4 Design of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

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4.1 Semantics of fuzzy sets: General observations

Semantics of fuzzy sets

- Generic constructs/building conceptual blocks to describe systems in a meaningful way

- Each fuzzy set comes with a well-delineated semantics (meaning)

  - Example: *small – medium – large* error

- Limited number of fuzzy sets

  - "magic" number of 7 +/- 2 (*Miller, 1956*) (short –term memory)

- Fuzzy sets require calibration
  - determination of their membership functions
- Two main approaches to the problem:
  - Expert –driven (designer, user, decision-maker…)
  - Data driven (from data to fuzzy sets)

4.2 Fuzzy sets as a descriptor of feasible solutions

Fuzzy sets as descriptor of feasible solutions (1)

Consider some function $f(x)$ defined in $\mathbf{\Omega}$,

$$f: \mathbf{\Omega} \rightarrow \mathbf{R}. \text{ where } \mathbf{\Omega} \subset \mathbf{R}$$

Determine its maximum

$$x^{\text{opt}} = \arg \max_x f(x).$$

Fuzzy set A of *optimal* solutions $\equiv$ a collection of feasible solutions that could be labeled as *optimal* with some degrees of membership.

$$A(x) = \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

Fuzzy sets as descriptor of feasible solutions (2)

Consider some function $f(x)$ defined in $\mathbf{\Omega}$,

$$f: \mathbf{\Omega} \rightarrow \mathbf{R}. \text{ where } \mathbf{\Omega} \subset \mathbf{R}$$

Determine its minimum

$$x^{opt} = \arg\max_x f(x).$$

Fuzzy set *A* of *optimal* solutions ≡ a collection of feasible solutions that could be labeled as *optimal* with some degrees of membership.

$$A(x) = 1 - \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

Fuzzy sets as descriptors of feasible solutions

Example

Linearization error

Linearize function $y = g(x) = \exp(-x)$ around $x_0=1$ and assess the quality of this linearization in the range [–1, 7].

Linearization formula: $y - y_0 = g'(x_0)(x - x_0)$

$y_0 = g(x_0)$ and $g'(x_0)$ is the derivative of $g(x)$ at x_0.

Linearized version of the function $\exp(-1)(2 - x)$.

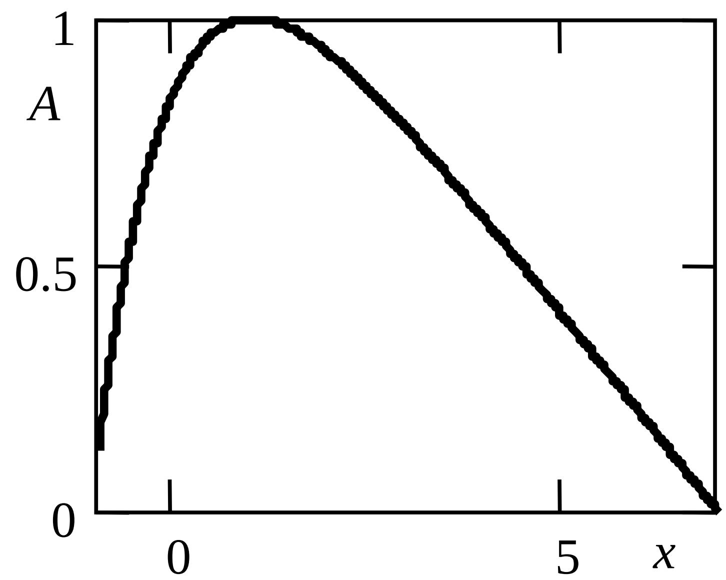
Quality of linearization $f(x) = |g(x) - \exp(-1)(2 - x)|$.

$$A(x) = 1 - \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

$f_{\max} = f(7) = 1.84$ and $f_{\min} = 0.0$

Fuzzy sets as descriptors of feasible solutions

Example



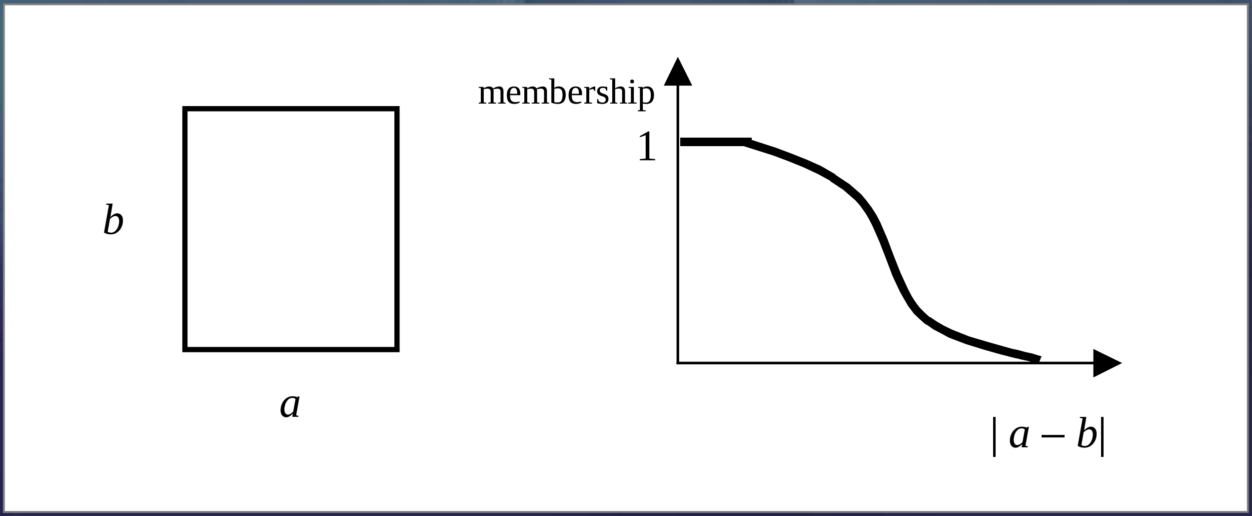
$$A(x) = 1 - \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}$$

4.3 Fuzzy sets as a descriptor of the notion of typicality

Fuzzy sets as notions of typicality

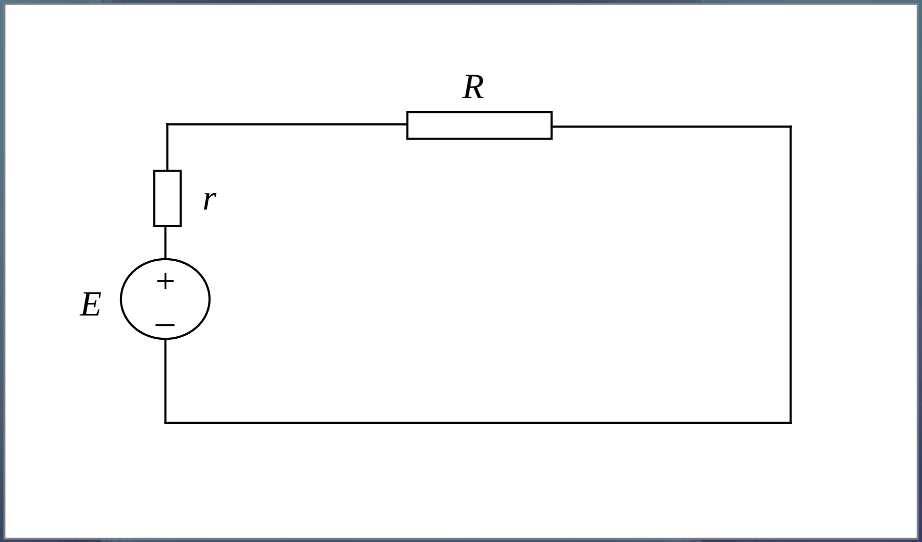
- Fuzzy set as collection of elements of varying degrees of typicality

- Geometric figures : squares, circles….

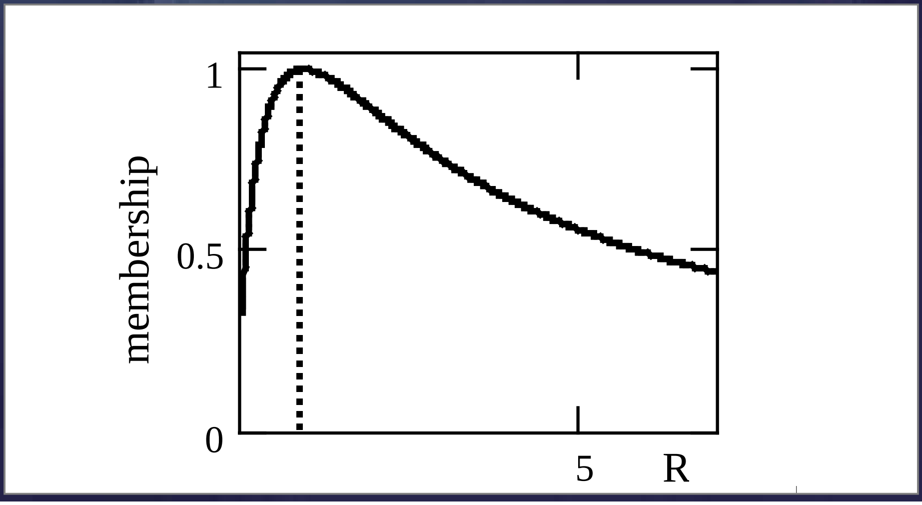


4.4 Membership functions in the visualization of preferences solutions

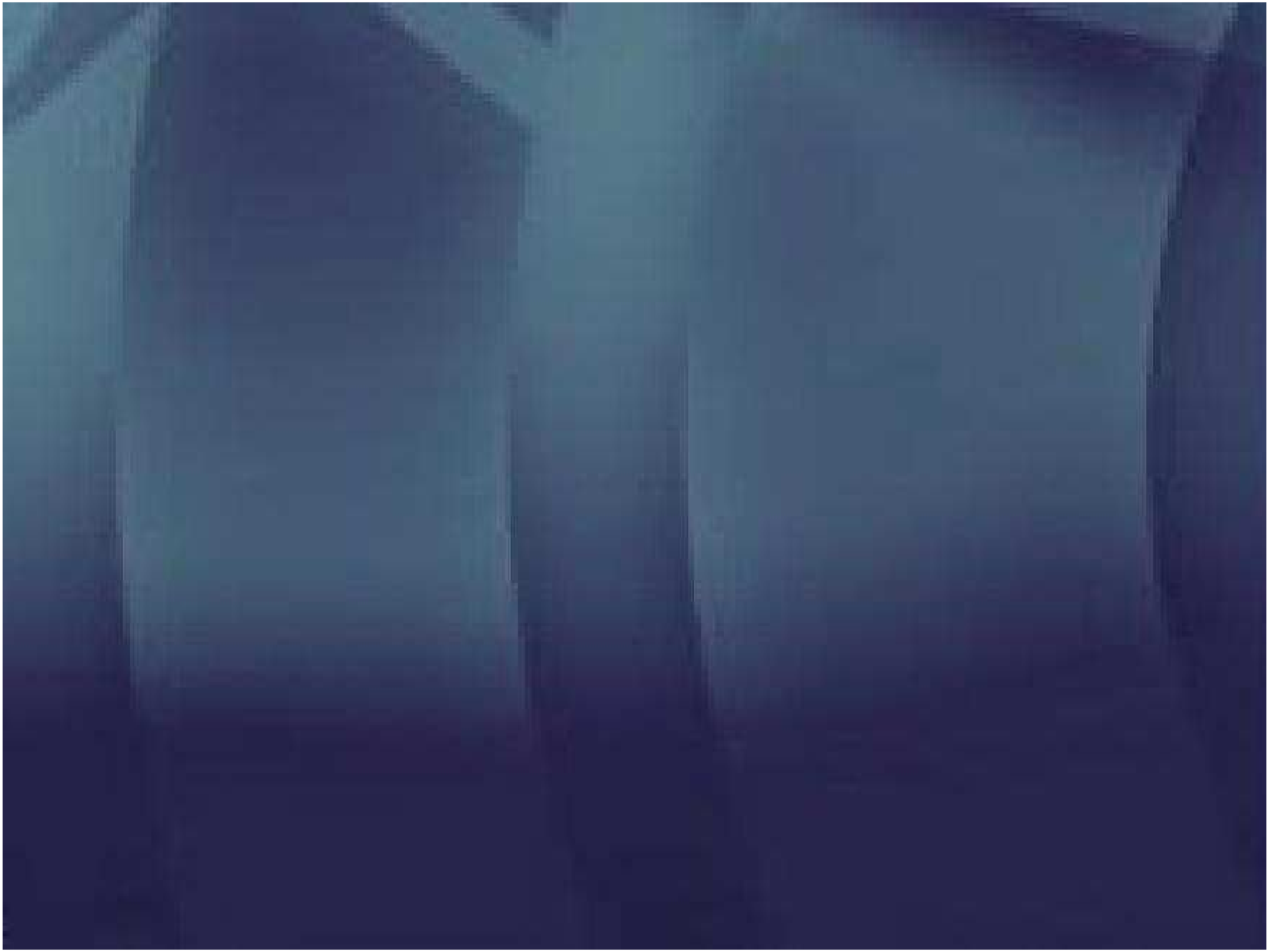
Fuzzy sets in visualization of preferences of solutions



$$P = i^2 R = \left(\frac{E}{R+r} \right)^2 R$$



4.5 Nonlinear transformations of fuzzy sets



4.6 Vertical and horizontal schemes of membership estimation

Horizontal scheme of membership estimation

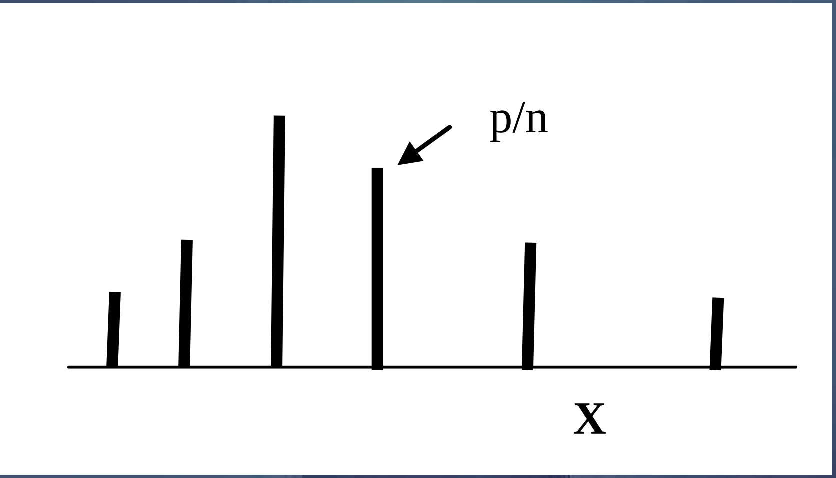
Finite elements of the universe of discourse X
Question of the form

-does x belong to concept A?

Accepted are binary answers (yes-no)

"n" experts – count of positive (yes) answers: p/n

Horizontal scheme of membership estimation



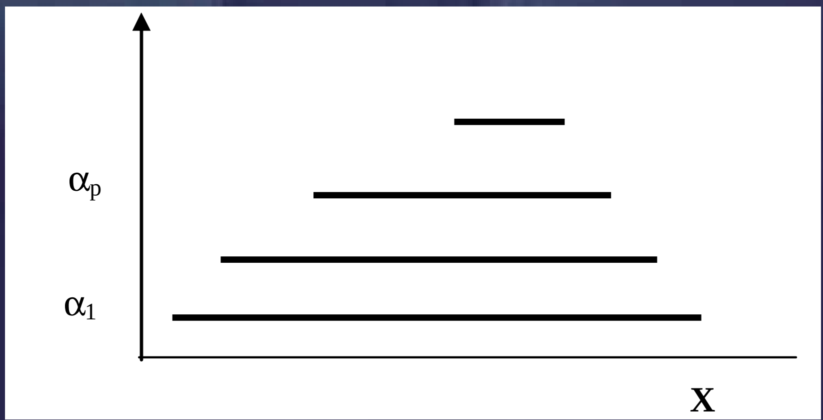
binary replies follow binomial distribution; we can determine confidence interval

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

Vertical scheme of membership estimation

Estimation of membership function by determining α-cuts and aggregating them (see representation theorem)

-What are the elements of **X** which belong to fuzzy set A at degree not lower than α?



Pedrycz and Gomide, FSE 2007

Horizontal and vertical schemes of membership estimation

Simple and intuitively appealing

Reflective of domain knowledge

Lack of continuity – elements of **X** considered independently

Saaty's priority method of pairwise comparison

Collection of elements $x_1, x_2, \ldots, x_n$

Membership degrees are given $A(x_1), A(x_2) \ldots. A(x_n)$

Reciprocal matrix R

$$R = [r_{ij}] = \begin{bmatrix} \frac{A(x_1)}{A(x_1)} & \frac{A(x_1)}{A(x_2)} & \ldots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & \frac{A(x_2)}{A(x_2)} & \ldots & \frac{A(x_2)}{A(x_1)} \\ & \ldots & \ldots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \ldots & \frac{A(x_n)}{A(x_n)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{A(x_1)}{A(x_2)} & \ldots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & 1 & \ldots & \frac{A(x_2)}{A(x_1)} \\ & & \ldots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \ldots & 1 \end{bmatrix}$$

Saaty's priority method of pairwise comparison

$$R = [r_{ij}] = \begin{bmatrix} \frac{A(x_1)}{A(x_1)} & \frac{A(x_1)}{A(x_2)} & \dots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & \frac{A(x_2)}{A(x_2)} & \dots & \frac{A(x_2)}{A(x_1)} \\ & \dots & \dots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \dots & \frac{A(x_n)}{A(x_n)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{A(x_1)}{A(x_2)} & \dots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & 1 & \dots & \frac{A(x_2)}{A(x_1)} \\ & & \dots & \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_n)} & \dots & 1 \end{bmatrix}$$

Reciprocal matrix R –main properties:

(a) reflexivity

(b) reciprocality

(c) transitivity

Saaty's priority method of pairwise comparison: computing

$$\left[\frac{A(x_i)}{A(x_1)}\ \frac{A(x_i)}{A(x_2)} \cdots \frac{A(x_i)}{A(x_n)}\right]\begin{bmatrix} A(x_1) \\ A(x_2) \\ \cdots \\ A(x_n) \end{bmatrix}$$

i-th row of R

RA = nA

n-the largest eigenvalue of R

Saaty's priority method of pairwise comparison

Estimation of reciprocal matrix:

Scale (typically 1-7 range, could be larger, 1-9)

- strong preference: high values on the scale (7-9)
- preference: 4-7
- weak preference or no preference 1-3

Solving the eigenvalue problem for R, max eigenvalue, λ_{max}

Saaty's priority method : consistency of results

$\nu = (\lambda_{max} - n)/(n-1)$

lack of consistency $\nu > 0.1$

Saaty's priority method : Example

high temperature

Universe of discourse: 10, 20, 30, 40, 45

Scale 1-5

$$R = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/5 \\ 2 & 1 & 1/3 & 1/4 \\ 4 & 3 & 1 & 1/3 \\ 5 & 4 & 3 & 1 \end{bmatrix}$$

max eigenvalue = 4.114 eigenvector [0.122 0.195 0.438 0.869]

after normalization [0.14 0.22 0.50 1.00].

Fuzzy sets as granular representation of numeric data

The principle of justifiable granularity

experiment-driven and intuitively appealing rationale:

(a) we expect that A reflects (or matches) the available experimental data to the highest extent, and

(b) second, the fuzzy set is kept specific enough so that it comes with a well-defined semantics.

The principle of justifiable granularity

(a) we expect that A reflects (or matches) the available experimental data to the highest extent, and

Maximize "coverage" of data

(b) second, the fuzzy set is kept specific enough so that it comes with a well-defined semantics.

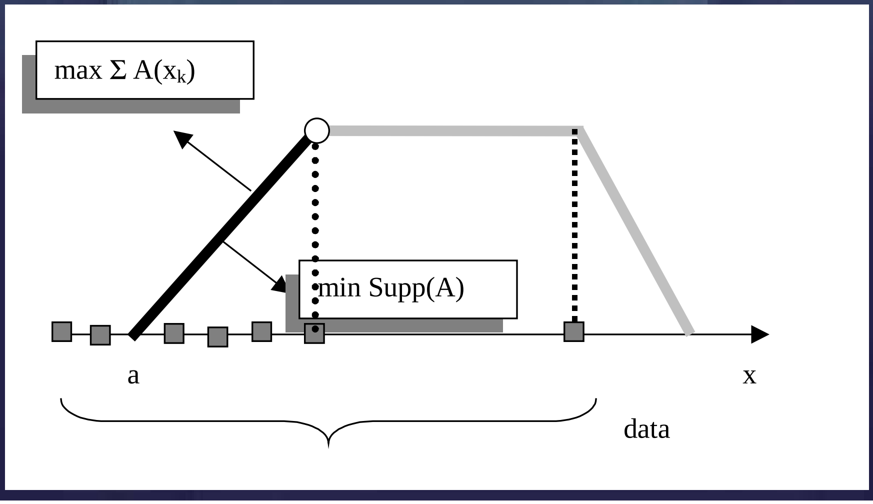
Minimize spread of fuzzy set

The principle of justifiable granularity: unimodal fuzzy set

Numeric data $x_1, x_2, \ldots, x_n$

Determine its "modified" median

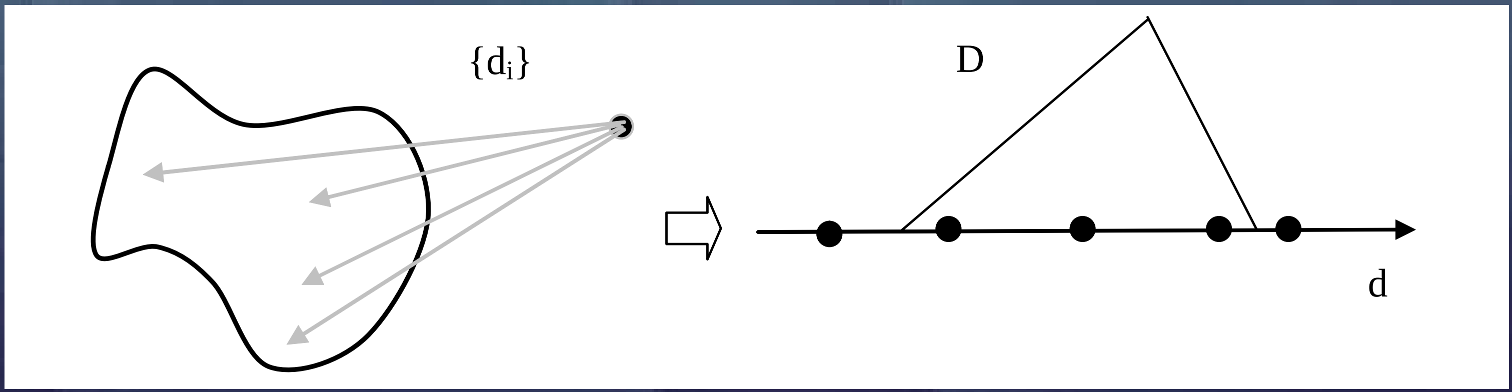
Consider separately data to the left and right from the median



$$\max_{a \neq m} \frac{\sum_k A(x_k)}{|m-a|}$$

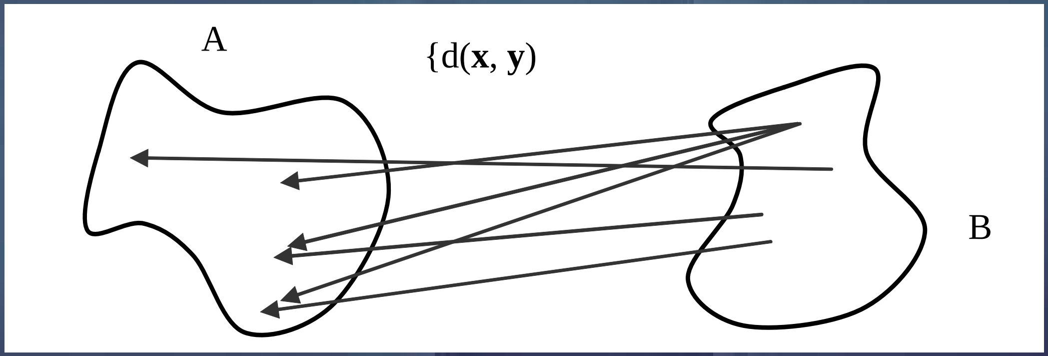
The principle of justifiable granularity: examples

Distance of point from geometric figure



The principle of justifiable granularity: examples

Distance between two geometric figures A and B



$$d_H(A, B) = \max\{\sup_{\mathbf{x}\in A}[\min_{\mathbf{y}\in B} d(\mathbf{x}, \mathbf{y})], \sup_{\mathbf{y}\in B}[\min_{\mathbf{x}\in A} d(\mathbf{x}, \mathbf{y})]\}$$

Clustering: Fuzzy C-Means (FCM)

Given a collection of n-dimensional data set $\{\mathbf{x}_k\}$, k=1,2,…,N,

determine its structure – a collection of “c” clusters.

Minimize the following objective function (performance index) Q

$$Q = \sum_{i=1}^{c} \sum_{k=1}^{N} u_{ik}^{m} \| \mathbf{x}_k - \mathbf{v}_i \|^2$$

Fuzzy clustering: structure representation

Partition matrix U

Prototypes $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c$

$$\sum_{i=1}^{c} u_{ik} = 1, \quad k = 1,2,\ldots, N$$

$$0 < \sum_{k=1}^{N} u_{ik} < N, \quad i = 1,2,\ldots, c$$

FCM – optimization procedure

Optimization with respect to

- Partition matrix U, and
- Prototypes $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c$

Optimization: partition matrix

use of Lagrange multipliers

$$V = \sum_{i=1}^{c} u_{ik}^{m} d_{ik}^{2} + \lambda\left(\sum_{i=1}^{c} u_{ik} - 1\right)$$

$$\frac{\partial V}{\partial u_{st}} = 0 \qquad \frac{\partial V}{\partial \lambda} = 0$$

Optimization: partition matrix

$$V = \sum_{i=1}^{c} u_{ik}^{m} d_{ik}^{2} + \lambda\left(\sum_{i=1}^{c} u_{ik} - 1\right)$$

$$\frac{\partial V}{\partial u_{st}} = 0 \qquad \frac{\partial V}{\partial \lambda} = 0$$

$$\frac{\partial V}{\partial u_{st}} = m u_{st}^{m-1} d_{st}^{2} + \lambda$$

$$u_{st} = -\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} d_{st}^{\frac{2}{m-1}}$$

$$-\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} \sum_{j=1}^{c} d_{jt}^{\frac{2}{m-1}} = 1$$

$$-\left(\frac{\lambda}{m}\right)^{\frac{1}{m-1}} = \frac{1}{\sum_{j=1}^{c} d_{jt}^{\frac{2}{m-1}}}$$

$$u_{st} = \frac{1}{\sum_{j=1}^{c} \left(\frac{d_{st}^{2}}{d_{jt}^{2}}\right)^{\frac{1}{m-1}}}$$

Optimization: prototypes

$$Q = \sum_{i=1}^{c} \sum_{k=1}^{N} u_{ik}^{m} \sum_{j=1}^{n} (x_{kj} - v_{ij})^2$$

Gradient of Q w.r.t. prototype $\mathbf{v}_s$

$$\sum_{k=1}^{N} u_{ik}^{m} (x_{kt} - v_{st}) = 0$$

$$v_{st} = \frac{\sum_{k=1}^{N} u_{ik}^{m} x_{kt}}{\sum_{k=1}^{N} u_{ik}^{m}}$$

FCM: an overview of the algorithm

procedure FCM-CLUSTERING ($\mathbf{x}$) **returns** prototypes and partition matrix

input : data $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}$

local: fuzzification parameter: m

threshold: ε

norm: ||.||

INITIALIZE-PARTITION-MATRIX

$t \leftarrow 0$

repeat

for i=1:c **do**

$$\mathbf{v}_i(t) \leftarrow \frac{\sum_{k=1}^{N} u_{ik}^m(t)\mathbf{x}_k}{\sum_{k=1}^{N} u_{ik}^m(t)} \quad \text{compute prototypes}$$

for i = 1:c **do**

for k = 1:N **do**

update partition matrix

$$u_{ik}(t+1) = \frac{1}{\sum_{j=1}^{c}\left(\frac{\|\mathbf{x}_k - \mathbf{v}_i(t)\|}{\|\mathbf{x}_k - \mathbf{v}_j(t)\|}\right)^{2/(m-1)}} \quad \text{update partition matrix}$$

$t \leftarrow t + 1$

until $\|U(t+1)-U(t)\| \leq \varepsilon$

return U, V

FCM and its parameters

Number of clusters (c)

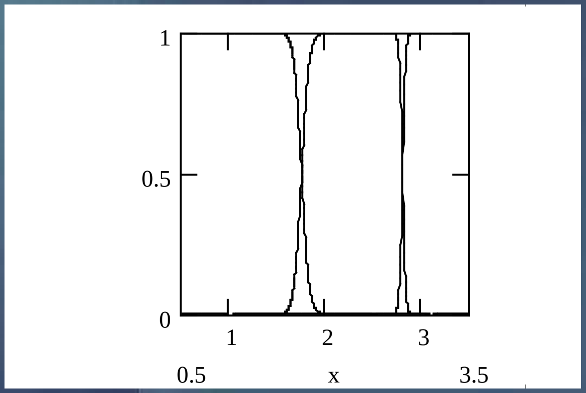
Objective function Q

Distance function ||.||

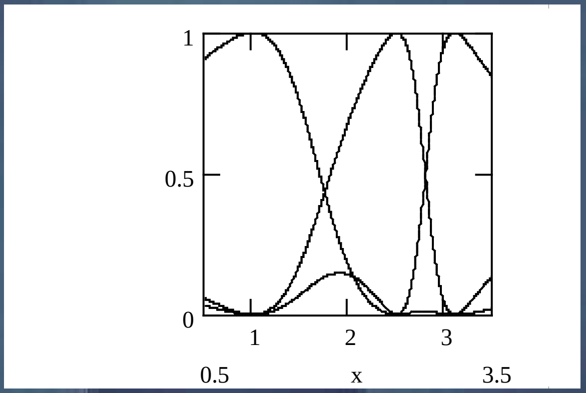
Fuzzification coefficient (m)

Termination criterion

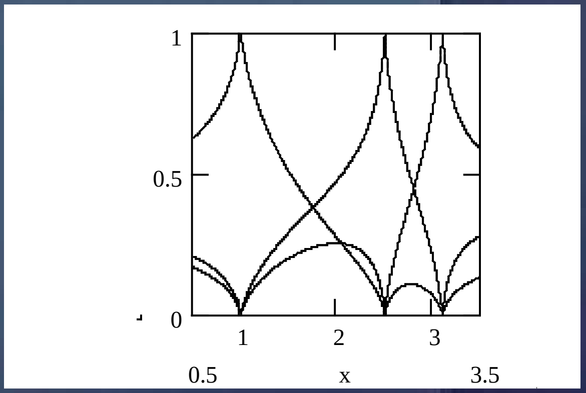
Geometry of clusters and fuzzification coefficient (m)



m =1.2



m =2.0



m =3.5

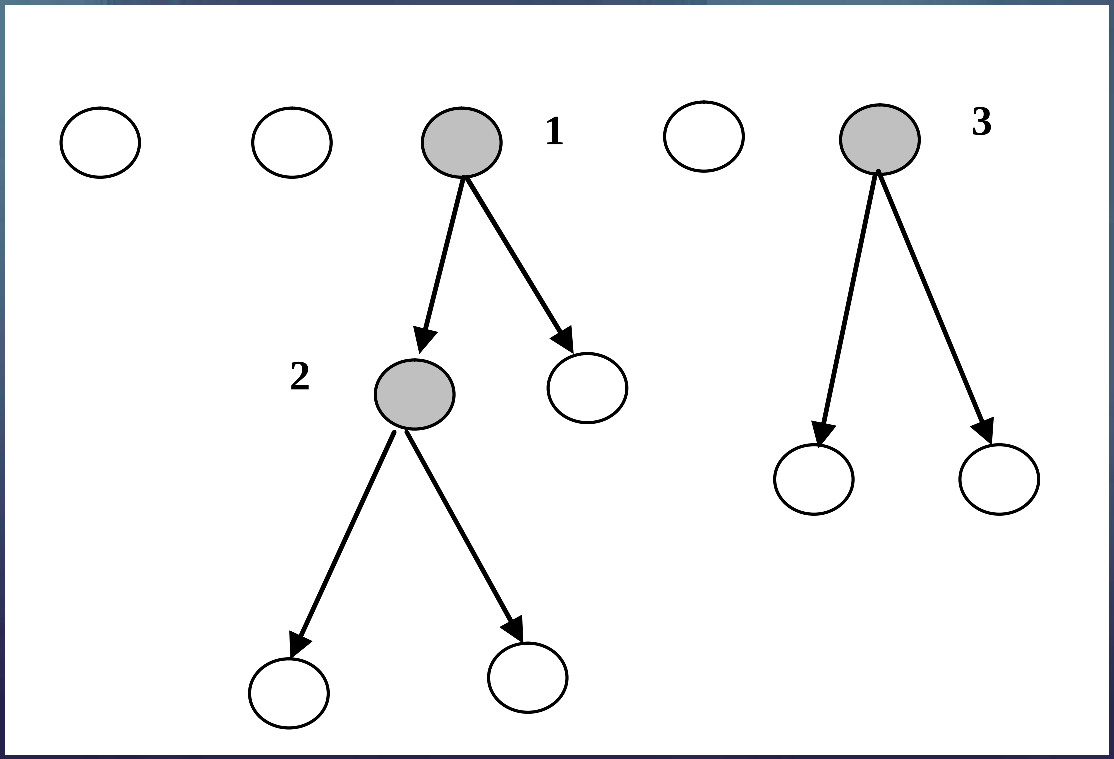
Cluster sharing: a separation measure

$$\varphi(u_1, u_2, \ldots, u_c) = 1 - c^c \prod_{i=1}^{c} u_i$$

Data fully belongs to a single cluster (1- 0)

Data belongs to all clusters at the same level (1/c)

Hierarchical format of FCM: Successive refinements of clusters

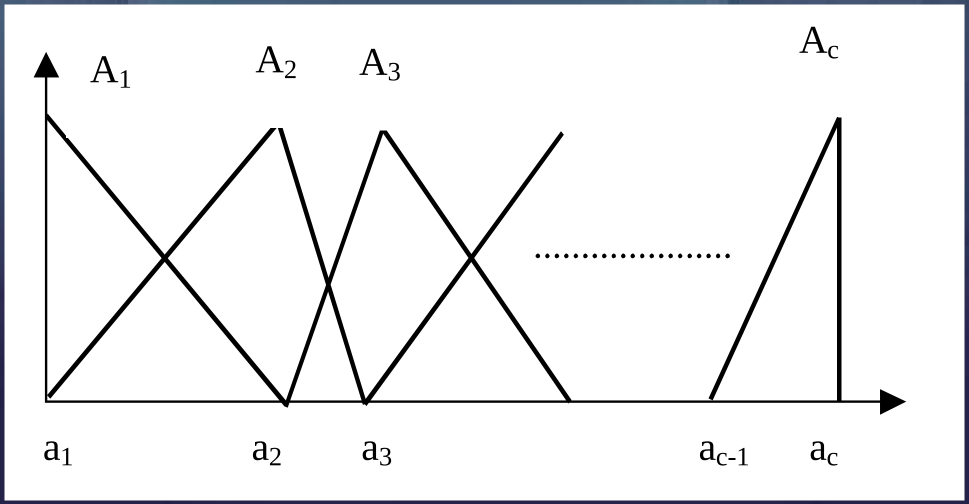


$$V_i = \sum_{k=1}^{N} u_{ik}^{m} \| \mathbf{x}_k - \mathbf{v}_i \|^2$$

$$\mathbf{X}(i_0) = \{ x_k \in \mathbf{X} | \; u_{i_0 k} = \max_i u_{ik} \}$$

Fuzzy equalization

Construct triangular fuzzy sets $A_1, A_2, \ldots, A_c$ defined in **R** such that they come with the same level of experimental evidence (support)



$$\sum_{k=1_k}^{N} A_1(x_k) = \frac{N}{2(c-1)}$$

$$\sum_{k=1_k}^{N} A_2(x_k) = \frac{N}{(c-1)}$$

$$\sum_{k=1_k}^{N} A_{c-1}(x_k) = \frac{N}{(c-1)}$$

$$\sum_{k=1_k}^{N} A_c(x_k) = \frac{N}{2(c-1)}$$

Linguistic approximation

Given is a family of reference fuzzy sets $\{A_i\}$ defined in some space **X**

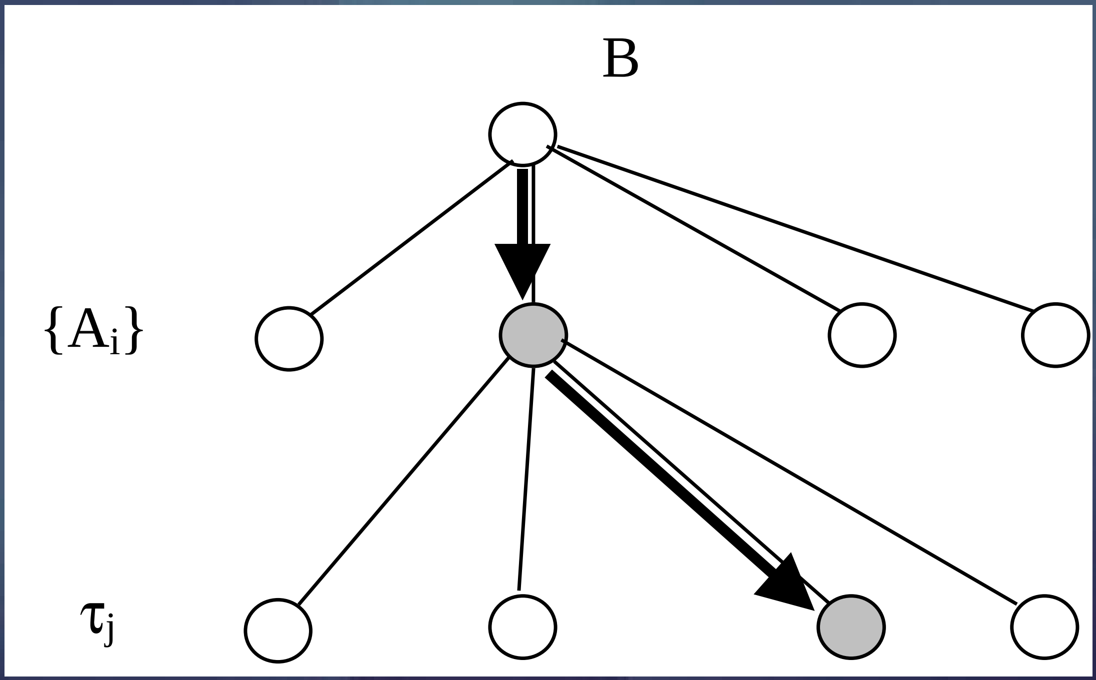
We have at disposal is a family of linguistic modifiers τ_j, say

more or less (dilution),

very (concentration)

Represent (approximate) B in **X** with the use of reference fuzzy sets and linguistic modifiers == linguistic approximation

Linguistic approximation: optimization



$$B \approx \tau_i(A_j)$$

Construction of fuzzy sets: Design guidelines (1)

Strive for highly visible and well-defined semantics of information granules. Keep the number of information granules quite low (7 $\pm$ 2 fuzzy sets).

There are several fundamental views at fuzzy sets and depending upon them, consider the use of various estimation techniques.

Fuzzy sets are context-sensitive constructs and require careful calibration. This The calibration mechanisms being used in the design of the membership function are reflective of human-centricity of fuzzy sets.

Construction of fuzzy sets: Design guidelines (2)

two major categories of approaches supporting the design of membership functions:

data-driven and expert (user)-driven.

The user-driven membership estimation uses the statistics of data yet in an *implicit* manner.

The granular term – fuzzy sets come into existence once there is some experimental evidence behind them

the development of fuzzy sets can be carried out in an stepwise manner.