3 Characterization of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing

Contents

3.1 Generic characterization of fuzzy sets: Some fundamental descriptors

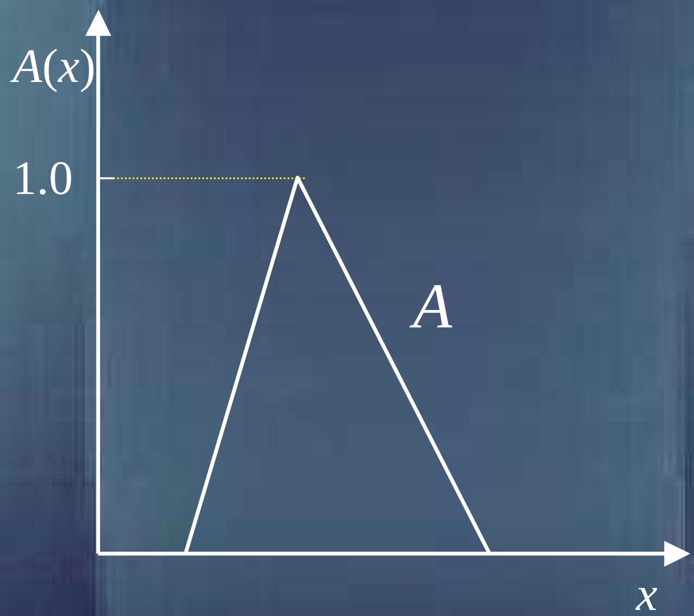
Fuzzy sets

- Fuzzy sets are membership functions $\boxed{A: \mathbf{X} \rightarrow [0, 1]}$

- In principle: any function is "eligible" to describe fuzzy sets

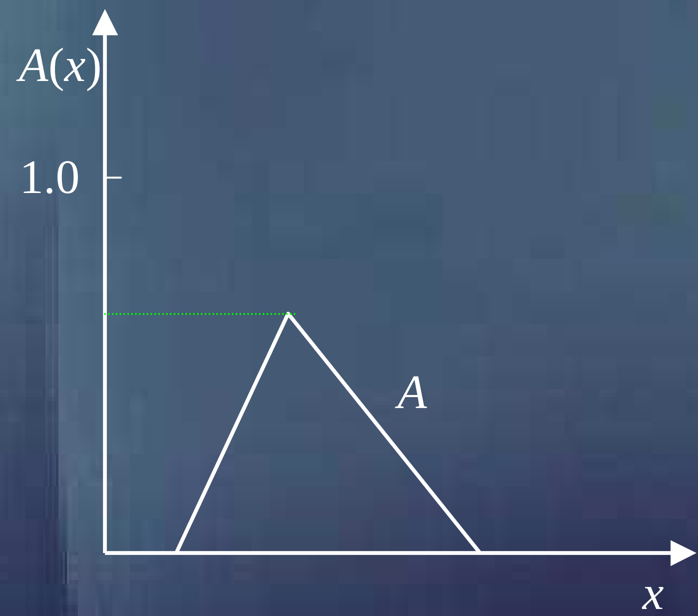
- In practice it is important to consider:
  - type, shape, and properties of the function
  - nature of the underlying phenomena
  - semantic soundness

Normality



Normal

$\text{hgt}(A) = 1$

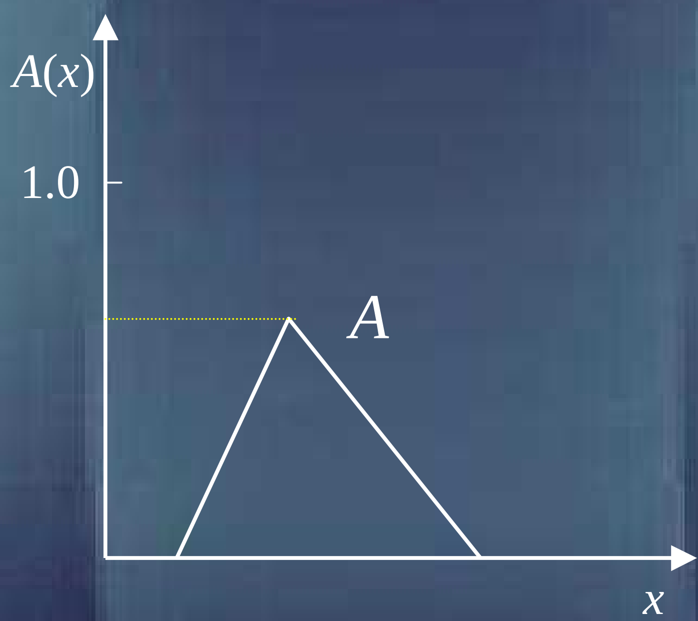


Subnormal

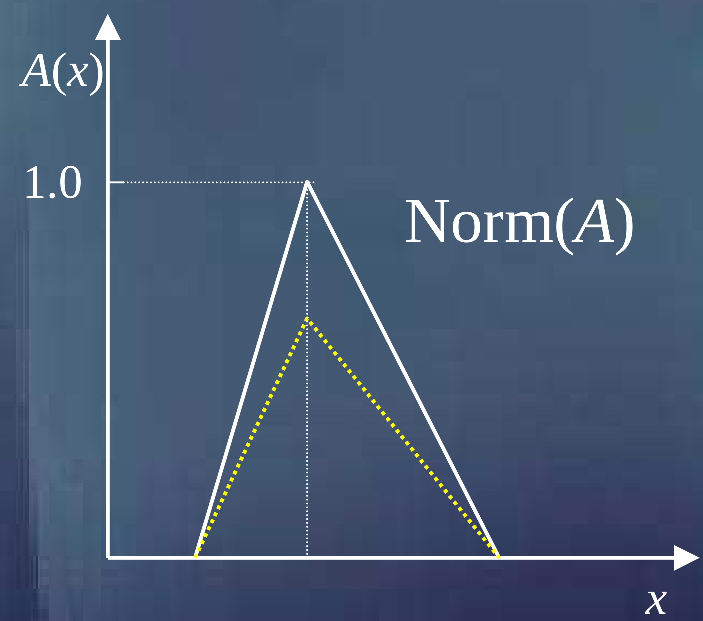
$\text{hgt}(A) < 1$

$$\text{hgt}(A) = \sup_{x \in \mathbf{X}} A(x)$$

Normalization



Subnormal



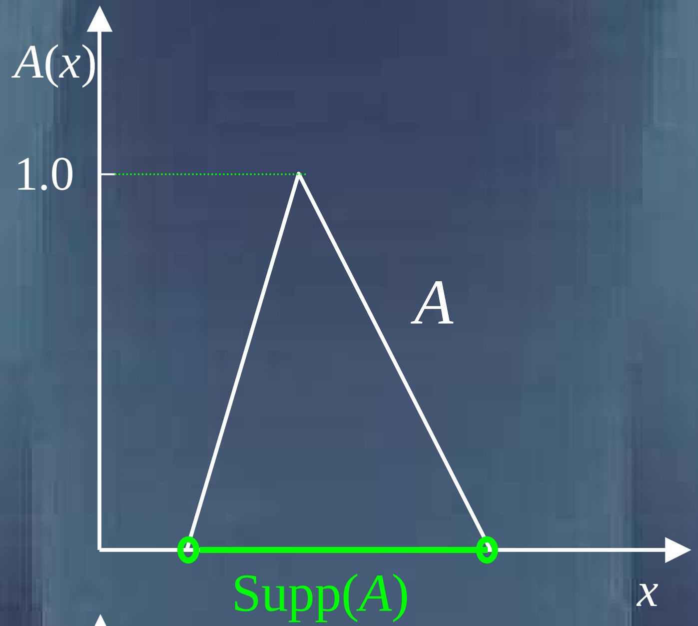
Normal

$\mathrm{hgt}(A) < 1$

$\mathrm{hgt}(\mathrm{Norm}(A)) = 1$

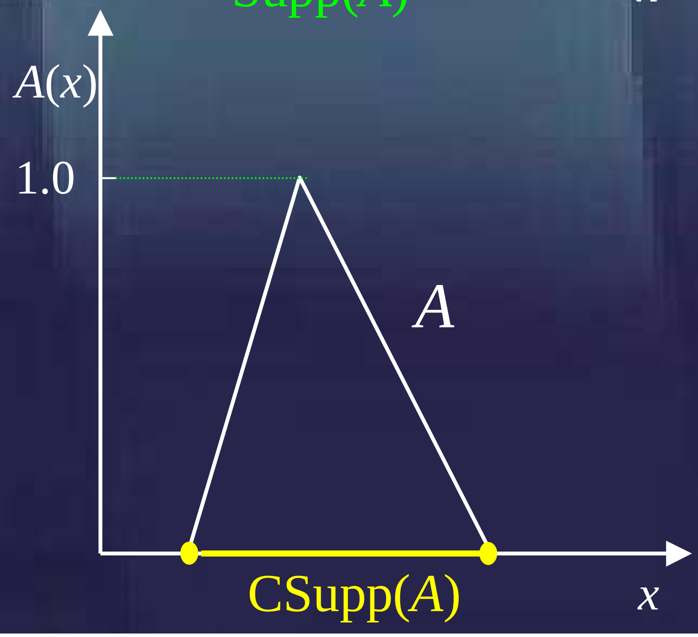
$$\boxed{\mathrm{Norm}(A)(x) = \frac{A(x)}{\mathrm{hgt}(A)}}$$

Support



$$\text{Supp}(A) = \{x \in \mathbf{X} \mid A(x) > 0\}$$

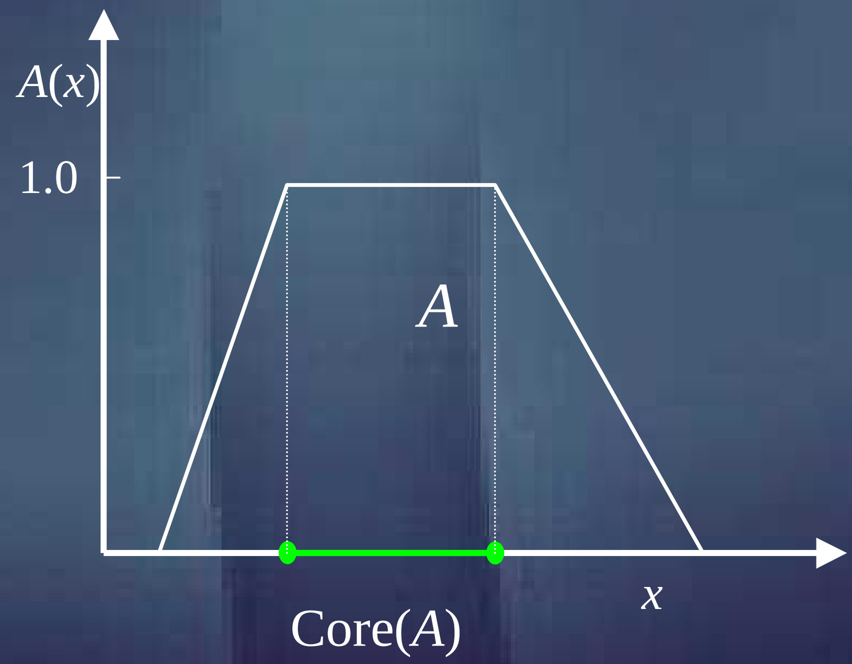
Open set



$$\text{CSupp}(A) = closure\{x \in \mathbf{X} \mid A(x) > 0\}$$

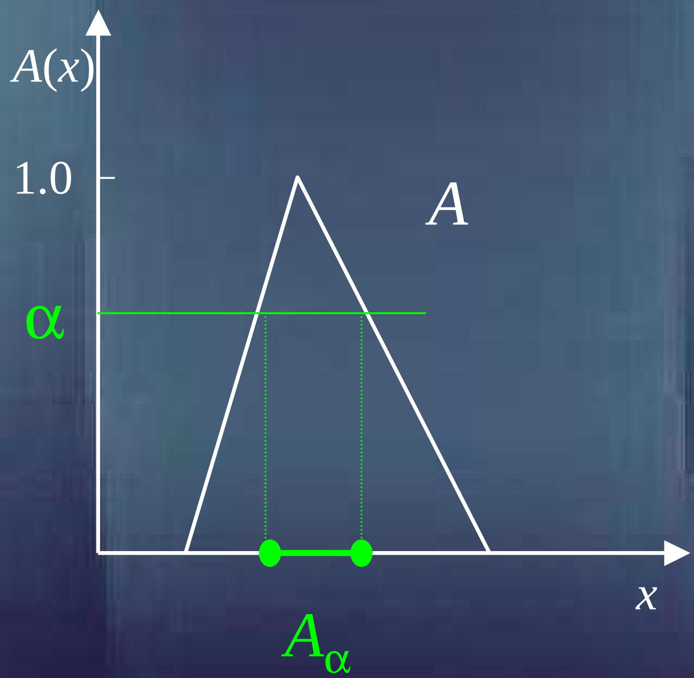
Closed set

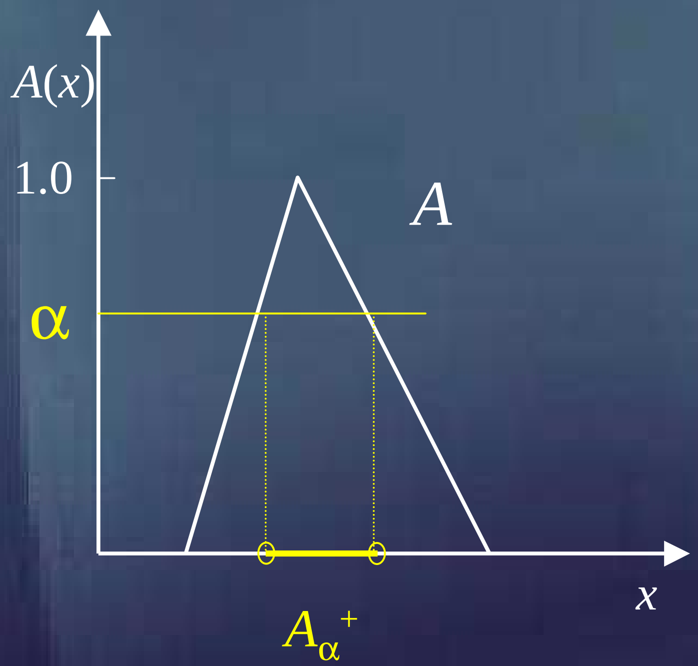
Core



$$\mathrm{Core}(A) = \{x \in \mathbf{X} \mid A(x) = 1\}$$

α–cut





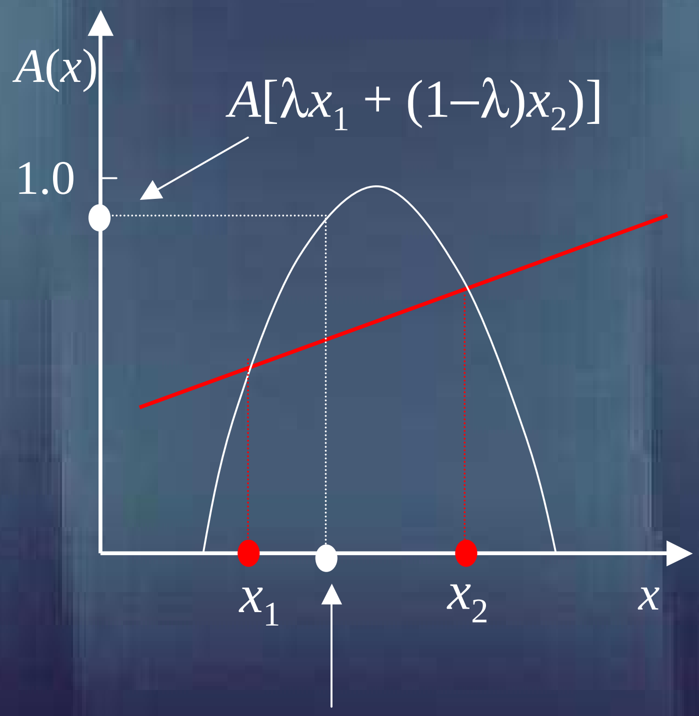
$$A_\alpha = \{x \in \mathbf{X} \mid A(x) \geq \alpha\}$$

$$A_\alpha = \{x \in \mathbf{X} \mid A(x) > \alpha\}$$

Stronger condition

Convexity

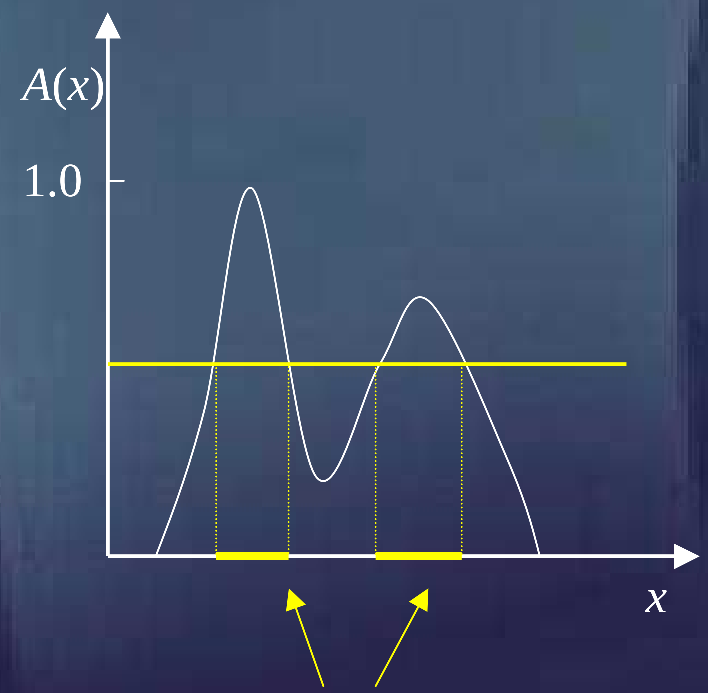
$$A[\lambda x_1 + (1-\lambda)x_2] \geq \min[A(x_1), A(x_2)]$$



$x = \lambda x_1 + (1-\lambda)x_2$

$0 \leq \lambda \leq 1$

Convex fuzzy set



$$A_\alpha = \{x \in \mathbf{X} \mid A(x) > \alpha\}$$

Nonconvex

Cardinality

$$\mathrm{Card}(A) = \sum_{x \in \mathbf{X}} A(x)$$

$\mathbf{X}$ finite or countable

$$\mathrm{Card}(A) = \int_{\mathbf{X}} A(x)\mathrm{d}x$$

$\mathrm{Card}(A) = |A|$ sigma count (σ–count)

3.2 Equality and inclusion relationships for fuzzy sets

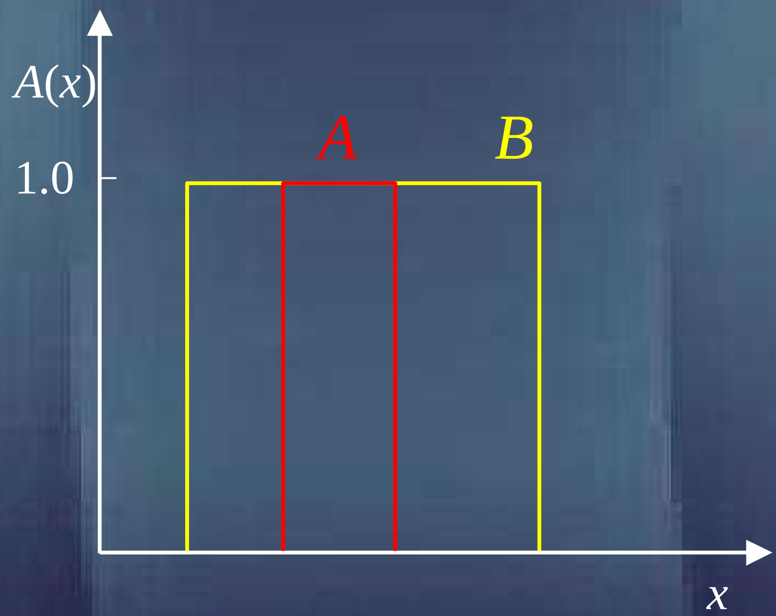
Equality

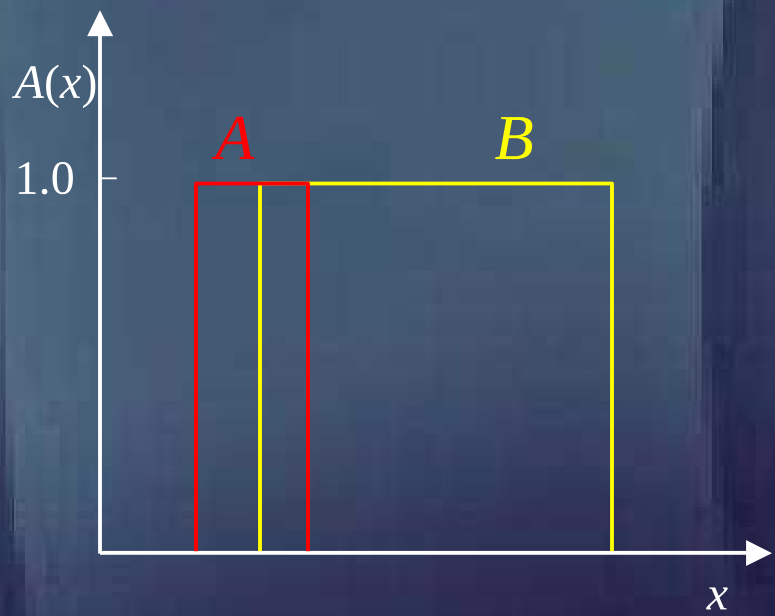
$A = B$ iff $A(x) = B(x) \quad \forall x \in \mathbf{X}$

Inclusion

$A \subseteq B$ iff $A(x) \leq B(x) \quad \forall x \in \mathbf{X}$

Sets





$A \subseteq B$ $\qquad$ $A \not\subset B$

Degree of inclusion

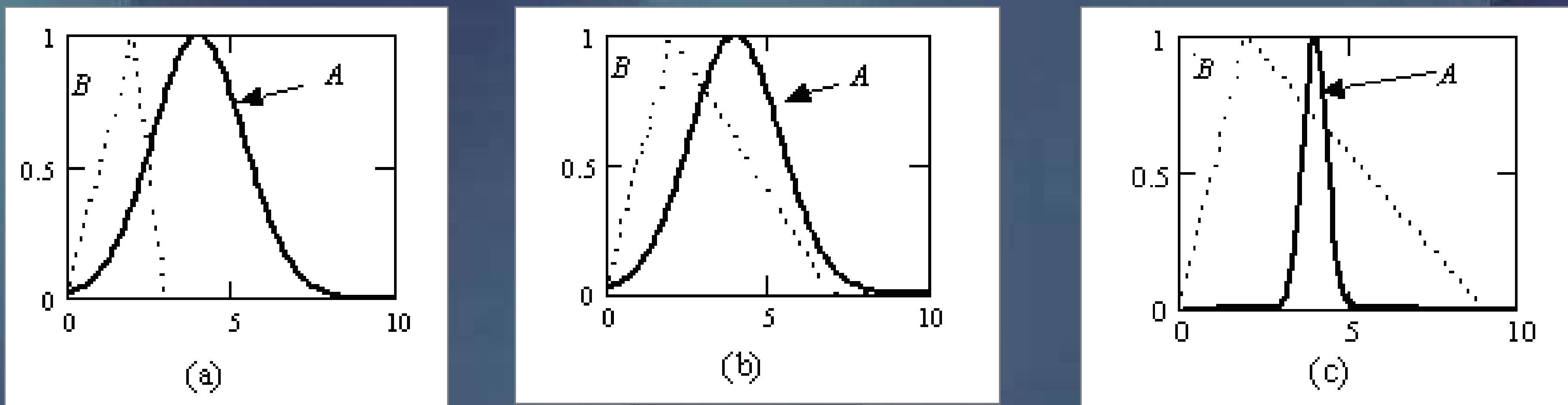
$$\left\| A(x) \subset B(x) \right\| = \frac{1}{\mathrm{Card}\,(\mathbf{X})} \int_X (A(x) \Rightarrow B(x))\mathrm{d}x$$

$$A(x) \Rightarrow B(x) = \begin{cases} 1 & \text{if } A(x) \le B(x) \\ 1\text{-}A(x) + B(x) & \text{otherwise} \end{cases}$$

Degree of equality

$$\left\| A(x) = B(x) \right\| = \frac{1}{\mathrm{Card}(\mathbf{X})} \int_{X} [\min(A(x) \Rightarrow B(x), B(x) \Rightarrow A(x))] \mathrm{d}x$$

Example



Examples of fuzzy sets *A* and *B* along with their degrees of inclusion:

(a) $a = 0, n = 2, b = 3; m = 4\ \sigma = 2; ||A = B|| = 0.637$

(b) $b = 7\quad ||A = B|| = 0.864$

(c) $a = 0, n = 2, b = 9,\ m = 4, \sigma = 0.5\ ||\ A = B\ || = 0.987$

3.3 Energy and entropy measures of fuzziness

Energy measure of fuzziness

$$E(A) = \sum_{i=1}^{n} e[A(x_i)]$$

$$E(A) = \int_{\mathbf{X}} e[A(x)]\mathrm{d}x$$

Card ($\mathbf{X}$) = n

$e : [0, 1] \rightarrow [0, 1]$ such that

$e(0) = 0$

$e(1) = 1$

e: monotonically increasing

Example

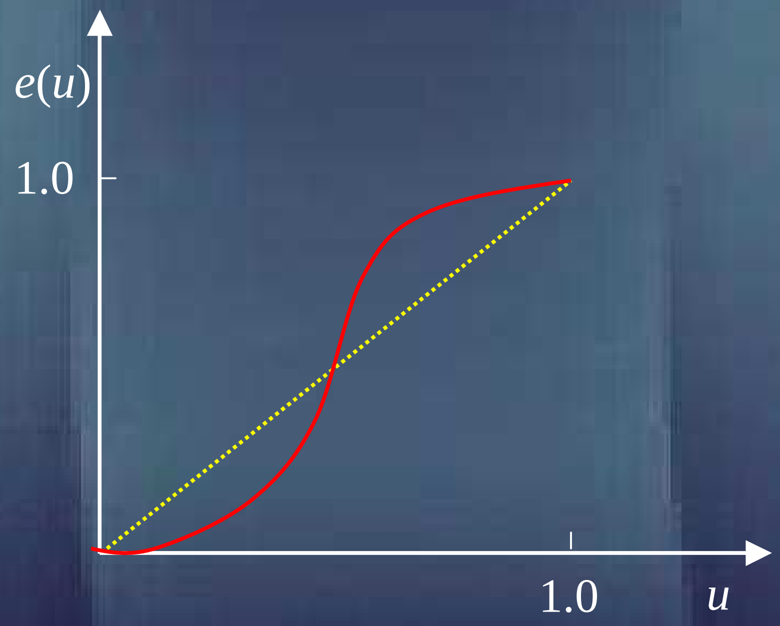
$e(u) = u \quad \forall u \in [0, 1]$ linear

$$E(A) = \sum_{i=1}^{n} A(x_i) = \mathrm{Card}(A)$$

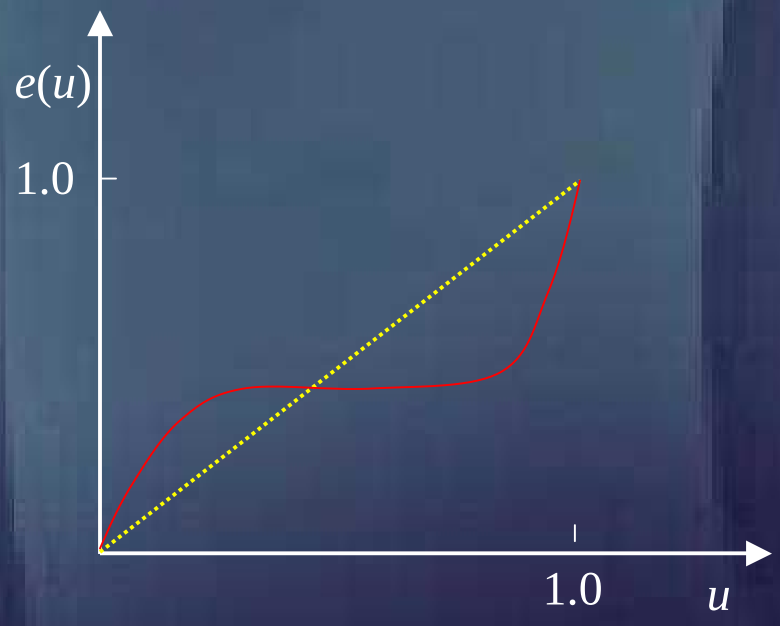
$$E(A) = \sum_{i=1}^{n} A(x_i) = \sum_{i=1}^{n} | A(x_i) - \phi(x_i) | = d(A, \phi)$$

d = Hamming distance

$e(u)$ non-linear



Emphasis on high membership values



Emphasis on low membership values

Inclusion of probabilistic information

$$E(A) = \sum_{i=1}^{n} p_i e[A(x_i)]$$

p_i: probability of x_i

$$E(A) = \int_{\mathbf{X}} p(x) e[A(x)] \mathrm{d}x$$

$p(x)$: probability density function

Entropy measure of fuzziness

$$H(A) = \sum_{i=1}^{n} h[A(x_i)]$$

$$\mathrm{H}(A) = \int_X h(A(x))\mathrm{d}x$$

$h : [0,1] \rightarrow [0,1]$

1-monotonically increasing $[0, \frac{1}{2}]$

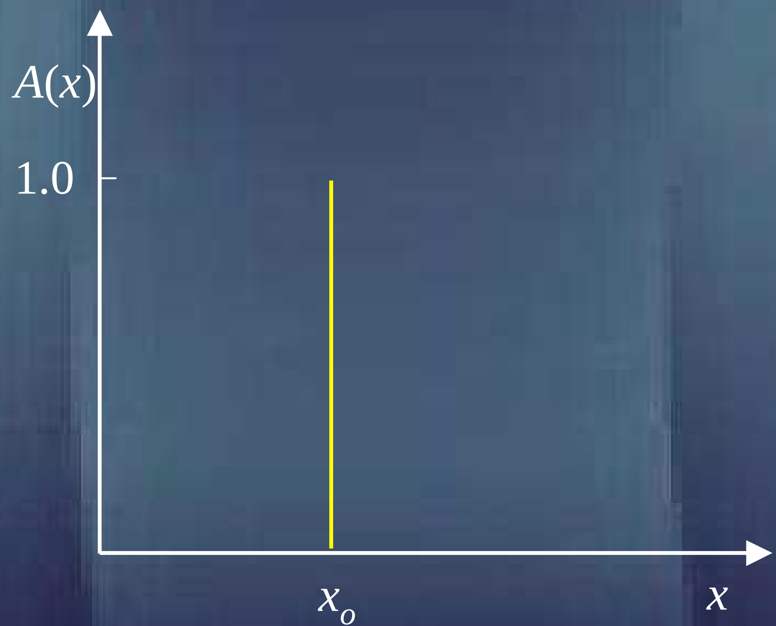
2-monotonically decreasing $(\frac{1}{2}, 1]$

3-boundary conditions:

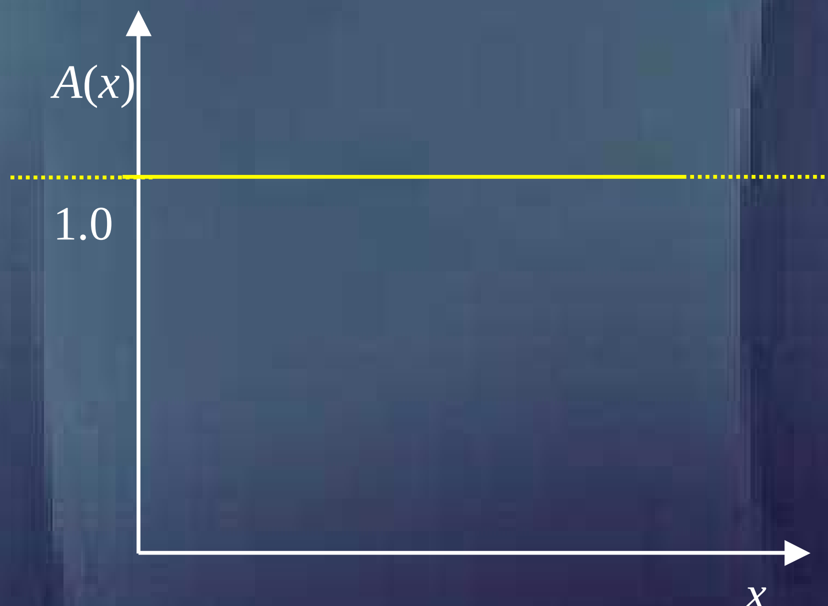
$h(0) = h(1) = 0$

$h(\frac{1}{2}) = 1$

Specificity of fuzzy sets



Specific fuzzy set



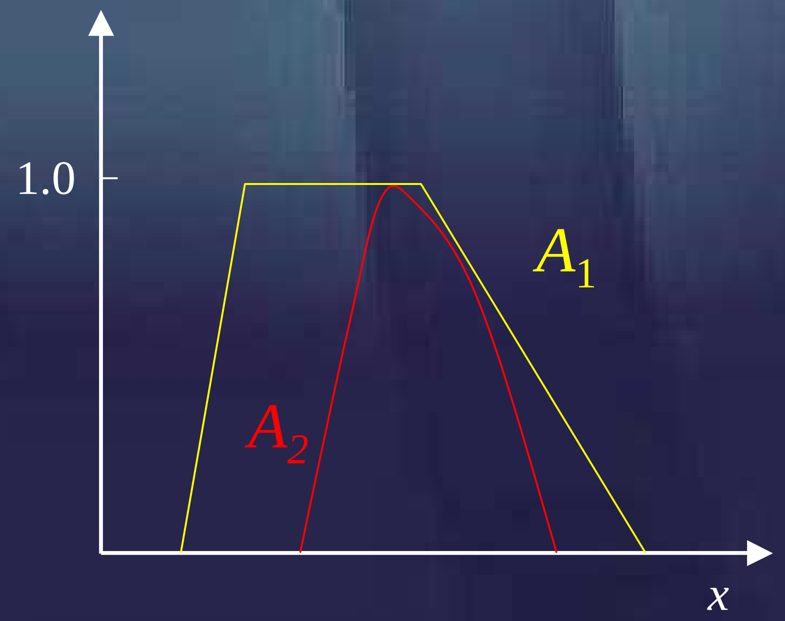
Lack of specificity

Specificity

1-Spec(A) = 1 if and only if $\exists x_0 \in A(x_0) = 1$, $A(x) = 0 \quad \forall x \neq x_0$

2-Spec(A) = 0 if and only if $A(x) = 0 \; \forall x \in \mathbf{X}$

3-Spec(A_1) $\leq$ Spec(A_2) if $A_1 \supset A_2$



Examples

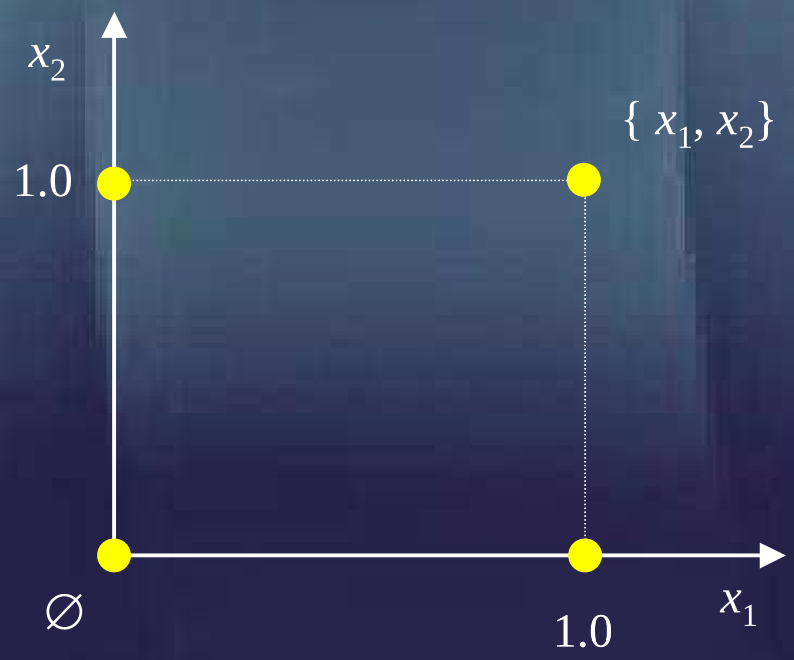
$$Spec(A) = \int_0^{\alpha_{\max}} \frac{1}{Card(A_\alpha)} \mathrm{d}\alpha$$

$$Spec(A) = \sum_{i=1}^{m} \frac{1}{Card(A_{\alpha_i})} \Delta\alpha_i$$

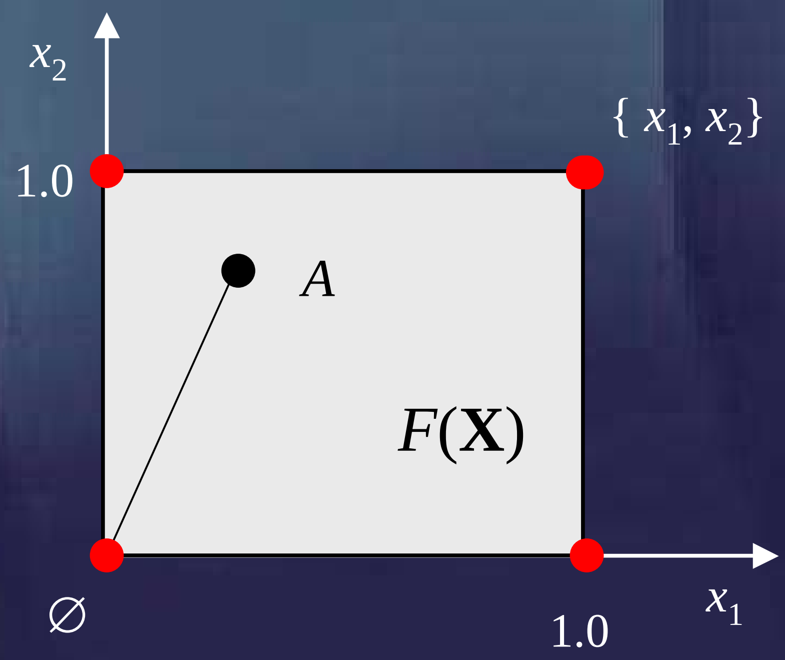
Yager (1993)

Geometric interpretation of sets and fuzzy sets

$\mathbf{X} = \{ x_1, x_2\} \quad P(\mathbf{X}) = \{\varnothing, \{x_1\}, \{x_2\}, \{ x_1, x_2\}\}$



Set



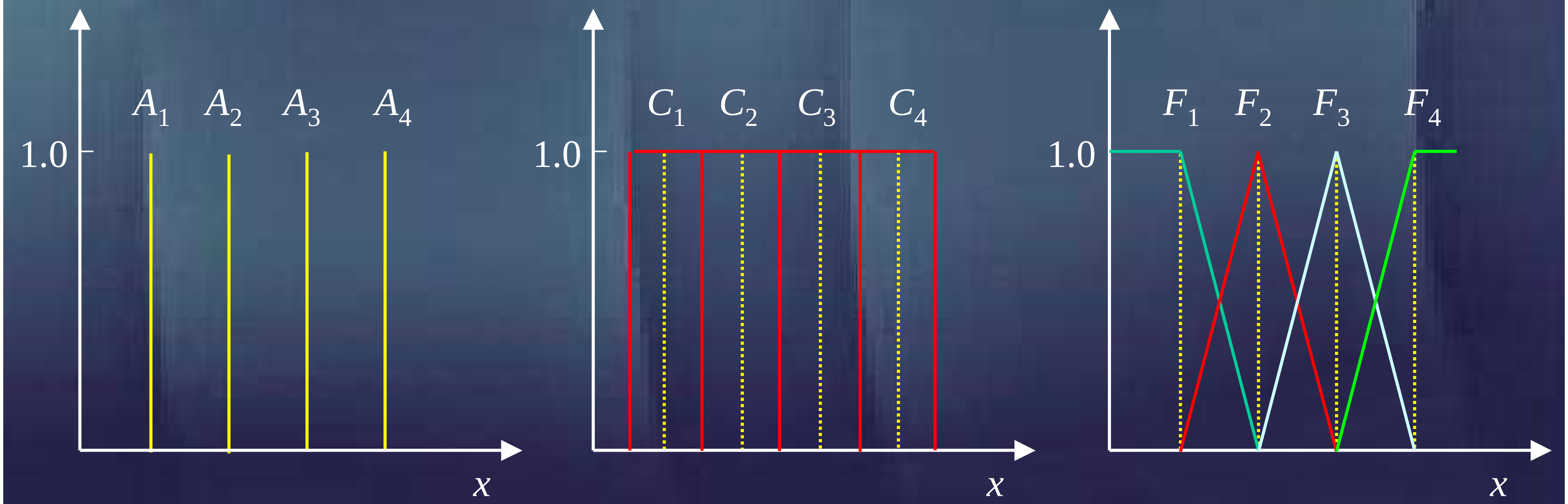
Fuzzy set

3.4 Granulation of information

Motivation

- **Need of granulation:**
  - abstract information
  - summarize information
- **Purpose:**
  - comprehension
  - decision making
  - description

Discretization, quantization, granulation



Discretatization Quantization Granulation

Pedrycz and Gomide, FSE 2007

Formal mechanisms of granulation

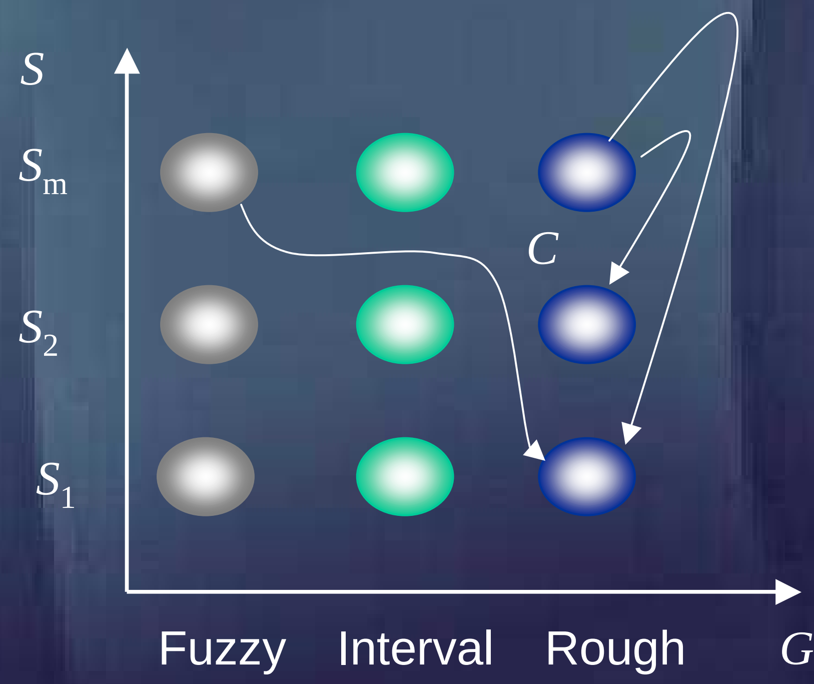
$\langle \mathbf{X}, G, S, C \rangle$

$\mathbf{X}$: universe

G: formal framework of granulation

S: collection of information granules

C: transformation



Pedrycz and Gomide, FSE 2007

3.5 Characterization of families of fuzzy sets

Frame of cognition

- Codebook of conceptual entities
  - family of linguistic landmarks

    $\Phi = \{A_1, A_2, \ldots ., A_m\}$

    A_i is a fuzzy set on $\mathbf{X}$, $i = 1, \ldots, m$

- Granulation that satisfies semantic constraints
  - coverage
  - semantic soundness

Coverage

$\Phi = \{A_1, A_2, \ldots, A_m\}$ covers $\mathbf{X}$ if, for any $x \in \mathbf{X}$

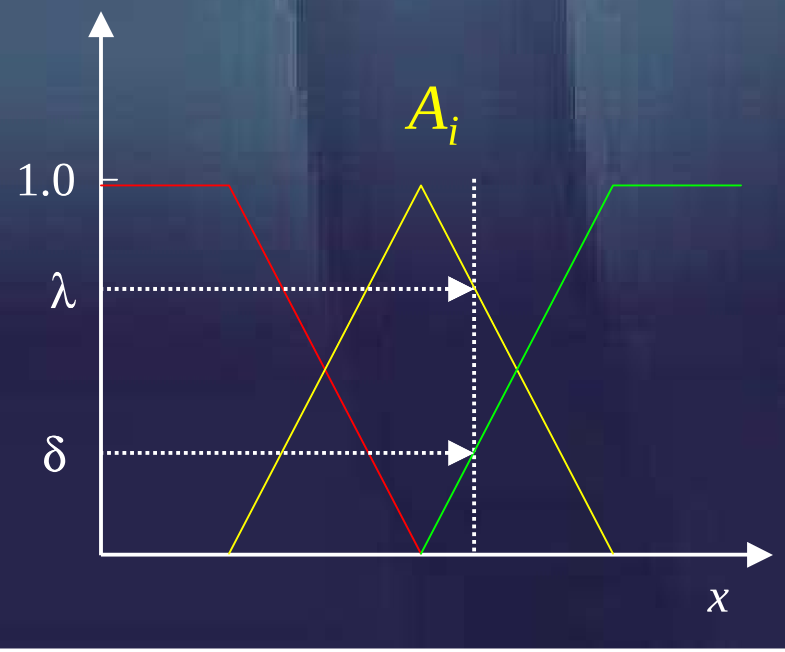
$\exists i \in \mathrm{I} \mid A_i(x) > 0$

$\exists i \in \mathrm{I} \mid A_i(x) > \delta$ (δ-level coverage) $\delta \in [0, 1]$

A_i´s are fuzzy set on $\mathbf{X}$, $i \in \mathrm{I} = \{1, \ldots, m\}$

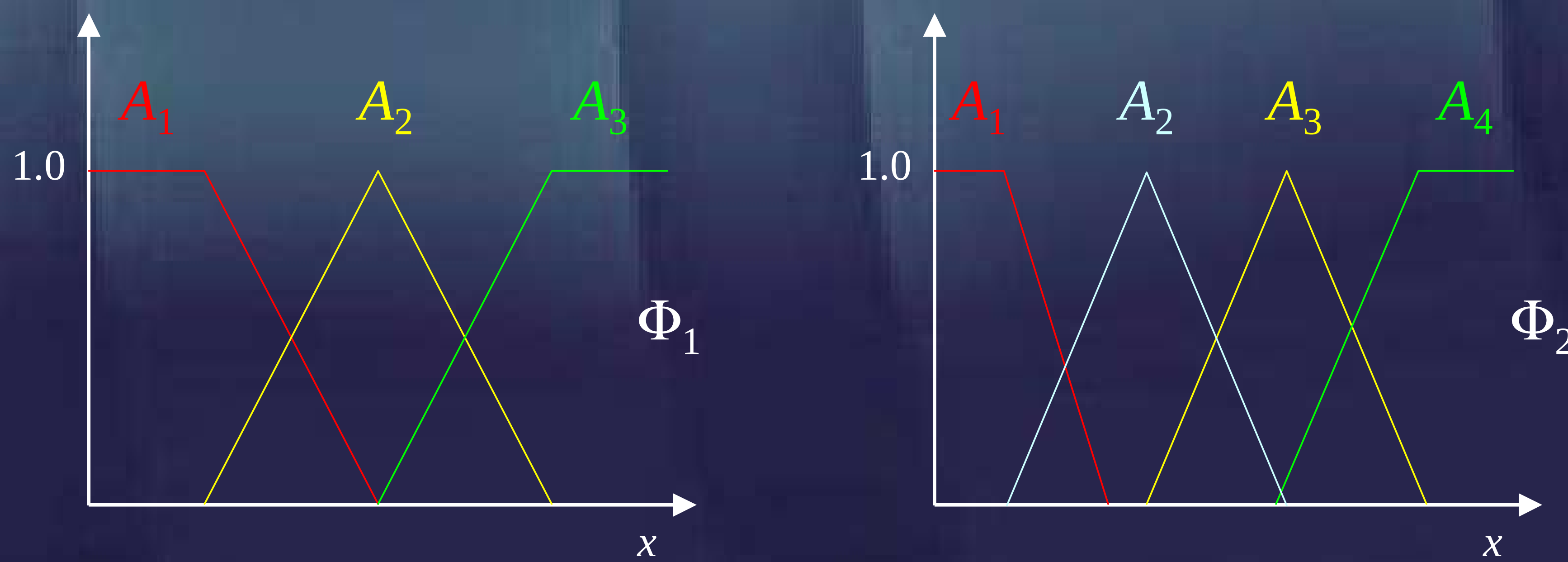
Semantic soundness

- Each A_i, $i \in \mathrm{I} = \{1, \ldots, m\}$ is unimodal and normal
- Fuzzy sets A_i are disjoint enough (λ-overlapping)
- Number of elements of Φ is low



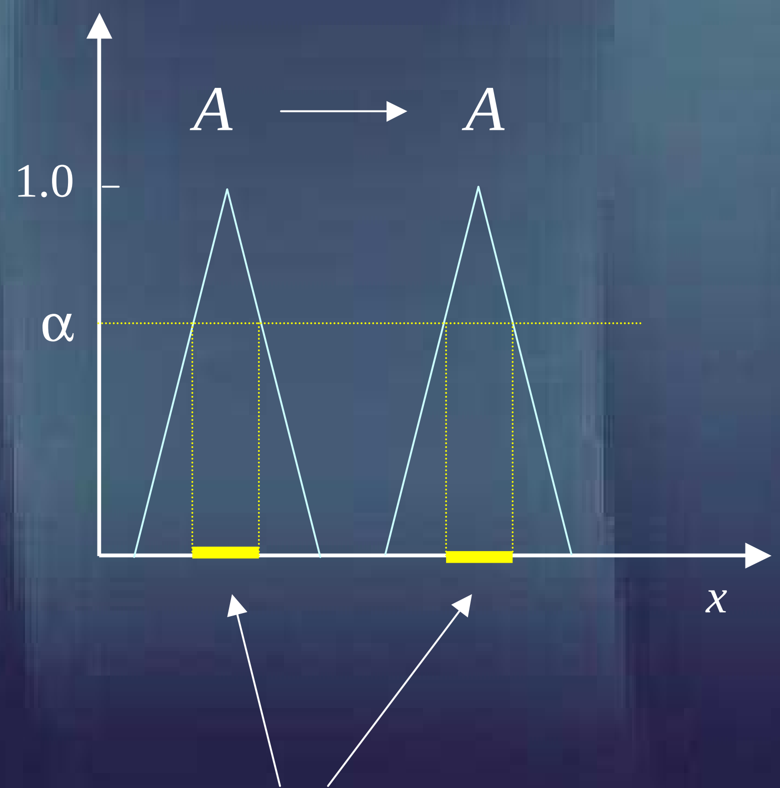
Characteristics of frames of Cognition

- Specificity: Φ_1 more specific than Φ_2 if $\mathrm{Spec}(A_{1i}) > \mathrm{Spec}(A_{2j})$
- Granularity: Φ_1 finer than Φ_2 if $|\Phi_1| > |\Phi_2|$



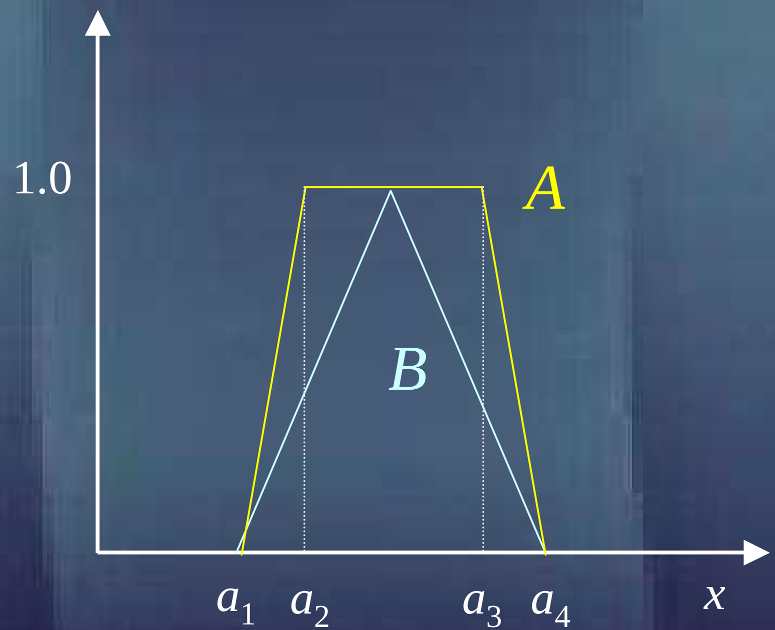
Pedrycz and Gomide, FSE 2007

- Focus of attention



Regions of focus of attention implied by the corresponding fuzzy sets

- Information hiding



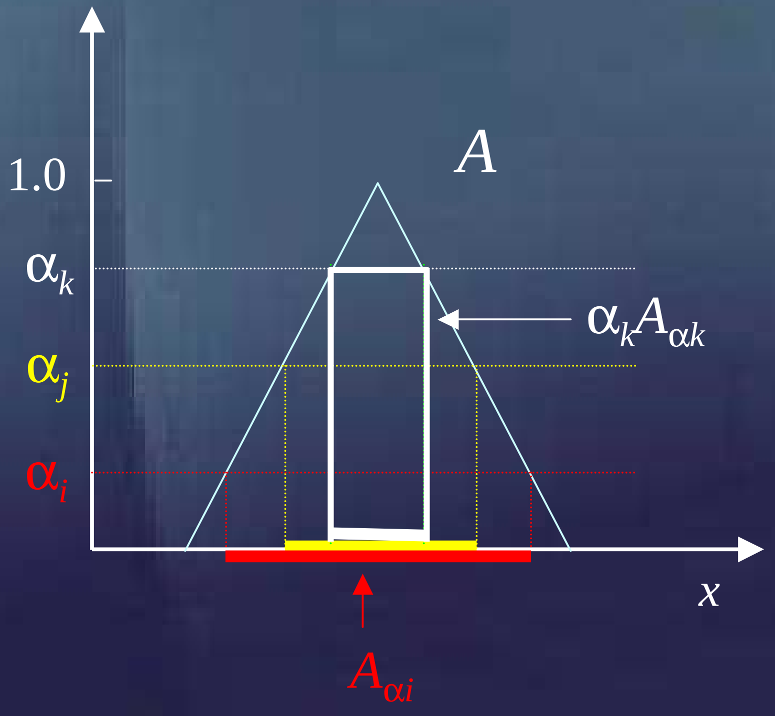
$x \in [a_2, a_3]$ indistinguishable for A, but not for B

3.6 Fuzzy sets, sets and the representation theorem

Any fuzzy set can be viewed as a family of sets:

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$$

$$A(x) = \sup_{\alpha \in [0,1]} \alpha A_\alpha(x)$$



Pedrycz and Gomide, FSE 2007

Example

$\mathbf{X} = \{1, 2, 3, 4\}$

$A = \{0/1, 0.1/2, 0.3/3, 1/4, 0.3/5\} = [0, 0.1, 0.3, 1, 0.3]$

$A_{0.1} = \{0/1, 1/2, 1/3, 1/4, 1/5\} = [0, 1, 1, 1, 1] \rightarrow 0.1A_{0.1} = [0, 0.1, 0.1, 0.1, 0.1]$

$A_{0.3} = \{0/1, 0/2, 1/3, 1/4, 1/5\} = [0, 0, 1, 1, 1] \rightarrow 0.3A_{0.3} = [0, 0 , 0.3, 0.3, 0.3]$

$A_1 = \{0/1, 0/2, 0/3, 1/4, 0/5\} = [0, 0, 0, 1, 0] \rightarrow 1.0A_1 = [0, 0, 0, 1, 0]$

$A = \max (0.1A_{0.1} , 0.3A_{0.3} , 1A_1)$

$A = [\max (0,0,0), \max(0.1,0,0), \max(0.1,0.3,0), \max(0.1,0.3,1), \max(0.1,0.3,0)]$

A = [0, 0.1, 0.3, 1, 0.3]