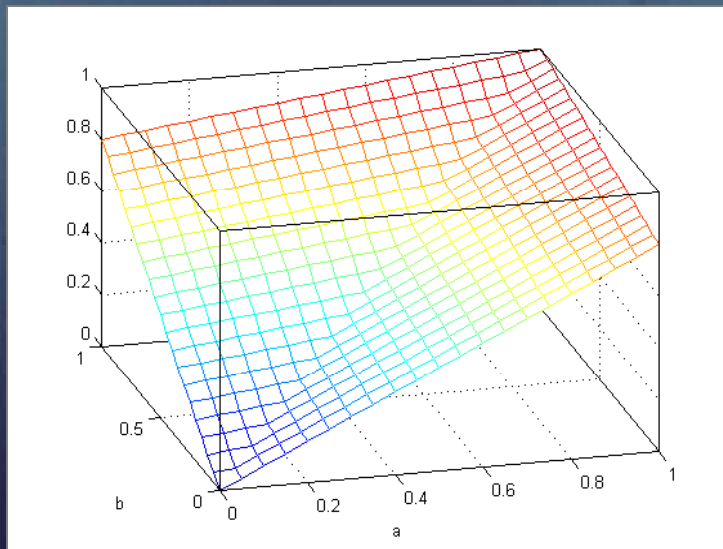
# Examples

1.  $\mathbf{w} = [1, 0, \dots, 0]$        $\text{OWA}(A, \mathbf{w}) = \min (A(x_1), A(x_2), \dots, A(x_n))$
2.  $\mathbf{w} = [0, 0, \dots, 1]$        $\text{OWA}(A, \mathbf{w}) = \max (A(x_1), A(x_2), \dots, A(x_n))$
3.  $\mathbf{w} = [1/n, 1/n, \dots, 1/n]$        $\text{OWA}(A, \mathbf{w}) = \text{arithmetic mean}$

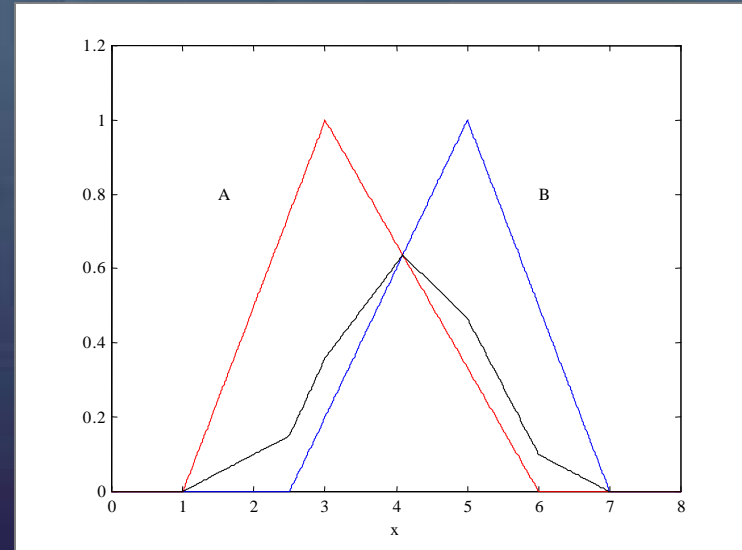
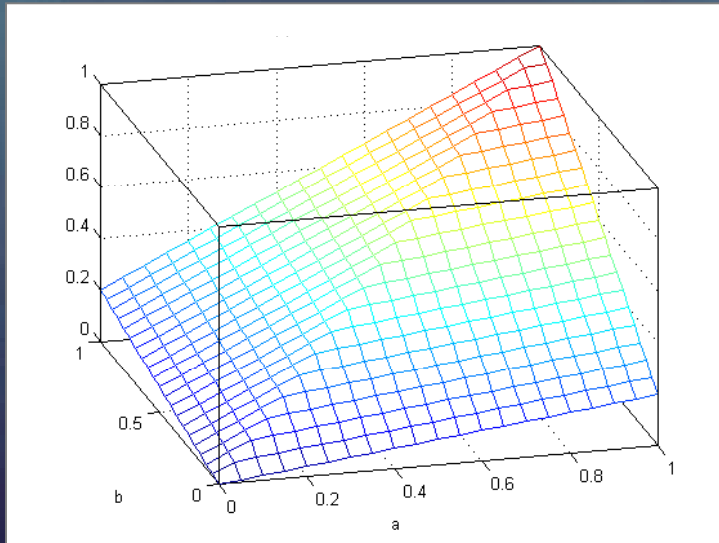
$$\min (A(x_1), A(x_2), \dots, A(x_n)) \leq \text{OWA}(A, \mathbf{w}) \leq \max (A(x_1), A(x_2), \dots, A(x_n))$$

# Examples

$$\mathbf{w} = [0.8, 0.2]$$



$$\mathbf{w} = [0.2, 0.8]$$



# Uninorms

$$u : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity:

$$a \ u \ b = b \ u \ a$$

- Associativity:

$$a \ u \ (b \ u \ c) = (a \ u \ b) \ u \ c$$

- Monotonicity:

$$\text{if } b \leq c \text{ then } a \ u \ b \leq a \ u \ c$$

- Identity:

$$a \ u \ e = a \quad \forall a \in [0, 1]$$

$e \in [0, 1]$ ,  $e = 1$   $u$  is a t-norm,  $e = 0$   $u$  is a t-conorm



# Results on uninorms

1.  $t_u, s_u: [0, 1] \times [0, 1] \rightarrow [0, 1]$  such that  $\forall e \in [0, 1]$

$$at_u b = \frac{(ea)u(eb)}{e}$$

$$as_u b = \frac{(e + (1-e)a)u(e + (1-e)b) - e}{1-e}$$

$t_u$  and  $s_u$  are t-norm and t-conorm

2. If  $a \leq e \leq b$  or  $a \geq e \geq b$  then

If  $a \leq e \leq b$  or  $a \geq e \geq b$  then  
 $\min(a, b) \leq a \cup b \leq \max(a, b)$

3. For any  $u$  with  $e \in [0, 1]$

$$au_w b \leq aub \leq au_s b$$

$$au_w b = \begin{cases} 0 & \text{if } 0 \leq a, b \leq e \\ \max(a, b) & \text{if } e \leq a, b \leq 1 \\ \min(a, b) & \text{otherwise} \end{cases}$$

$$au_s b = \begin{cases} \min(a, b) & \text{if } 0 \leq a, b \leq e \\ 1 & \text{if } e \leq a, b \leq 1 \\ \max(a, b) & \text{otherwise} \end{cases}$$

## 4. Conjunctive and disjunctive uninorm

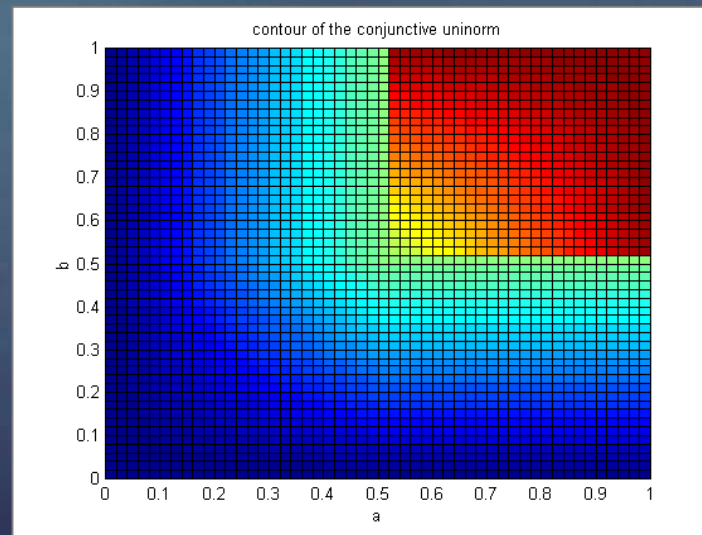
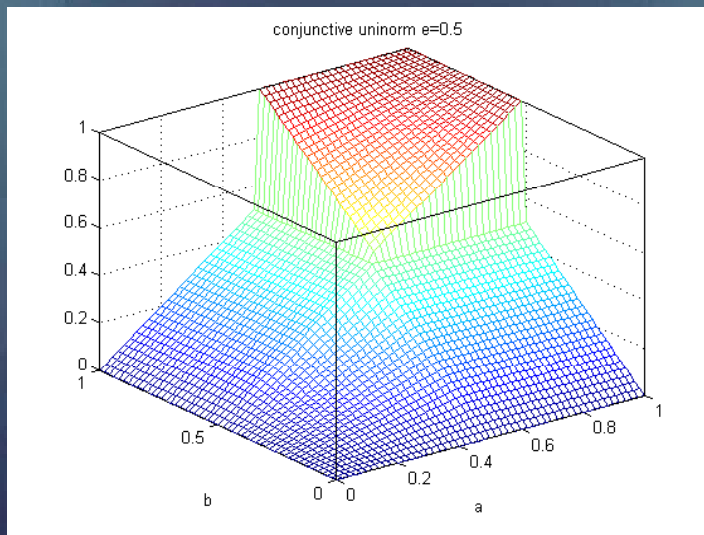
if  $(0u1) = 0$  then

$$au_c b = \begin{cases} e\left(\frac{a}{e}\right)t\left(\frac{b}{e}\right) & \text{if } 0 \leq a, b \leq e \\ e + (1-e)\left(\frac{a-e}{1-e}\right)s\left(\frac{b-e}{1-e}\right) & \text{if } e \leq a, b \leq 1 \\ \min(a, b) & \text{otherwise} \end{cases}$$

if  $(0u1) = 1$  then

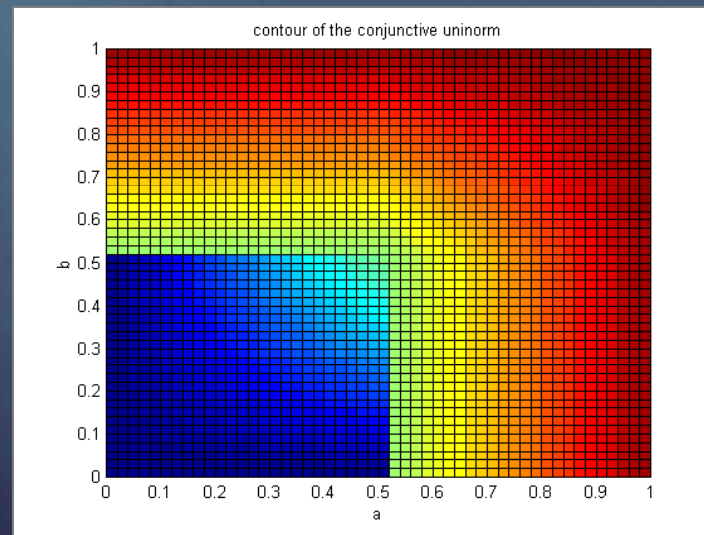
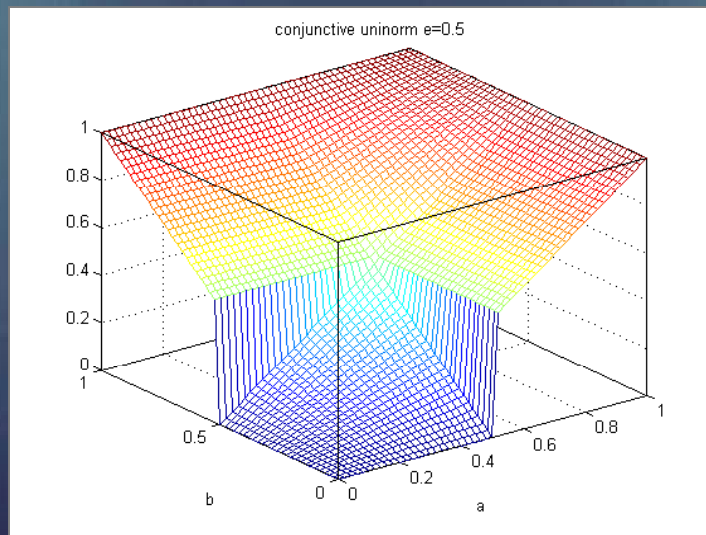
$$au_d b = \begin{cases} e\left(\frac{a}{e}\right)t\left(\frac{b}{e}\right) & \text{if } 0 \leq a, b \leq e \\ e + (1-e)\left(\frac{a-e}{1-e}\right)s\left(\frac{b-e}{1-e}\right) & \text{if } e \leq a, b \leq 1 \\ \max(a, b) & \text{otherwise} \end{cases}$$

# Conjunctive uninorm



$$e = 0.5$$

# Disjunctive uninorm



$$e = 0.5$$

## 5. Almost continuous Archimedean uninorms

$$a \mathop{u} a < a \text{ for } 0 < a < e$$

$$a \mathop{u} a > a \text{ for } e < a < 1$$

## 6. Additive and multiplicative generators of almost continuous uninorms

$$a \mathop{u}_f b = f^{-1}(f(a) + f(b))$$

$$a \mathop{u}_g b = g^{-1}(g(a)g(b))$$

$$g(x) = e^{-f(x)}$$

$f$  strictly increasing

$g$  strictly decreasing

## 7. Ordinal sum

$$u_{\text{co}}(a, b, l, \tau, \sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)t_k \left( \frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a, b] \in \iota_1 \\ \alpha_k + (\beta_k - \alpha_k)s_k \left( \frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a, b] \in \iota_2 \\ \max(a, b) & \text{if } a, b \notin [\alpha_k, \beta_k] \\ \min(a, b) & \text{and } a, b \geq e \\ & \text{otherwise} \end{cases}$$

$$l = \{[\alpha_k, \beta_k], k \in K\}$$

$$l_1 = \{[\alpha_k, \beta_k] \in l \mid \beta_k \leq e\}$$

$$l_2 = \{[\alpha_k, \beta_k] \in l \mid \alpha_k \geq e\}$$

$$\tau = \{t_k, k \in K\}, \quad \sigma = \{s_k, k \in K\}$$



$$u_{\text{co}}(a, b, l, \tau, \sigma) = \begin{cases} \alpha_k + (\beta_k - \alpha_k)t_k \left( \frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a, b] \in \iota_1 \\ \alpha_k + (\beta_k - \alpha_k)s_k \left( \frac{a - \alpha_k}{\beta_k - \alpha_k}, \frac{b - \alpha_k}{\beta_k - \alpha_k} \right) & \text{if } [a, b] \in \iota_2 \\ \min(a, b) & \text{if } a, b \notin [\alpha_k, \beta_k] \\ \max(a, b) & \text{and } a, b \leq e \\ & \text{otherwise} \end{cases}$$

# Nullnorms

$$\nu : [0,1] \times [0,1] \rightarrow [0,1]$$

- Commutativity:

$$a \nu b = b \nu a$$

- Associativity:

$$a \nu (b \nu c) = (a \nu b) \nu c$$

- Monotonicity:

$$\text{if } b \leq c \text{ then } a \nu b \leq a \nu c$$

- Absorbing element:

$$a \nu e = e \quad \forall a \in [0, 1]$$

- Boundary conditions:

$$a \nu 0 = a \quad \forall a \in [0, e]$$

$$a \nu 1 = a \quad \forall a \in [e, 1]$$

$$as_{\nu}b = \frac{(ea)\nu(eb)}{e}$$

$$at_{\nu}b = \frac{(e + (1 - e)a)\nu(e + (1 - e)b) - e}{1 - e}$$

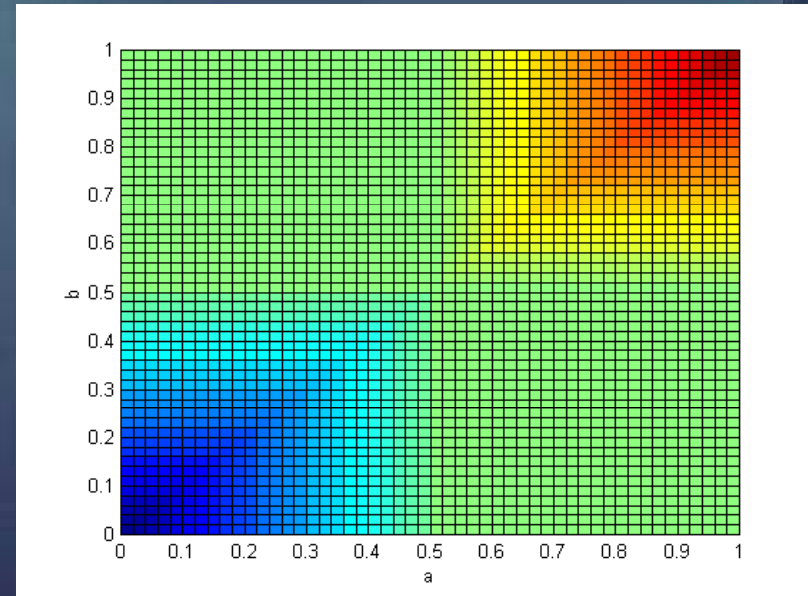
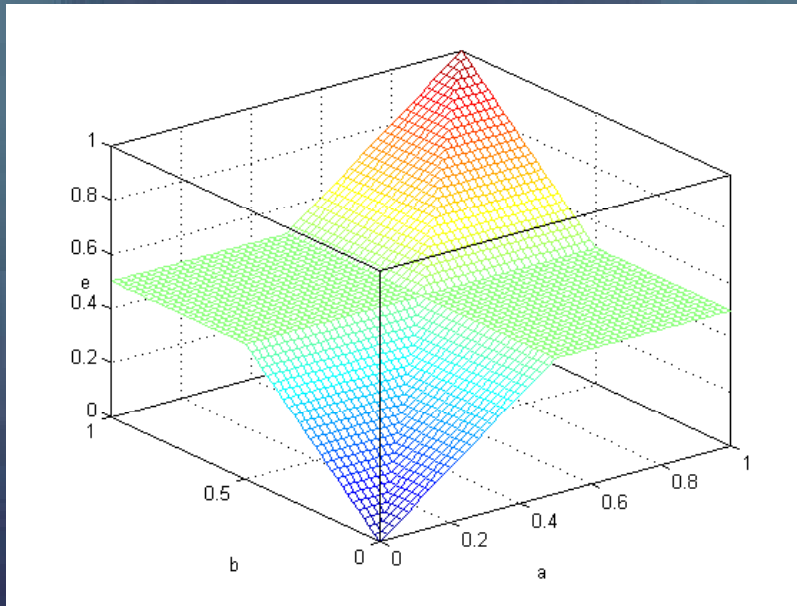
$e \in [0, 1]$

$\nu$  behaves as a t-norm in  $[0, e] \times [0, e]$

$\nu$  behaves as a t-conorm in  $[e, 1] \times [e, 1]$

$\nu = e$  in the rest of the unit square

# Example



$e = 0.5$ , t-norm = min, t-conorm = max

# Symmetric sums

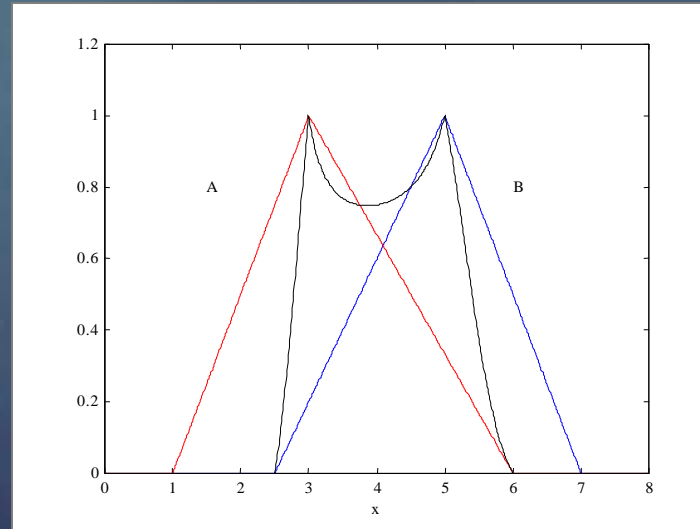
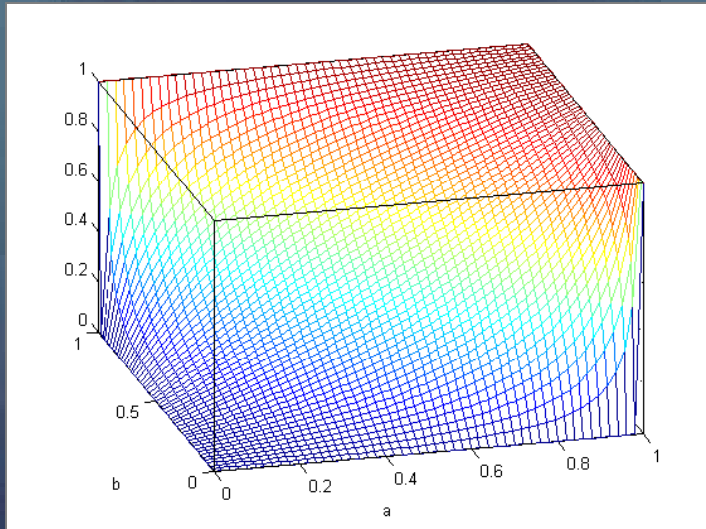
$$\sigma_s(a_1, a_2, \dots, a_n) = 1 - \sigma_s(1 - a_1, 1 - a_2, \dots, 1 - a_n)$$

$$\sigma_s(a_1, a_2, \dots, a_n) = \left[ 1 + \frac{f(1 - a_1, 1 - a_2, \dots, a_n)}{f(a_1, a_2, \dots, a_n)} \right]^{-1}$$

$f$  increasing, continuous

$$f(0, 0, \dots, 0) = 0$$

# Example



$$f(a,b) = a^2 + b^2$$

# Compensatory operations

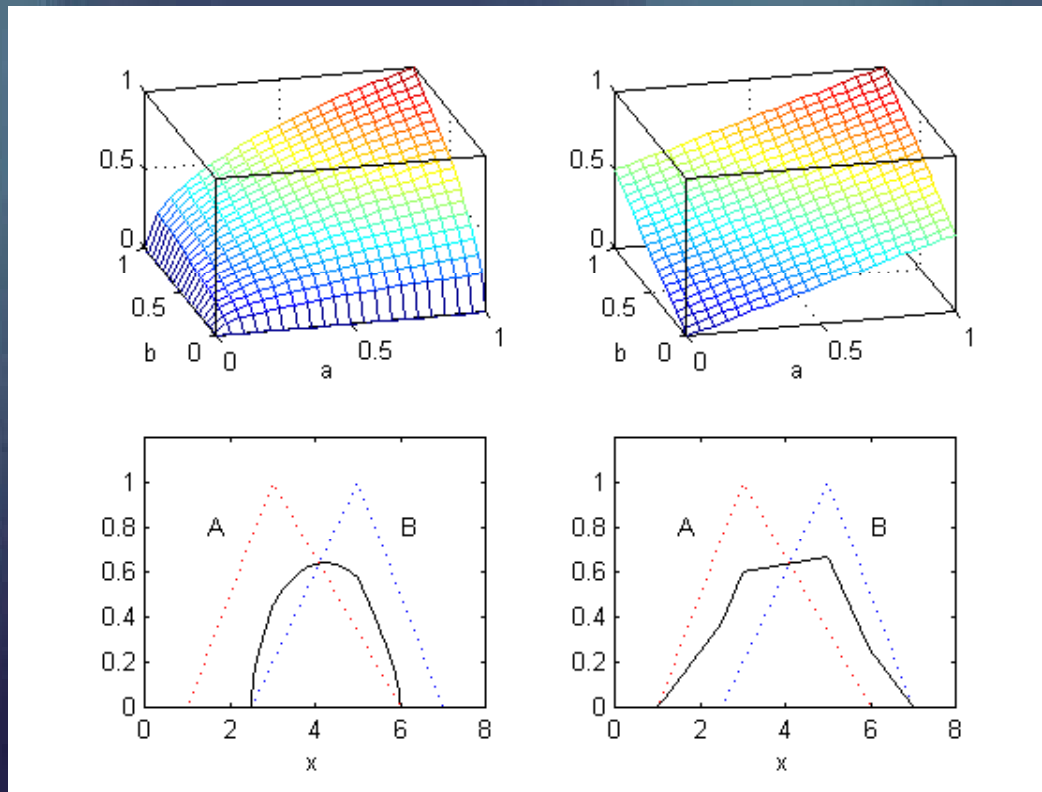
$$a \odot b = (a \text{ t } b)^{1-\gamma} (a \text{ s } b)^\gamma$$

compensatory product

$$a \oplus b = (1 - \gamma)(a \text{ t } b) + \gamma(a \text{ t } b)$$

compensatory sum

# Example



(a)

(b)

$\gamma = 0.5$ , t-norm = min, t-conorm = max



# 5.7 Fuzzy measure and integral

# Fuzzy measure

$$g : \Omega \rightarrow [0,1]$$

- Boundary conditions:  $g(\emptyset) = 0$   
 $g(\mathbf{X}) = 1$
- Monotonicity: if  $A \subset B$  then  $g(A) \leq g(B)$

# $\lambda$ -fuzzy measure

$$g(A \cup B) = g(A) + g(B) + \lambda g(A) g(B), \quad \lambda > -1$$

- $\lambda = 0$        $g(A \cup B) = g(A) + g(B)$       additive
- $\lambda > 0$        $g(A \cup B) \geq g(A) + g(B)$       super-additive
- $\lambda < 0$        $g(A \cup B) \leq g(A) + g(B)$       sub-additive

# Fuzzy integral

$h : \mathbf{X} \rightarrow [0,1]$      $\Omega$  measurable

fuzzy integral of  $h$  with respect to  $g$  over  $A$

$$\int_A h(x) \circ g() = \sup_{\alpha \in [0,1]} \{ \min[\alpha, g(A \cap H_\alpha)] \}$$

$$H_\alpha = \{x \mid h(x) \geq \alpha\}$$

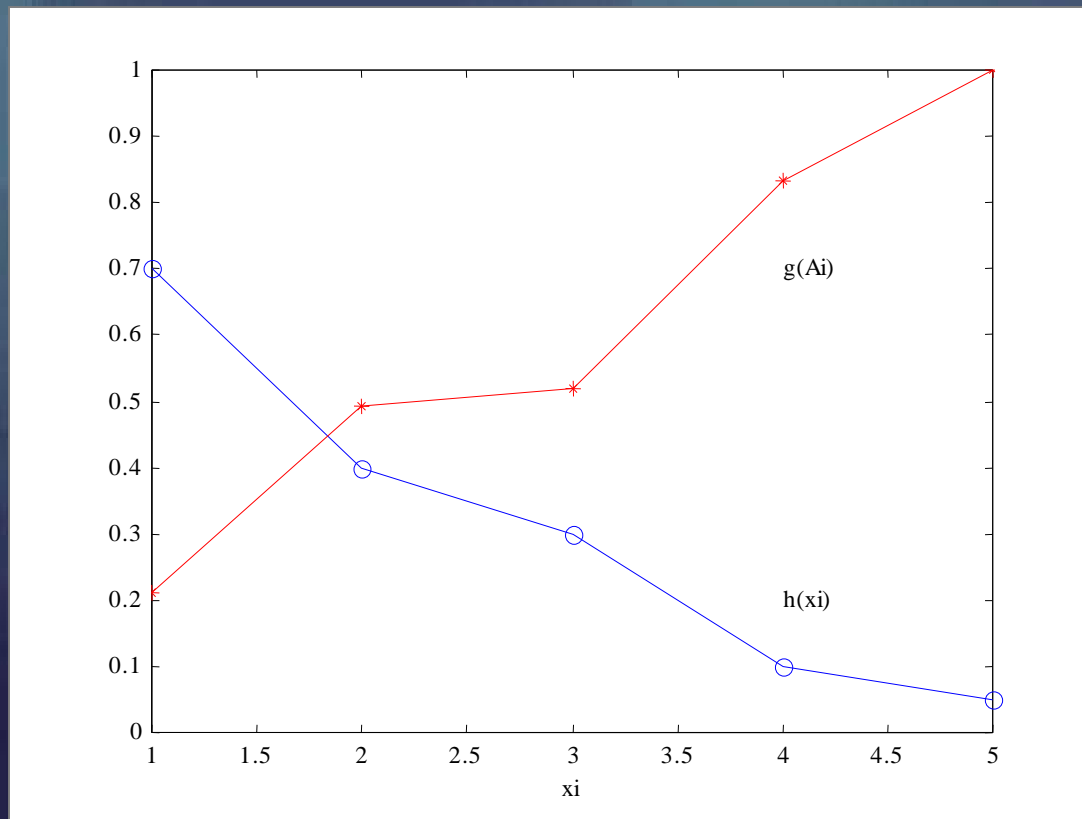
$$\mathbf{X} = \{x_1, x_2, \dots, x_n\}$$

$$h(x_1) \geq h(x_2) \geq \dots \geq h(x_n)$$

$$A_1 = \{x_1\}, A_2 = \{x_1, x_2\}, \dots, A_n = \{x_1, x_2, \dots, x_n\} = \mathbf{X}$$

$$\int_A h(x) \circ g(\cdot) = \max_{i=1, \dots, n} \{\min[h(x_i), g(A_i)]\}$$

# Example



# Choquet integral

$$(Ch) \int f \circ g = \sum_{i=1}^n [h(x_i) - h(x_{i+1})] g(A_i)$$

$$h(x_{n+1}) = 0$$

# 5.8 Negations



# Definition

$$N : [0,1] \rightarrow [0,1]$$

1. Monotonicity:

$N$  is nonincreasing

2. Boundary conditions:

$$N(0) = 1$$

$$N(1) = 0$$

### 3. Continuity:

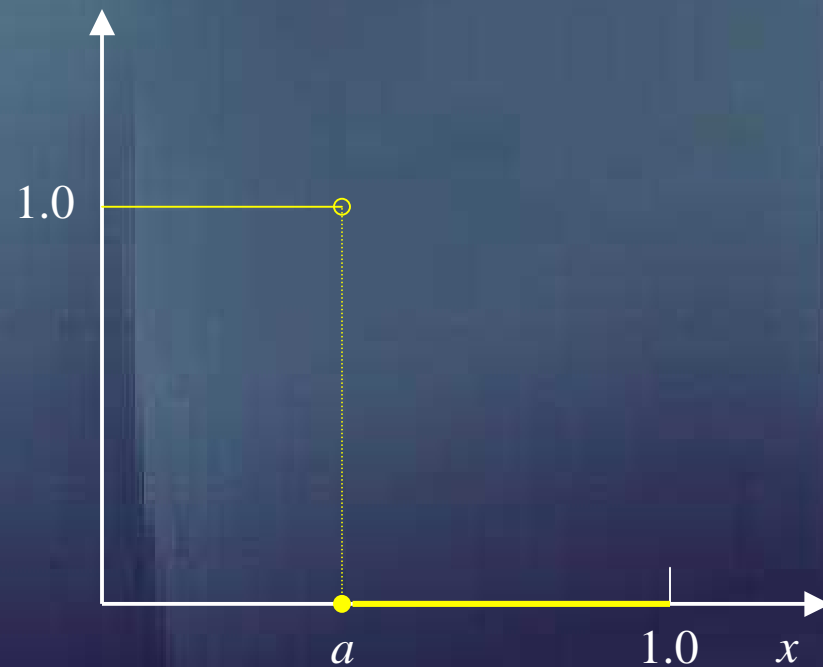
$N$  is a continuous function

### 4. Involution:

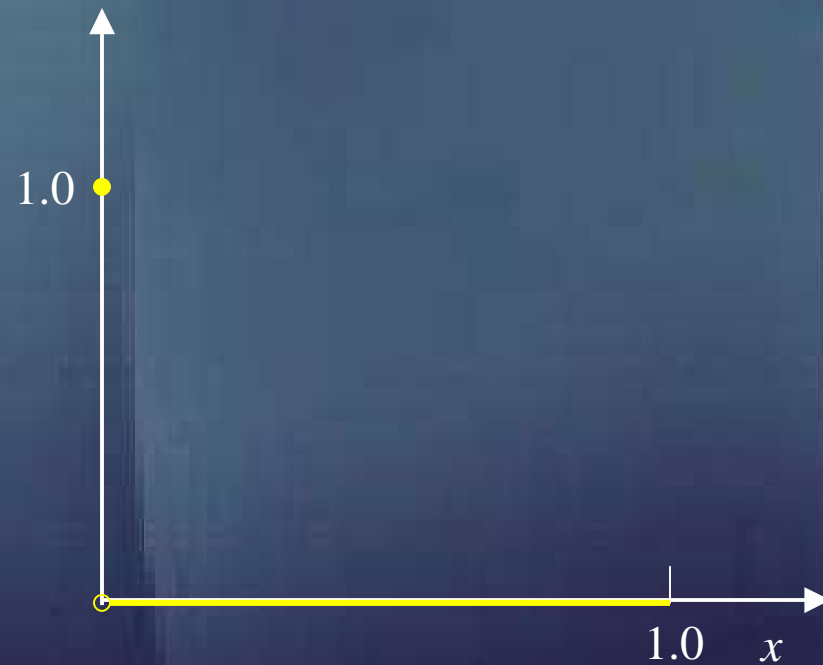
$$N(N(x)) = x \quad \forall x \in [0, 1]$$

# Examples

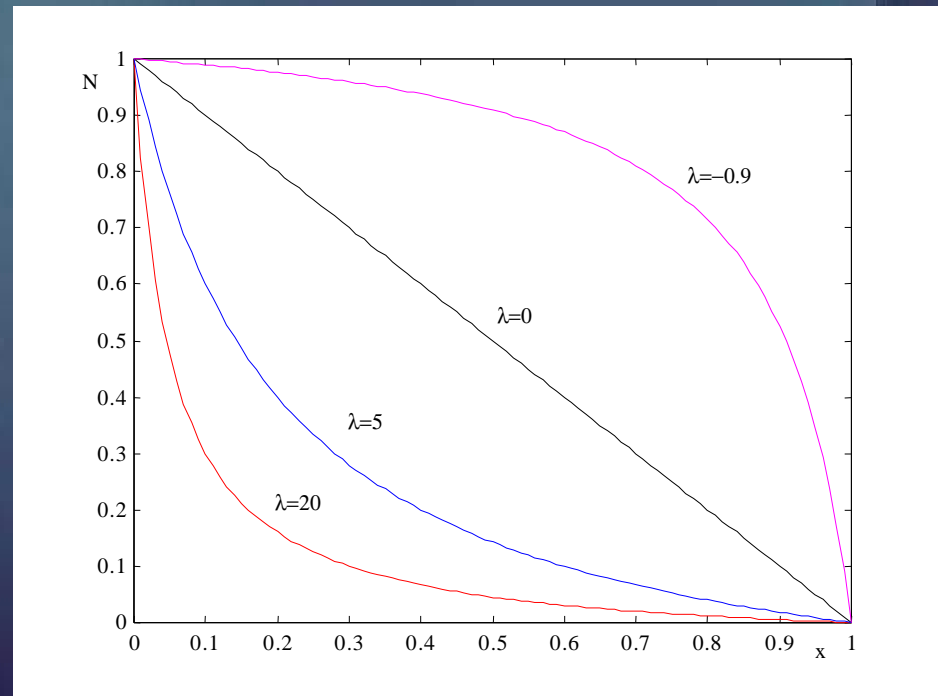
$$N(x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } x \geq a \end{cases}$$



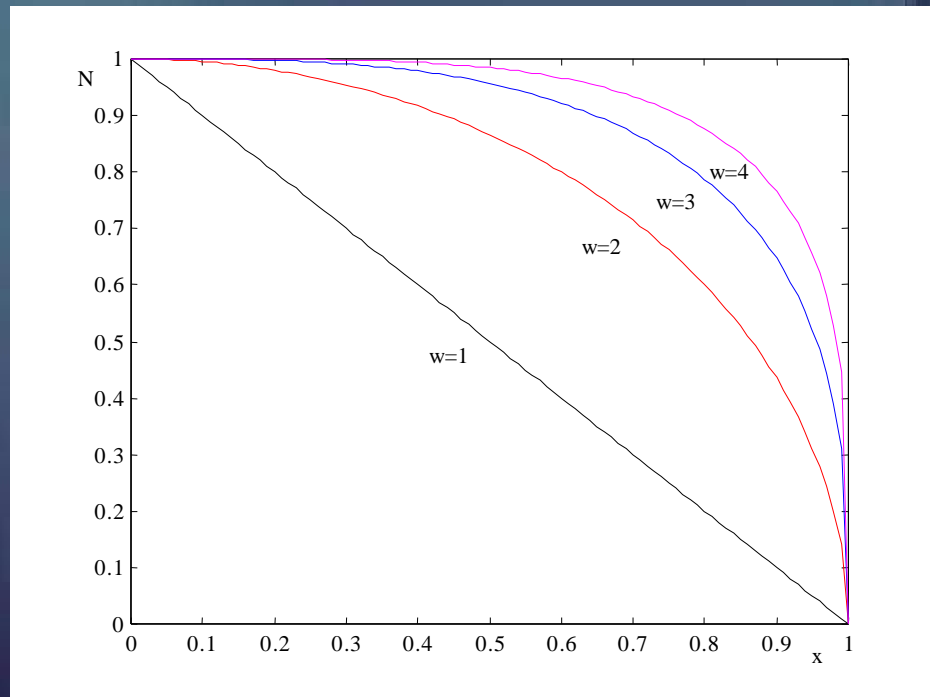
$$N(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$



$$N(x) = \frac{1-x}{1+\lambda x} \quad \lambda \in (-1, \infty)$$



$$N(x) = \sqrt[w]{1-x^w} \quad w \in (0, \infty)$$



## $(t, s, N)$ system

$$x s y = N (N(x) t N(y))$$

$$x t y = N (N(x) s N(y))$$

$$\forall x, y \in [0, 1]$$

# Examples

$$xty = \max\left(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right)$$

$$xsy = \min(1, x + y - 1 + \lambda xy)$$

$$N(x) = \frac{1 - x}{1 + \lambda x} \quad \lambda \in (-1, \infty)$$

$$xty = \min(x, y)$$

$$xsy = \max(x, y)$$

$$N(x) = 1 - x$$