

2 Notions and Concepts of Fuzzy Sets

*Fuzzy Systems Engineering
Toward Human-Centric Computing*

Contents

2.1 Sets and fuzzy sets: A departure from the principle of dichotomy

2.2 Interpretation of fuzzy sets

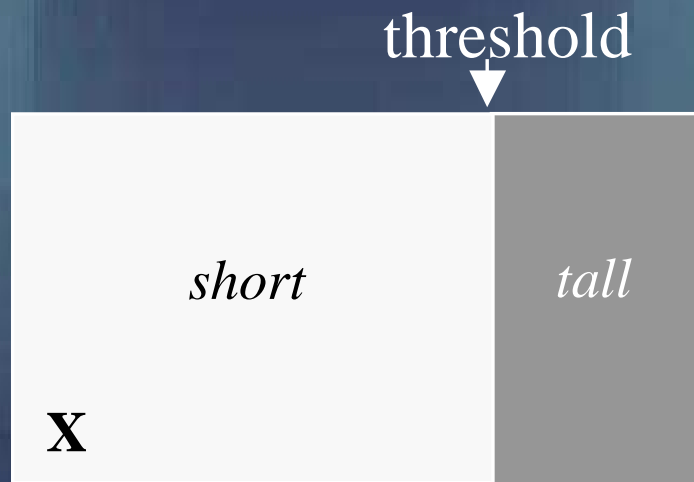
2.3 Membership functions and their motivation

2.4 Fuzzy numbers and intervals

2.5 Linguistic variables

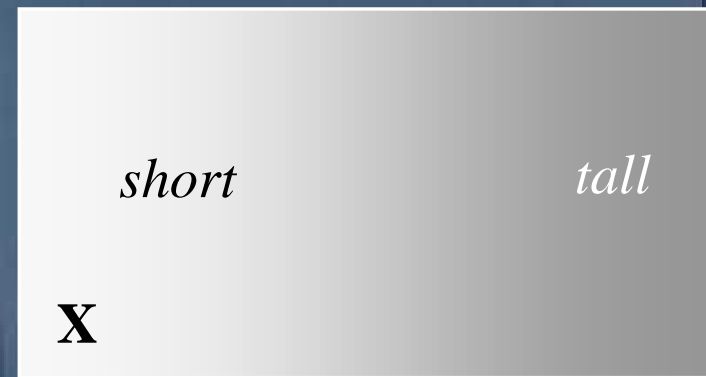
2.1 Sets and fuzzy sets: A departure from the principle of dichotomy

Dichotomy



(a)

Set and the principle
of dichotomy



(b)

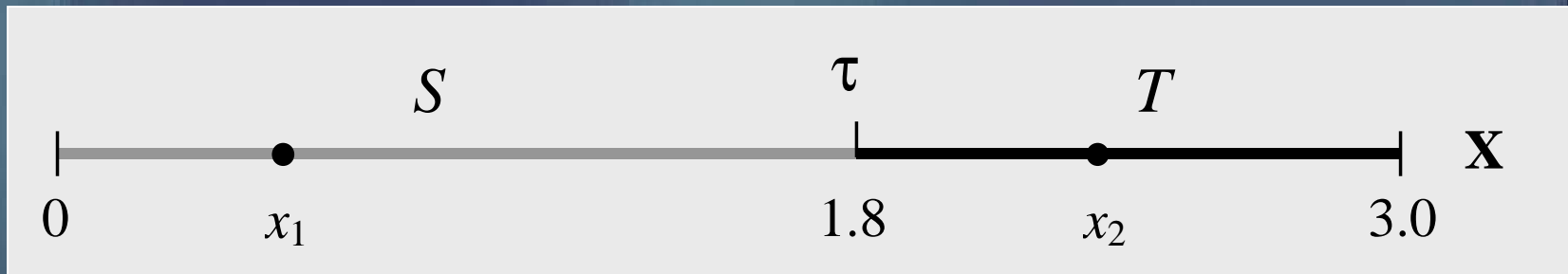
Relaxation of complete
inclusion and exclusion

Inherent problems of dichotomization

One seed does not constitute a pile nor two or three. From the other side, everybody will agree that 100 million seeds constitutes a pile. What is therefore the appropriate limit?

E. Borel, 1950

Sets



Threshold $\tau = 1.8$

$$S = \{x \in \mathbf{X} \mid 0 \leq x \leq 1.8\}$$

$$T = \{x \in \mathbf{X} \mid 1.8 < x \leq 3.0\}$$

Dichotomy

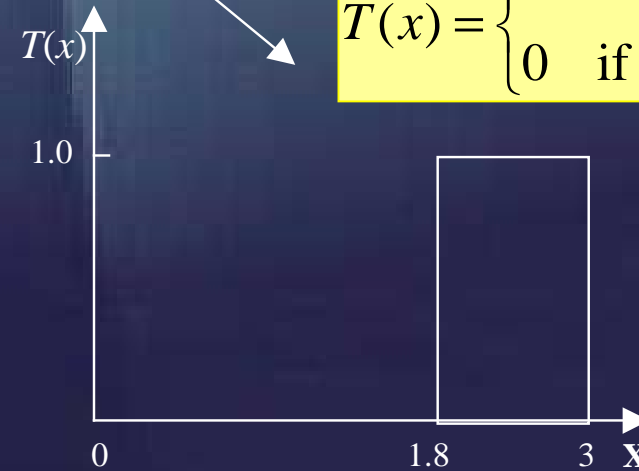
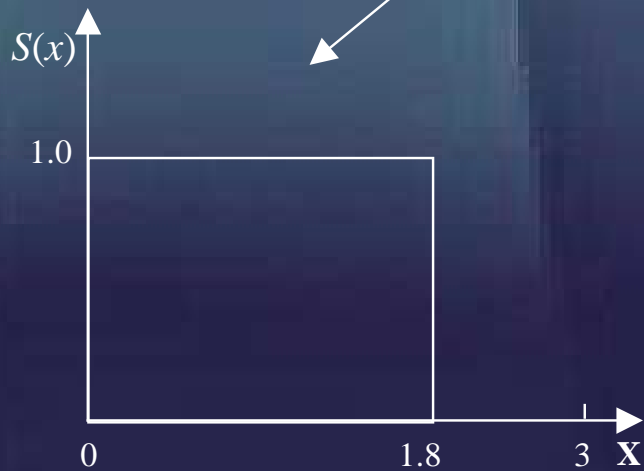
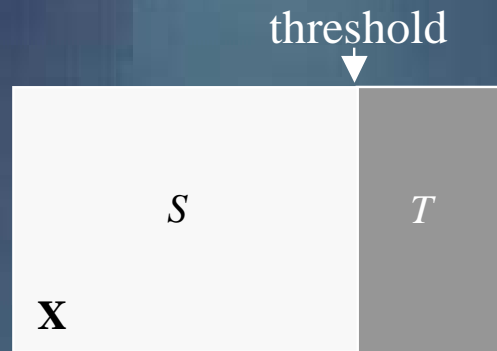
$$x_1 \in S, \quad x_1 \notin T$$

$$x_2 \in T, \quad x_2 \notin S$$

Characteristic function

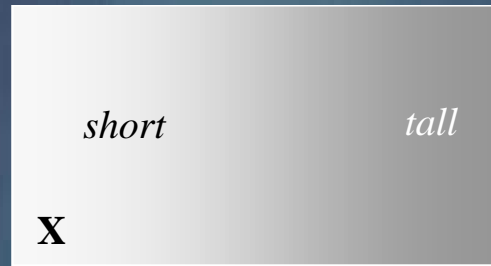
$$A : X \rightarrow \{0,1\}$$

$$A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

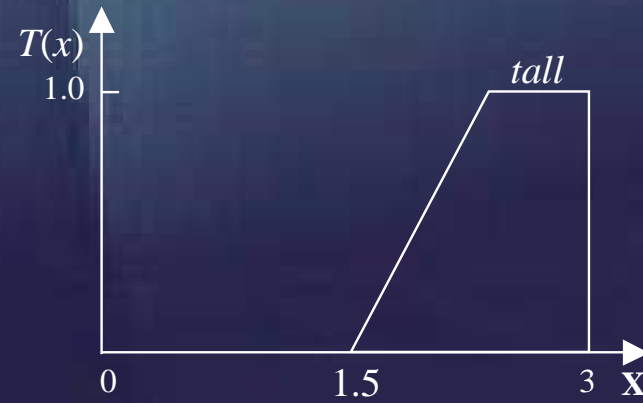


$$T(x) = \begin{cases} 1 & \text{if } x \in [1.8, 3.0] \\ 0 & \text{if } x \notin [1.8, 3.0] \end{cases}$$

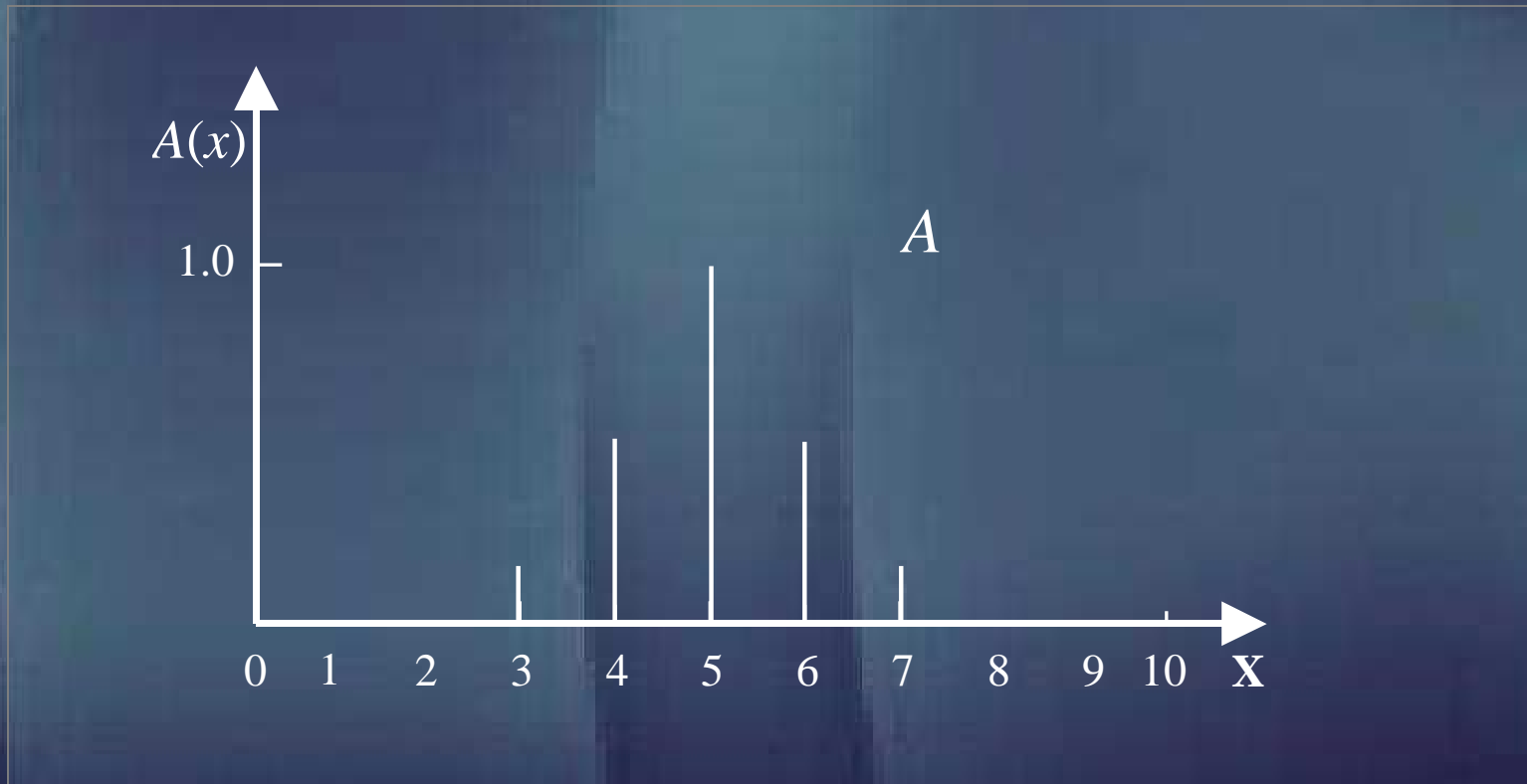
Fuzzy set: Membership function



$$A : \mathbf{X} \rightarrow [0,1]$$



Fuzzy sets in discrete universes



$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{(A(x), x)\}$$

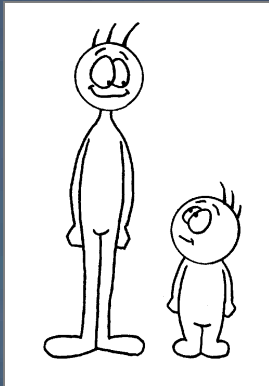
$$A = \{0/0, 0/1, 0/2, 0.2/3, 0.5/4, 1.0/5, 0.5/6, 0.2/7, 0/8, 0/9, 0/10\}$$

$$A = [0, 0, 0, 0.2, 0.5, 1.0, 0.5, 0.2, 0, 0, 0]$$

2.2 Interpretation of fuzzy sets

Fuzziness \neq Probability

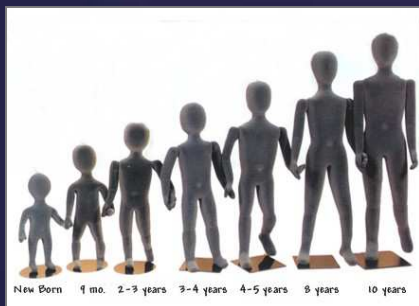
John is tall



Head or tail ?



Height of people



Fuzziness

$$A : \mathbf{X} \rightarrow [0,1]$$

\mathbf{X} : universe

A : membership **function**

Probability

$$P(A) : \mathbf{F} \rightarrow [0,1]$$

P : probability **(set) function**

A : set

\mathbf{X} : universe

\mathbf{F} : σ -algebra, a set of subsets of \mathbf{X}

Membership grades

- **Similarity:** degree of compatibility
(data analysis and processing)
- **Uncertainty:** possibility
(reasoning under uncertainty)
- **Preference:** degree of satisfaction
(decision-making, optimization)

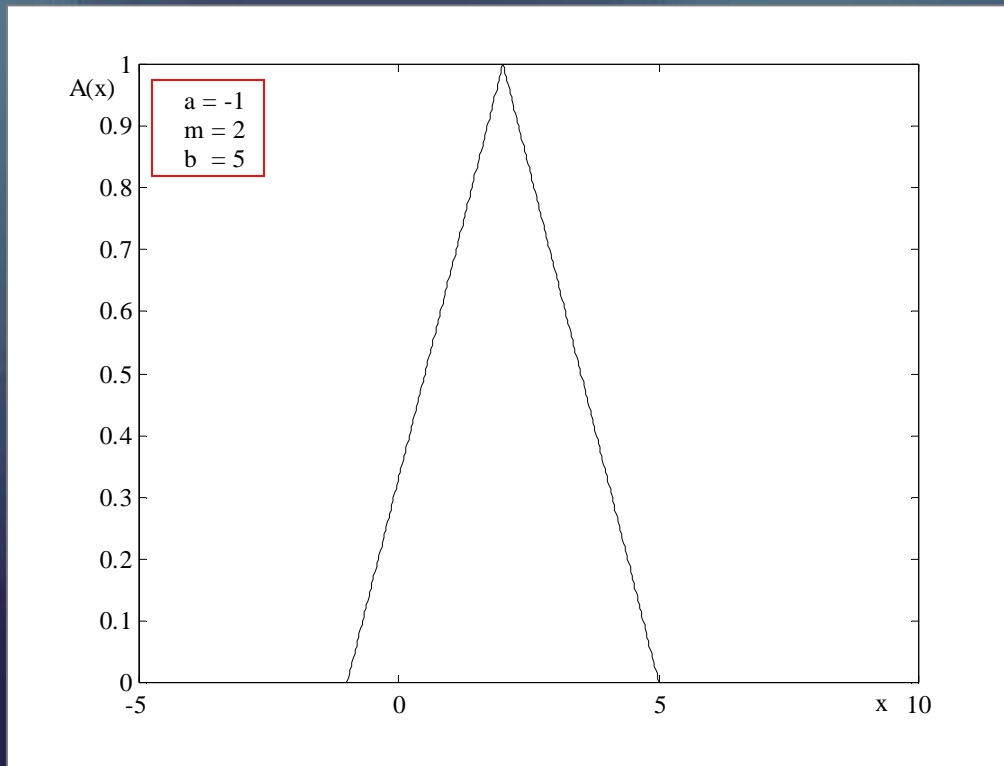
2.3 Membership functions and their motivation

Choosing membership functions

Criteria should reflect:

- Nature of the problem at hand
- Perception of the concept to represent
- Level of details to be captured
- Context of application
- Suitability for design and optimization

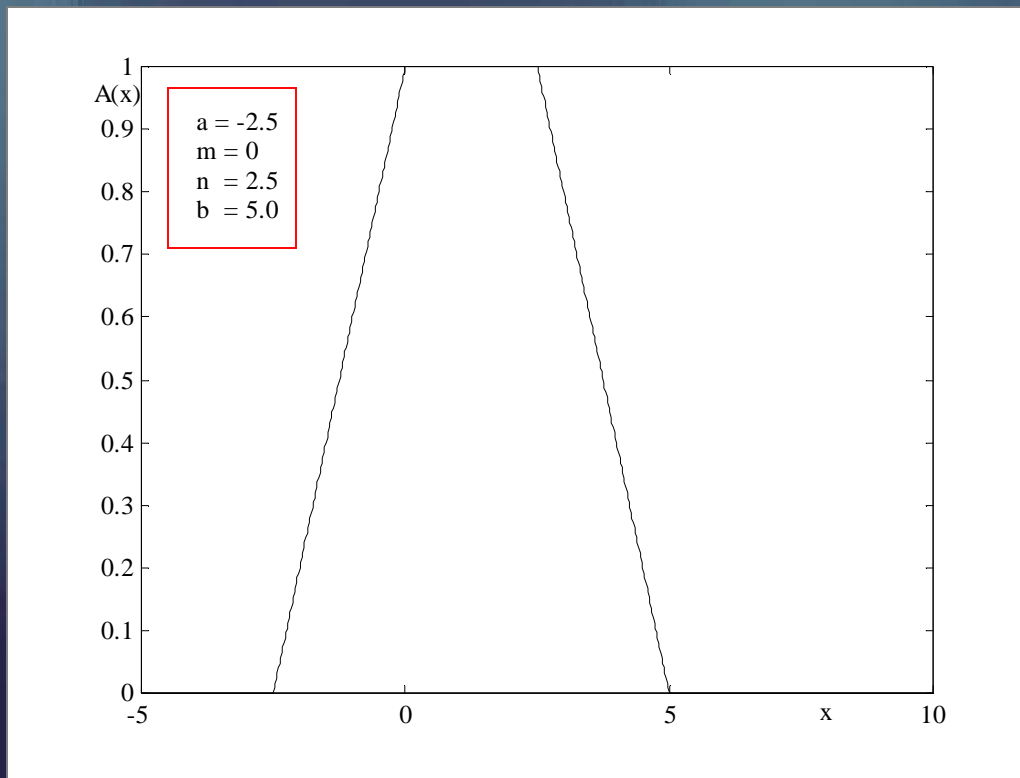
Triangular membership function



$$A(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{m-a} & \text{if } x \in [a, m] \\ \frac{b-x}{b-m} & \text{if } x \in [m, b] \\ 0 & \text{if } x \geq b \end{cases}$$

$$A(x, a, m, b) = \max\{\min[(x-a)/(m-a), (b-x)/(b-m)], 0\}$$

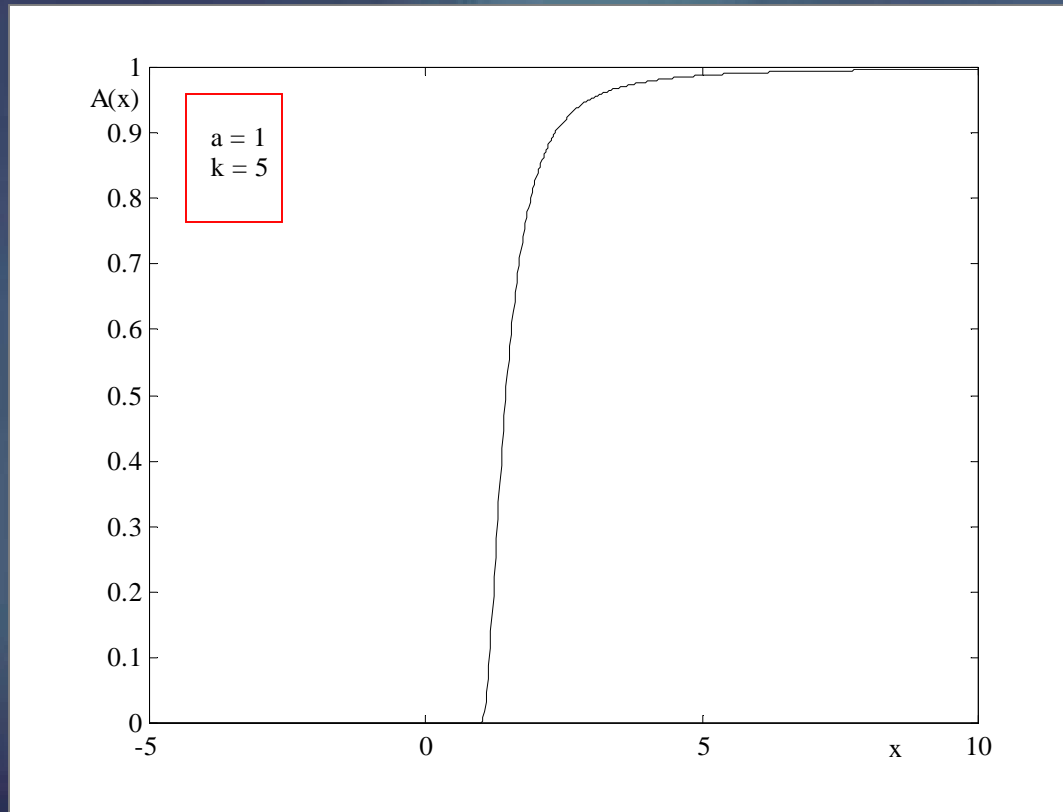
Trapezoidal membership function



$$A(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{m-a} & \text{if } x \in [a, m) \\ 1 & \text{if } x \in [m, n) \\ \frac{b-x}{b-n} & \text{if } x \in [n, b] \\ 0 & \text{if } x > b \end{cases}$$

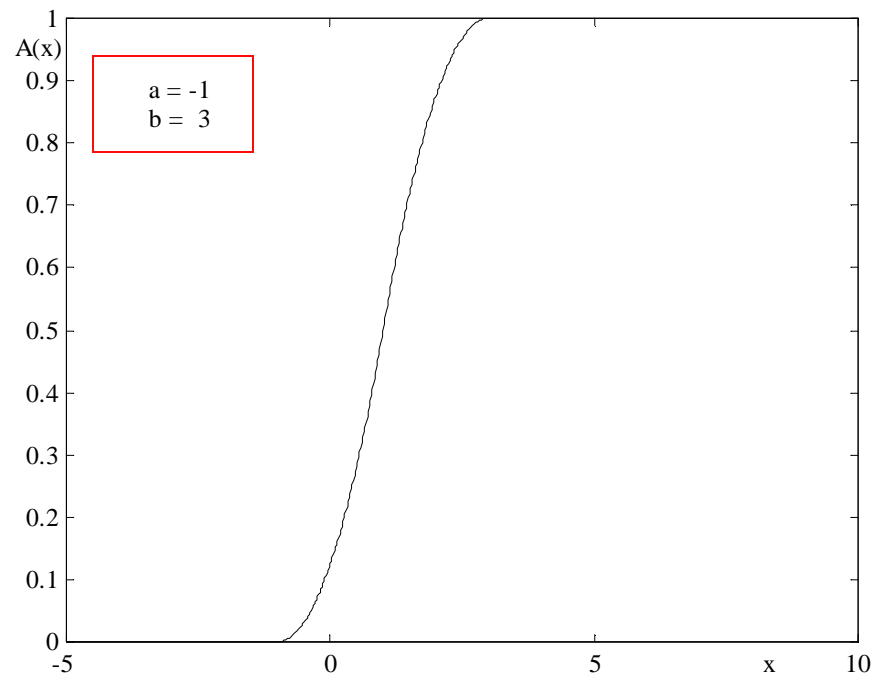
$$A(x, a, m, n, b) = \max\{\min[(x-a)/(m-a), 1, (b-x)/(b-n)], 0\}$$

Γ -membership function



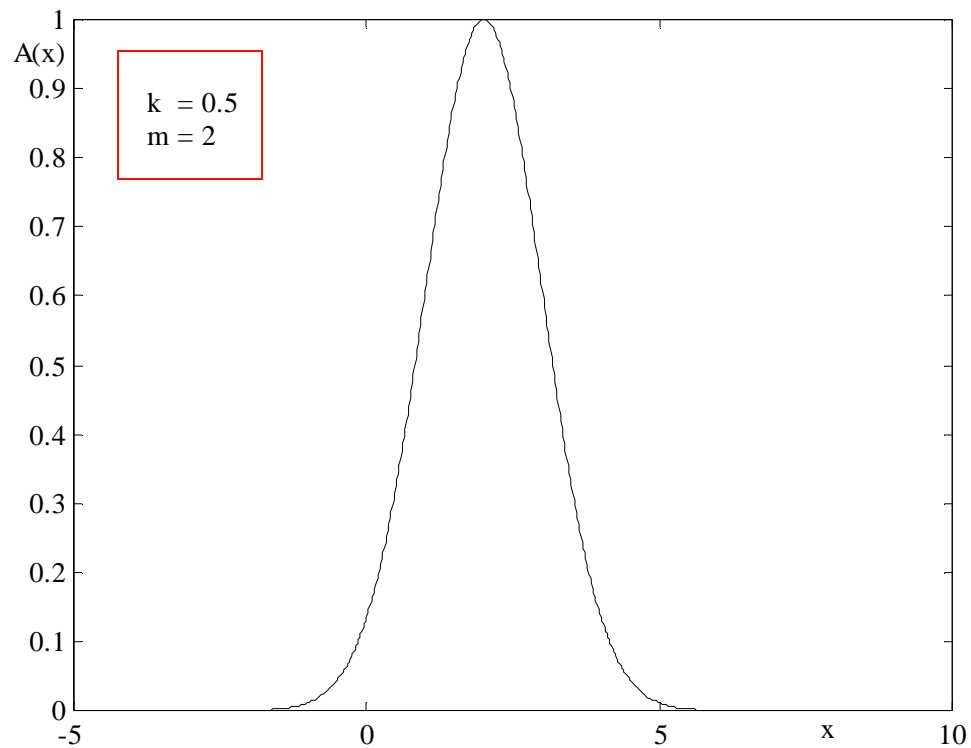
$$A(x) = \begin{cases} 0 & \text{if } x \leq a \\ 1 - e^{-k(x-a)^2} & \text{if } x > a \end{cases} \quad \text{or} \quad A(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{k(x-a)^2}{1+k(x-a)^2} & \text{if } x > a \end{cases}$$

S-membership function



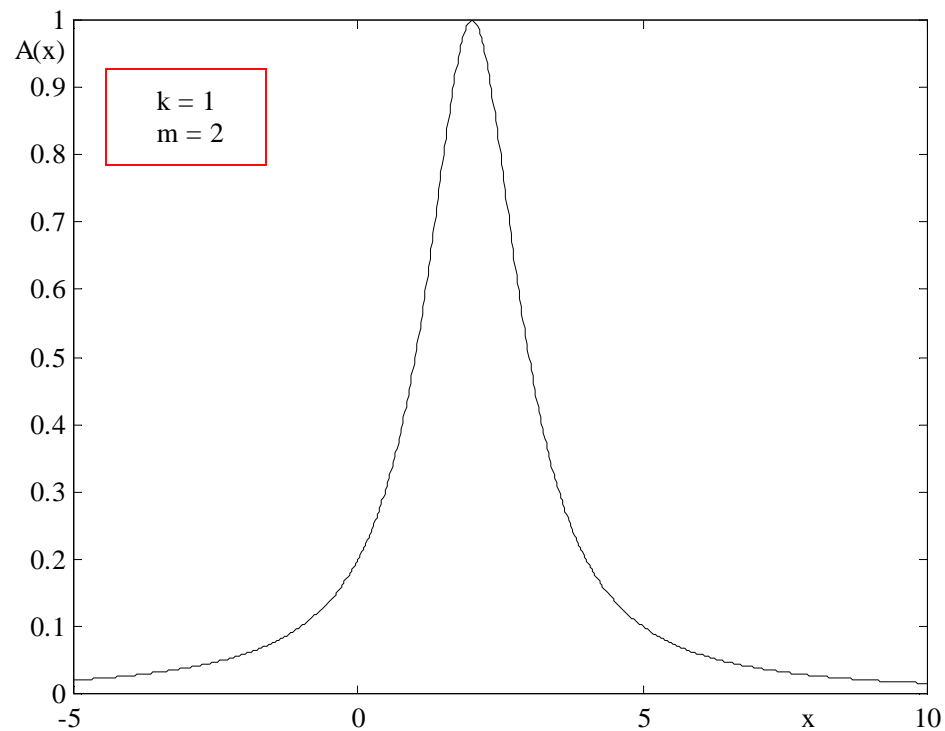
$$A(x) = \begin{cases} 0 & \text{if } x \leq a \\ 2\left(\frac{x-a}{b-a}\right)^2 & \text{if } x \in [a, m) \\ 1 - 2\left(\frac{x-b}{b-a}\right)^2 & \text{if } x \in (m, b] \\ 1 & \text{if } x > b \end{cases}$$

Gaussian membership function



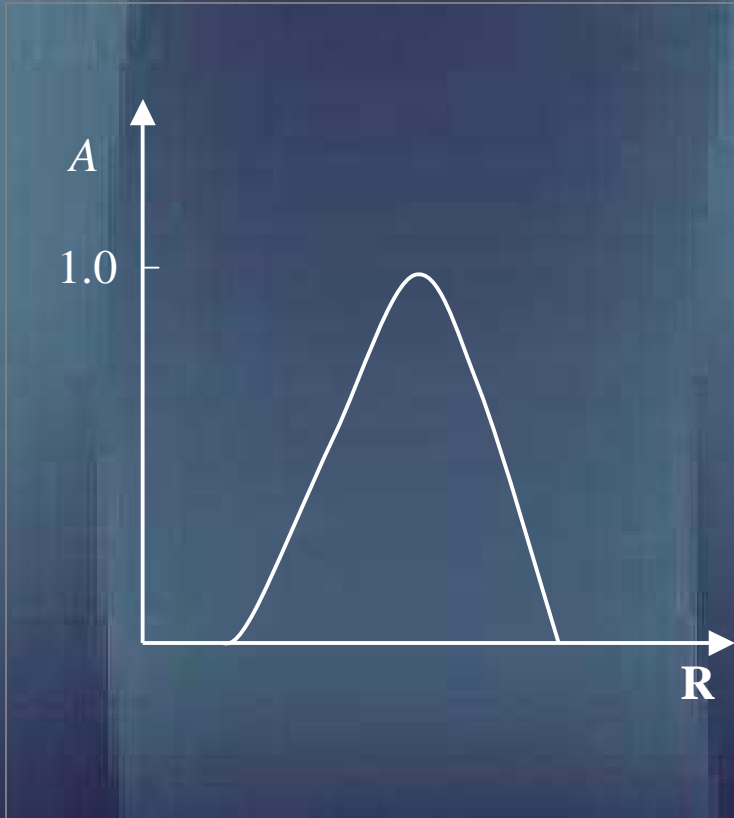
$$A(x) = \exp\left(-\frac{(x-m)^2}{\sigma^2}\right)$$

Exponential-like membership function

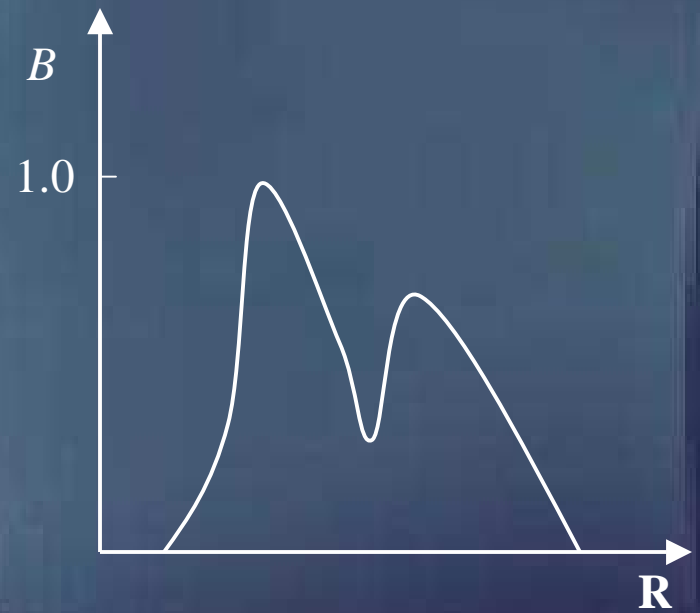


$$A(x) = \frac{1}{1 + k(x - m)^2} \quad k > 0$$

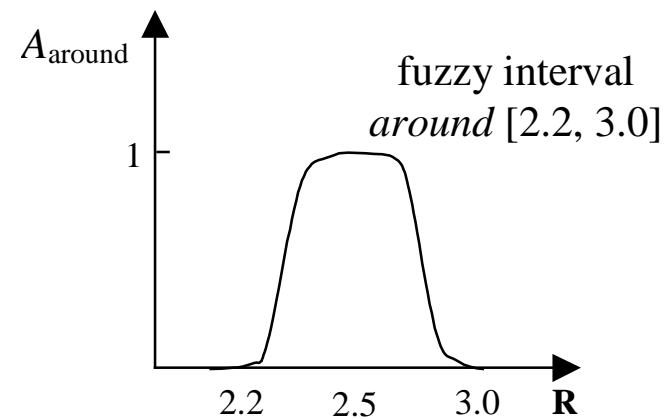
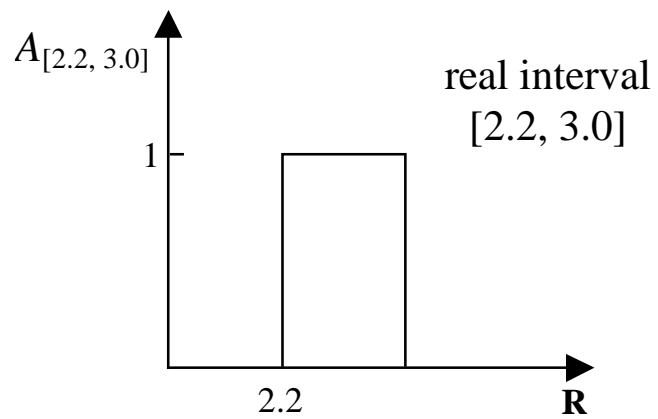
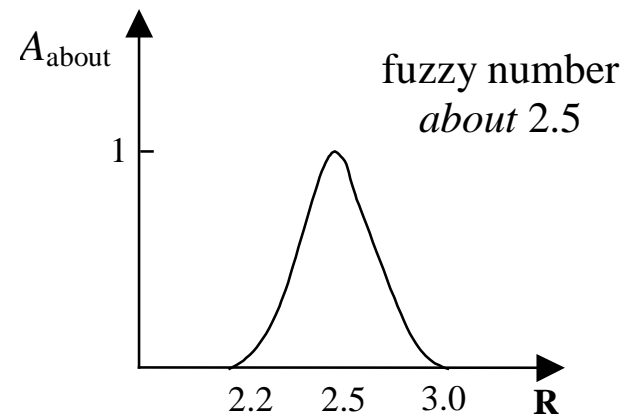
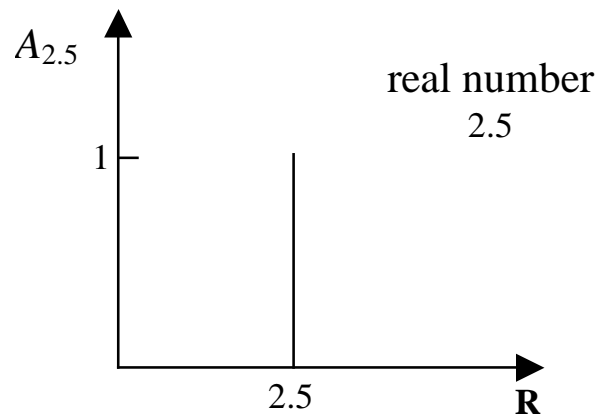
2.4 Fuzzy numbers and intervals



A is a fuzzy number



B is not a fuzzy number



2.5 Linguistic variables

Linguistic variables

- A certain variable (attribute) can be quantified in terms of a small number of information granules
 - temperature is {*low, high*}
 - speed is { *low, medium, high, very high*}
- Each information granule comes with a well-defined meaning (semantics)

Linguistic variables: A definition

$\langle X, T(X), \mathbf{X}, G, M \rangle$

X : is the name of the variable

$T(X)$: is term set of X ; elements are labels L of linguistic values of X

G : grammar that generates the names of X

M : semantic rule that assigns to each label $L \in T(X)$ a meaning whose realization is a fuzzy set on \mathbf{X} with base variable x

Example

$\langle X, T(X), X, G, M \rangle$

X : temperature

X : $[0, 40]$

$T(X)$: {*cold, comfortable, warm*}

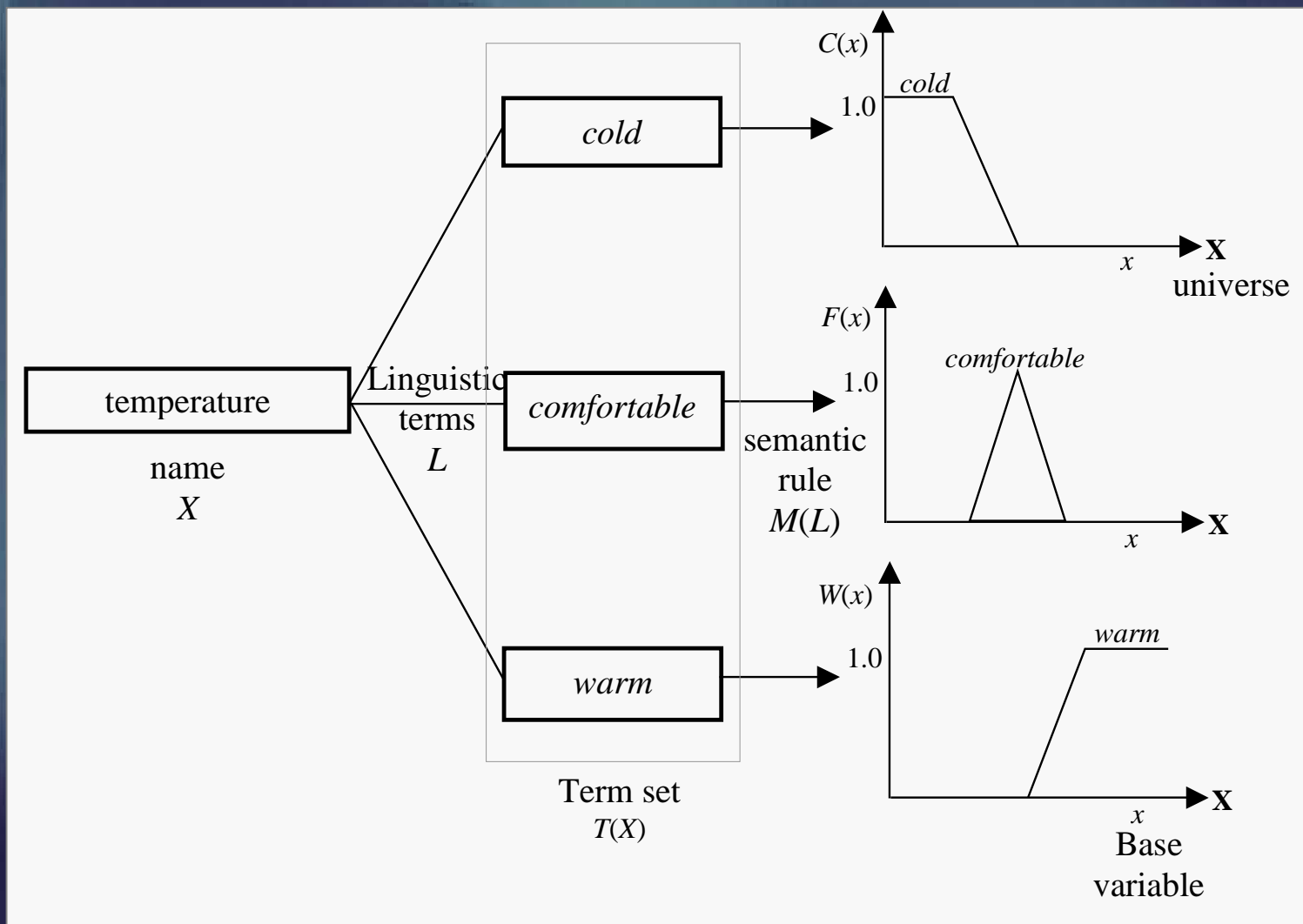
G : only terminal symbols, the terms of $T(X)$

M (*cold*) $\rightarrow C$

M (*comfortable*) $\rightarrow F$

M (*warm*) $\rightarrow W$

C, F and W are fuzzy sets on $[0, 40]$



$$\langle X, T(X), X, G, M \rangle$$