11 Fuzzy Rule-Based Models

Fuzzy Systems Engineering
Toward Human-Centric Computing

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11.1 Fuzzy rules as a vehicle of knowledge representation

Rule ≡ conditional statement

- **If** ⟨ input variable is *A* ⟩ **then** ⟨ output variable is *B* ⟩
  - *A* and *B:* descriptors of pieces of knowledge
  - rule: expresses a relationship between inputs and outputs
- Example
  - **If** ⟨ the temperature is *high* ⟩ **then** ⟨ the electricity demand is *high* ⟩
- **If** and **then** parts ⟨.......⟩ formed by information granules
  - sets
  - rough sets
  - fuzzy sets

Rule-based system/model (FRBS)

- FRBS is a family of rules of the form

If $\langle$ input variable is A_i $\rangle$ **then** $\langle$ output variable is B_i $\rangle$

$i = 1, 2, \ldots, c$

A_i and B_i are information granules

- More complex rules

If $\langle$ input variable$_1$ is A_i $\rangle$ ***and*** $\langle$ input variable$_2$ is B_i $\rangle$ ***and*** .....
then $\langle$ output variable is Z_i $\rangle$

– multidimensional input space (Cartesian product of inputs)
– individual inputs aggregated by the ***and*** connective
– highly parallel, modular granular model

11.2 General categories of fuzzy rules and their semantics

Multi-input multi-output fuzzy rules

- **If** X_1 is A_1 ***and*** X_2 is A_2 ***and*** ..... ***and*** X_n is A_n
  then Y_1 is B_1 ***and*** Y_2 is B_2 ***and*** ..... ***and*** Y_m is B_m

  X_i = variables whose values are fuzzy sets A_i

  Y_j = variables whose values are fuzzy sets B_j

  A_i on $\mathbf{X}_i$, $i = 1, 2,...,n$

  B_j on $\mathbf{Y}_j$, $j = 1, 2,...,m$

- No loss of generality if we assume rules of the form

  If X is A ***and*** Y is B **then** Z is C

Certainty-qualified rules

- **If** X is A ***and*** Y is B **then** Z is C with certainty μ

  $\mu \in [0,1]$

  μ : degree of certainty of the rule

  $\mu = 1$ rule is certain

Gradual rules

- the ***more*** X is A the ***more*** Y is B
  - relationships between changes in X and Y
  - captures tendency between information granules
- Examples:

  the *higher* the income, the *higher* the taxes

  the *lower* the temperature, the *higher* energy consumption

Functional fuzzy rules

- **If** X is Ai **then** $\boldsymbol{y = f}$ **(x,a$_i$)**

  $f : \mathbf{X} \to \mathbf{Y}$

  $\mathbf{x} \in \mathrm{R}^n$

- Rule: confines the function to the support of granule A_i

  f : linear or nonlinear (neural nets, etc..)

- Highly modular models

11.3 Syntax of fuzzy rules

Backus-Naur form (BNF)

⟨ If_then_rule⟩ ::= if ⟨antecedent⟩ then ⟨consequent⟩{⟨certainty⟩}
⟨gradual_rule ⟩ ::= ⟨word⟩ ⟨antecedent⟩⟨word⟩ ⟨consequent⟩
⟨word⟩ ::= ⟨more⟩ {⟨less⟩}
⟨antecedent⟩ ::= ⟨expression⟩
⟨consequent⟩ ::= ⟨expression⟩
⟨expression⟩ ::= ⟨disjunction⟩{and ⟨disjunction⟩}
⟨disjunction⟩ ::= ⟨variable⟩{or⟨variable⟩}
⟨variable⟩ ::= ⟨attribute⟩ is ⟨value⟩
⟨certainty⟩ ::= ⟨none⟩{certainty $\mu \in [0,1]$}

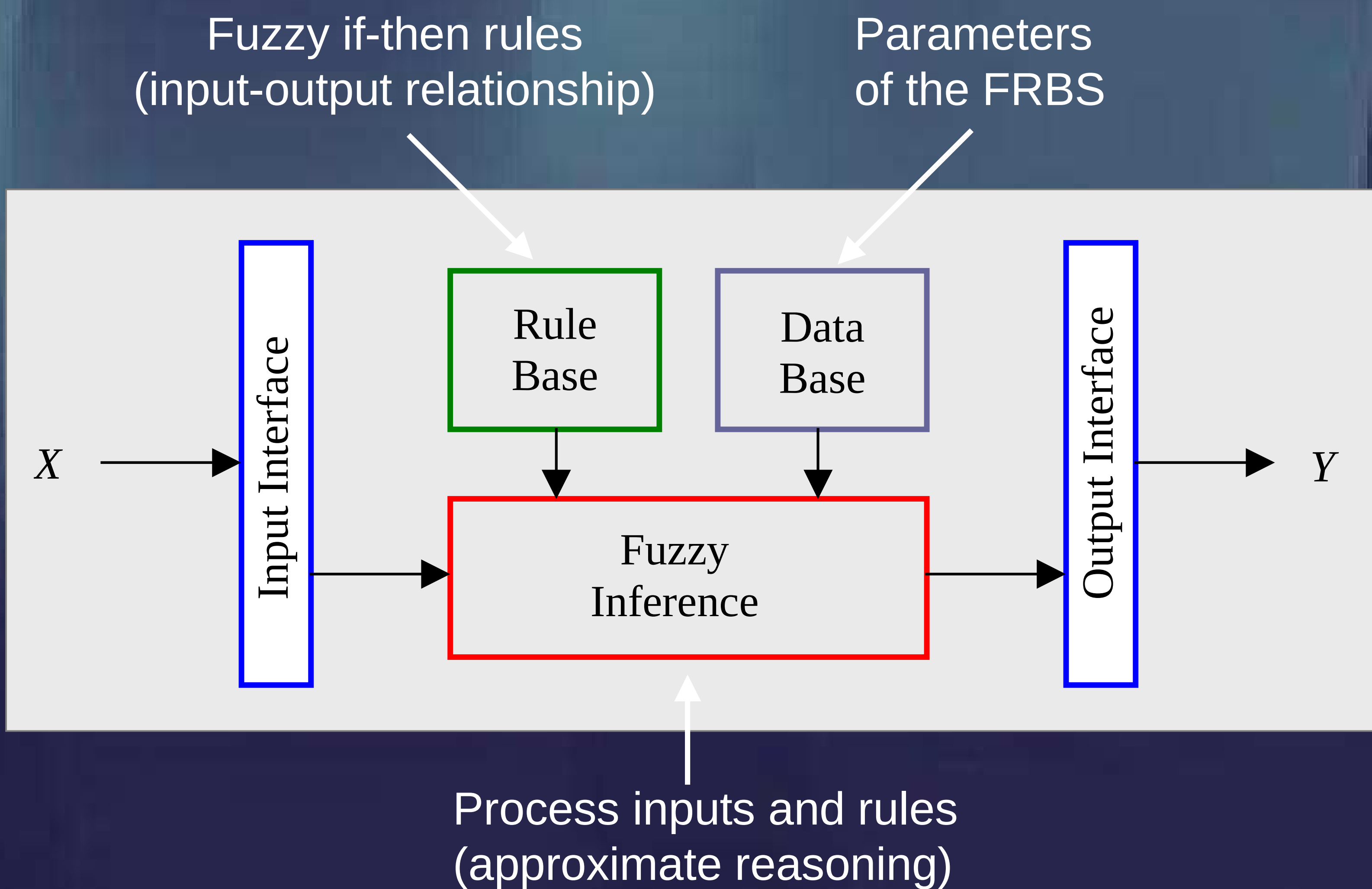
Construction of computable representations

Main steps:

1. specification of the fuzzy variables to be used
2. association of the fuzzy variables using fuzzy sets
3. computational formalization of each rule using fuzzy relations and definition of aggregation operator to combine rules together

11.4 Basic functional modules of FRBS

General architecture of FRBS



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Input interface

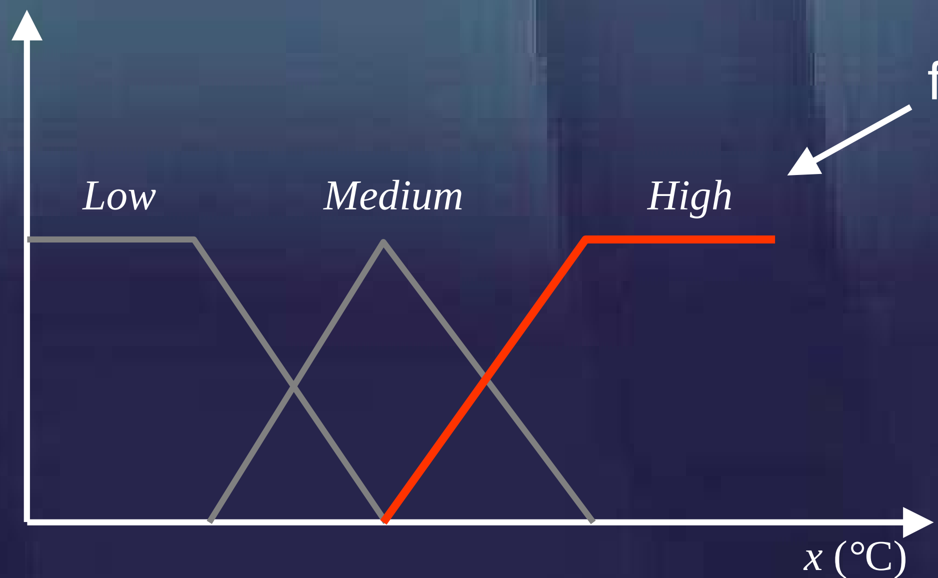
- (attribute) of (input) is (value)

  the temperature of the motor is *high*

- Canonical (atomic) form

  p: X is A $\quad\quad \underbrace{\text{temperature (motor)}}_{X}$ is $\underbrace{\textit{high}}_{A}$

fuzzy set



Multiple fuzzy inputs: conjunctive canonical form

p : X_1 is A_1 *and* X_2 is A_2 *and* ..... *and* X_n is A_n conjunctive canonical form

X_i are fuzzy (linguistic) variables

A_i : fuzzy sets on $\mathbf{X}_i$

$i = 1, 2, ..., n$

Compound proposition induces a fuzzy relation P on $\mathbf{X}_1 \times \mathbf{X}_1 \times ... \mathbf{X}_n$

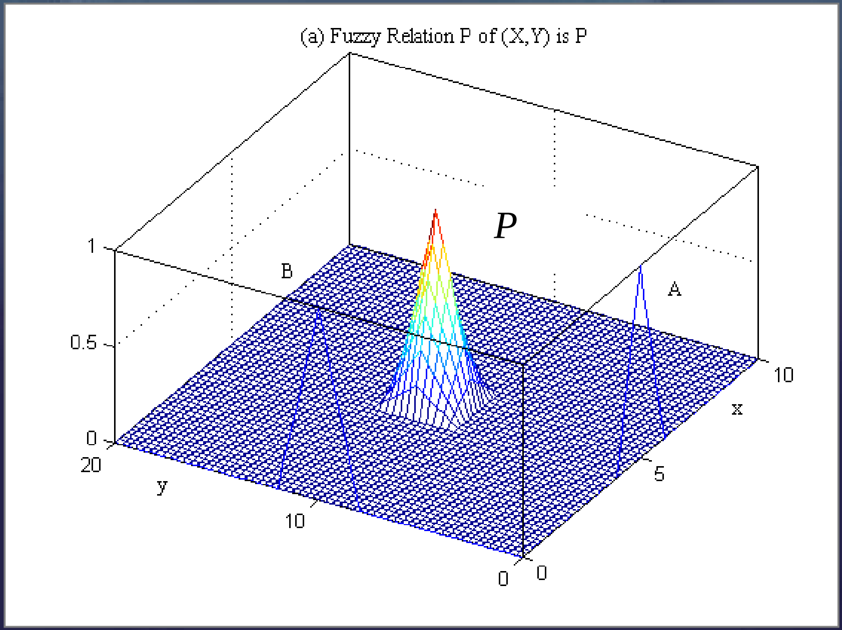
$$P(x_1, x_2, \ldots, x_n) = A_1(x_1) t A_2(x_2) t \ldots t A_n(x_n) = \underset{i=1}{\overset{n}{T}} A_i(x_i)$$

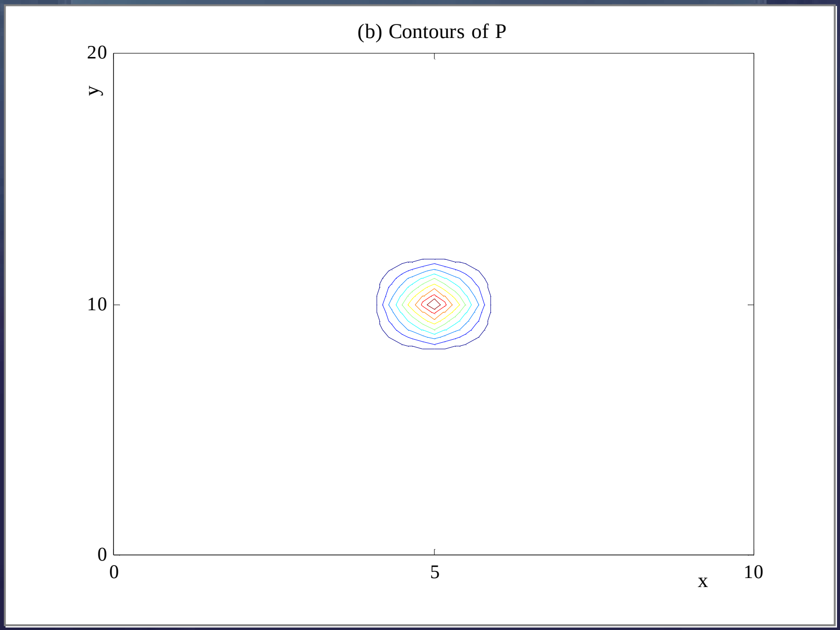
t (T) = t-norm

p : (X_1, X_2 , ..., X_n) is P

Example

- Fuzzy relation associated with (X,Y) is P
- Triangular fuzzy sets $A_1(x,4,5,6) = A$, $A_2(y,8,10,12) = B$
- t-norm: algebraic product





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Multiple fuzzy inputs: disjunctive canonical form

q : X_1 is A_1 *or* X_2 is A_2 *or* ..... *or* X_n is A_n disjunctive canonical form

X_i are fuzzy (linguistic) variables

A_i : fuzzy sets on $\mathbf{X}_i$

$i = 1, 2, ..., n$

Compound proposition induces a fuzzy relation Q on $\mathbf{X}_1 \times \mathbf{X}_1 \times ... \mathbf{X}_n$

$$Q(x_1, x_2, \ldots, x_n) = A_1(x_1) s A_2(x_2) s \ldots s A_n(x_n) = \mathop{S}_{i=1}^{n} A_i(x_i)$$

s (S) = t-conorm

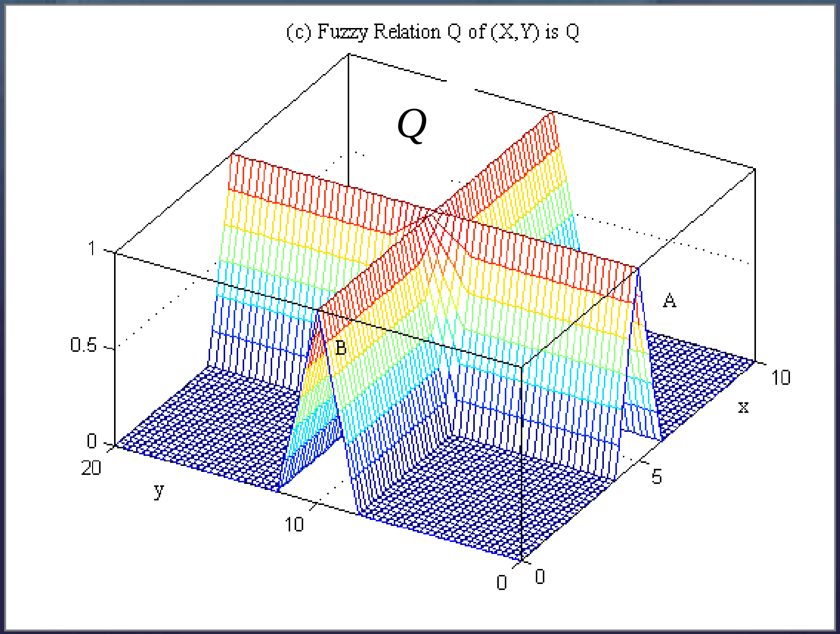
q : (X_1, X_2 , ..., X_n) is Q

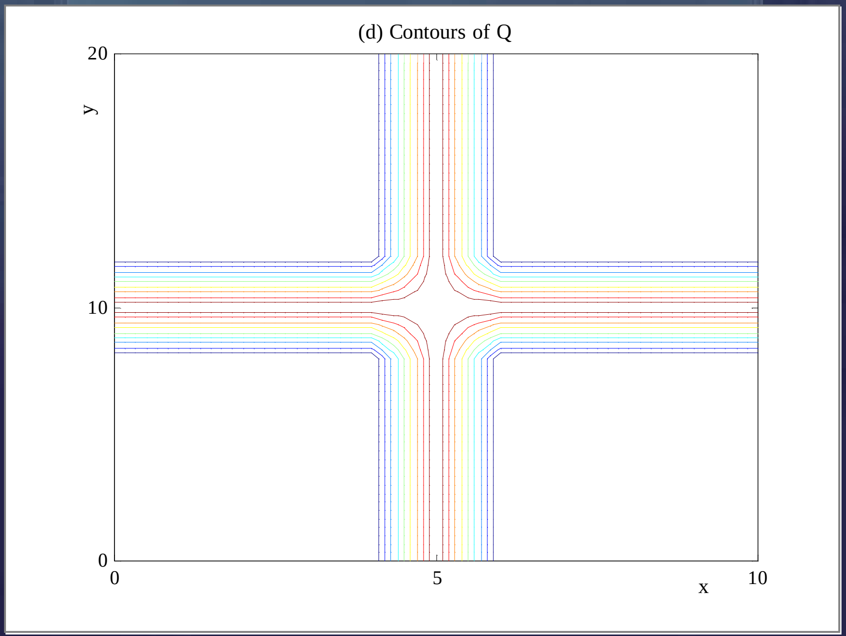
Example

- Fuzzy relation associated with (X,Y) is Q

- Triangular fuzzy sets $A_1(x,4,5,6) = A,\quad A_2(y,8,10,12) = B$

- t-conorm: probabilistic sum





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Rule base

- Fuzzy rule: **If** X is A **then** Y is B $\equiv$ relationship between X and Y

- Semantics of the rule is given by a fuzzy relation R on $\mathbf{X}\times\mathbf{Y}$

- R determined by a relational assignment

$$R(x,y) = f\,(A(x),B(y)) \quad \forall (x, y) \in \mathbf{X}\times\mathbf{Y}$$

$$f : [0,1]^2 \rightarrow [0,1]$$

- In general f can be

  – fuzzy conjunction: f_t
  – fuzzy disjunction: f_s
  – fuzzy implication: f_i

Fuzzy conjunction

Choose a t-norm t and define:

$$R(x,y) \equiv f_t(x,y) = A(x)\ t\ B(y) \quad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

Examples:

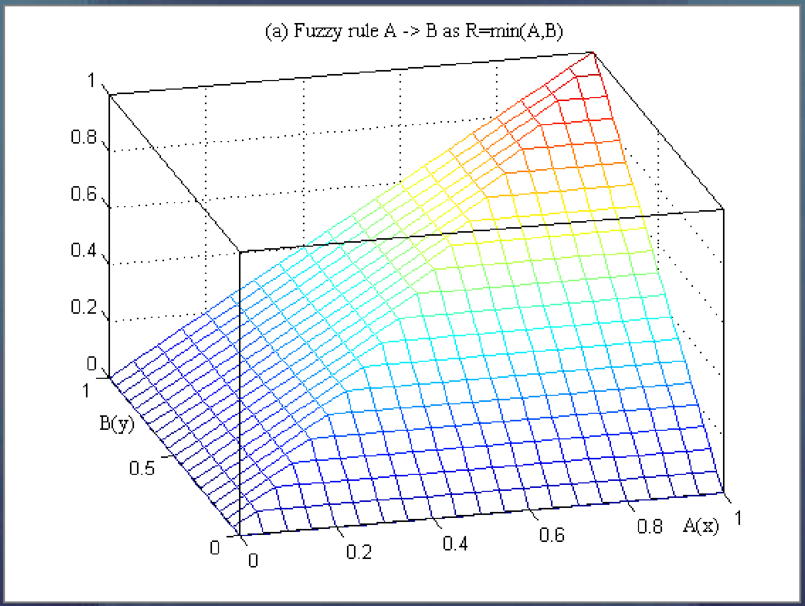
- t = min

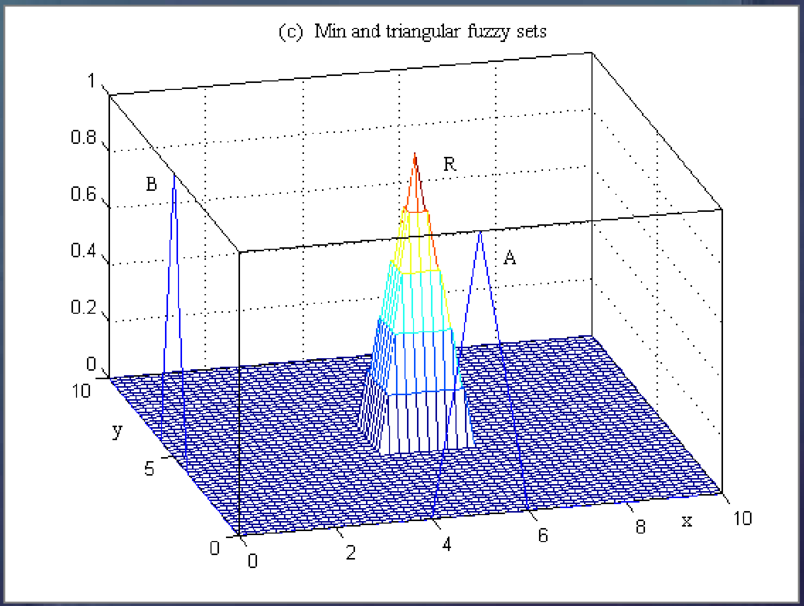
$$R_c(x,y) \equiv f_c(x,y) = \min[A(x)\ t\ B(y)]$$ (Mamdani)

- t = algebraic product

$$R_p(x,y) \equiv f_p(x,y) = A(x)B(y)$$ (Larsen)

Example: t = min





$R_c(x,y) = \min\{A(x), B(y)\}$

$\forall\ (A(x), B(y)) \in [0,1]^2$

$R_c(x,y) = \min\{A(x), B(y)\}$

$A(x) = A(x,4,5,6)$

$B(y) = B(y,4,5,6)$

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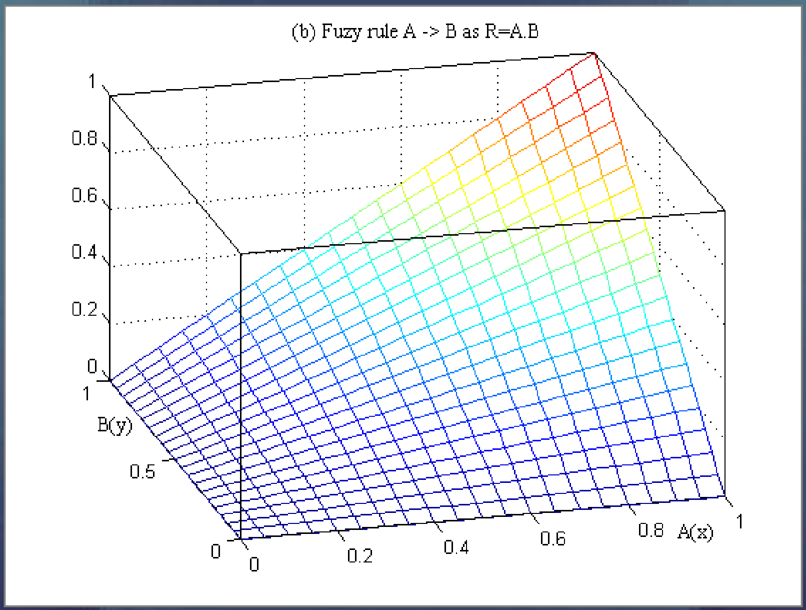
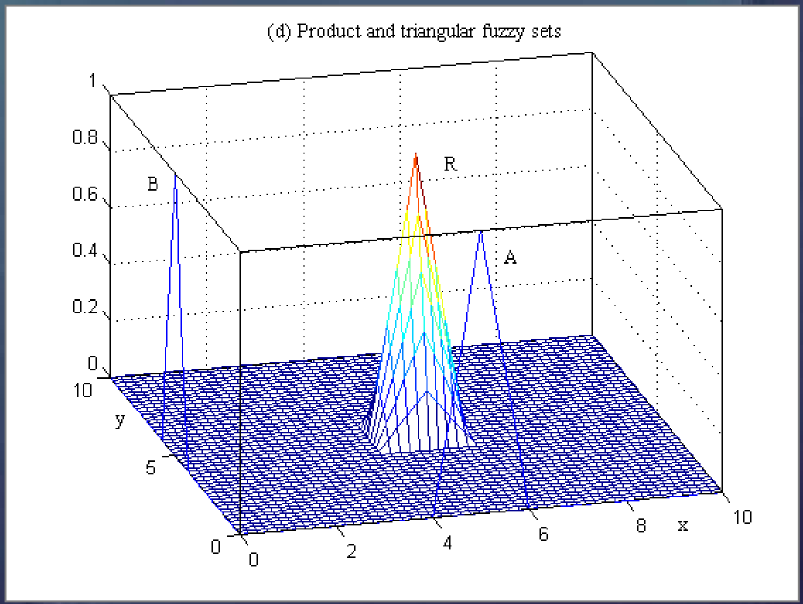
Example: t = algebraic product





$R_p(x,y) = A(x)B(y)$

$\forall\ (A(x), B(y)) \in [0,1]^2$

$R_p(x,y) = A(x)B(y)$

$A(x) = A(x,4,5,6)$

$B(y) = B(y,4,5,6)$

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Fuzzy disjunction

Choose a t-conorm s and define:

$$R_s(x,y) \equiv f_s(x,y) = A(x)\ s\ B(y) \quad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

Examples:

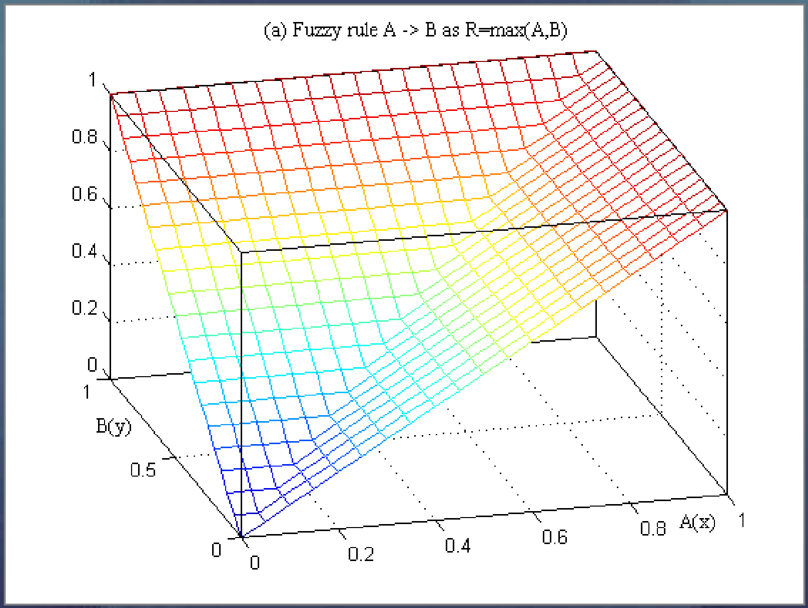
- s = max

$$R_m(x,y) \equiv f_m(x,y) = \max[A(x)\ t\ B(y)]$$

- s = Lukasiewicz t-conorm

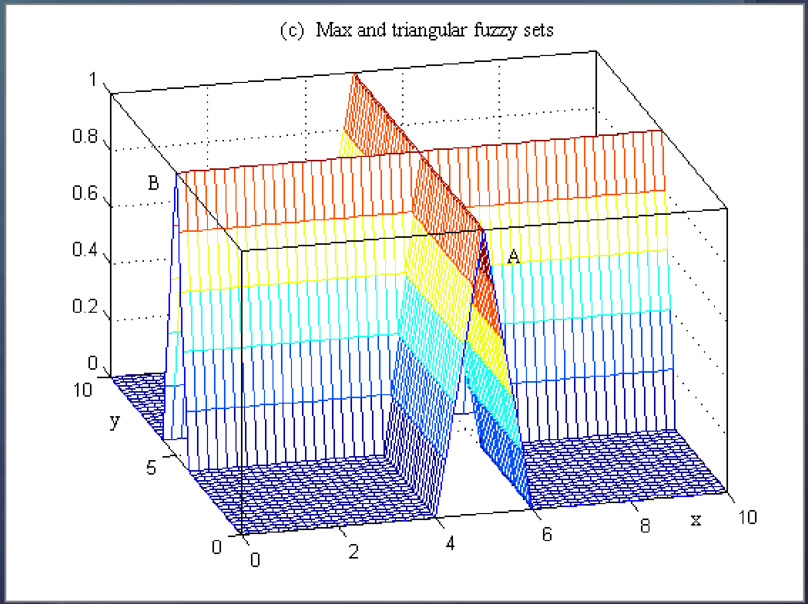
$$R_\ell(x,y) \equiv f_\ell(x,y) = \min[1, A(x) + B(y)]$$

Example: $s = \max$



$$R_\ell(x,y) = \max\{A(x), B(y)\}$$

$$\forall\ (A(x), B(y)) \in [0,1]^2$$

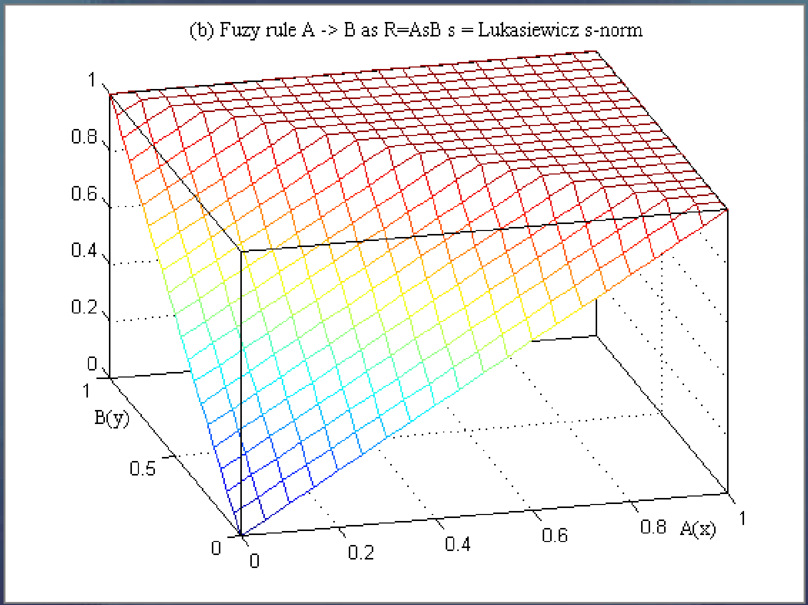


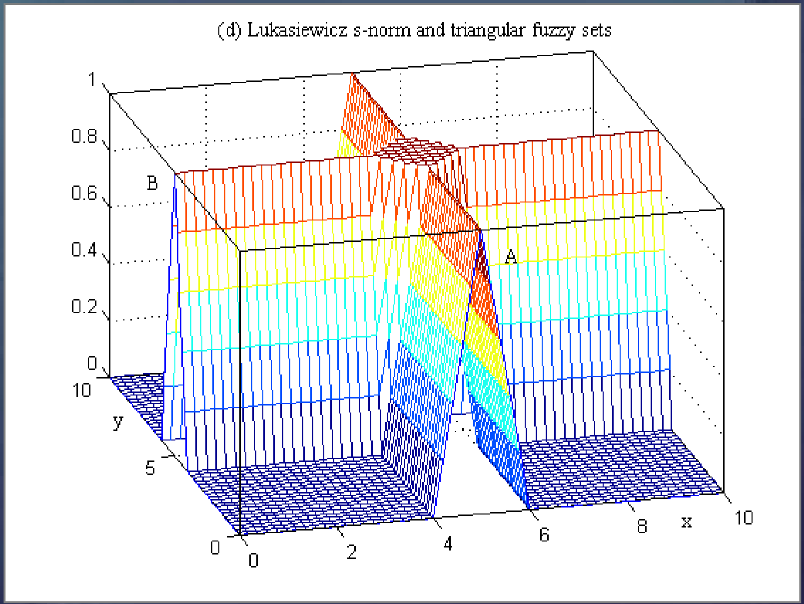
$$R_\ell(x,y) = \max\ \{A(x), B(y)\}$$

$$A(x) = A(x,4,5,6)$$

$$B(y) = B(y,4,5,6)$$

Example: *s* = Lukasiewicz





$R_c(x,y) = \min\{1, A(x)+B(y)\}$

$\forall\ (A(x), B(y)) \in [0,1]^2$

$R_c(x,y) = \min\{1, A(x)+B(y)\}$

$A(x) = A(x,4,5,6)$

$B(y) = B(y,4,5,6)$

Fuzzy implication

- Choose a fuzzy implication f_i and define:

$$R_i(x,y) \equiv f_i(x,y) \quad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

- $f_i : [0,1]^2 \to [0,1]$ is a fuzzy implication if:

1. $B(y_1) \leq B(y_2) \Rightarrow f_i(A(x), B(y_1)) \leq f_i(A(x), B(y_2))$ — monotonicity 2nd argument
2. $f_i(0, B(y)) = 1$ — dominance of falsity
3. $f_i(1, B(y)) = B(y)$ — neutrality of truth

- Further requirements may include:

4. $A(x_1) \le A(x_2) \Rightarrow f_i(A(x_1), B(y)) \ge f_i(A(x_2), B(y))$ — monotonicity 1st argument

5. $f_i(A(x_1), f_i(A(x_2), B(y)) = f_i(A(x_2), f_i(A(x_1), B(y))$ — exchange

6. $f_i(A(x), A(x)) = 1$ — identity

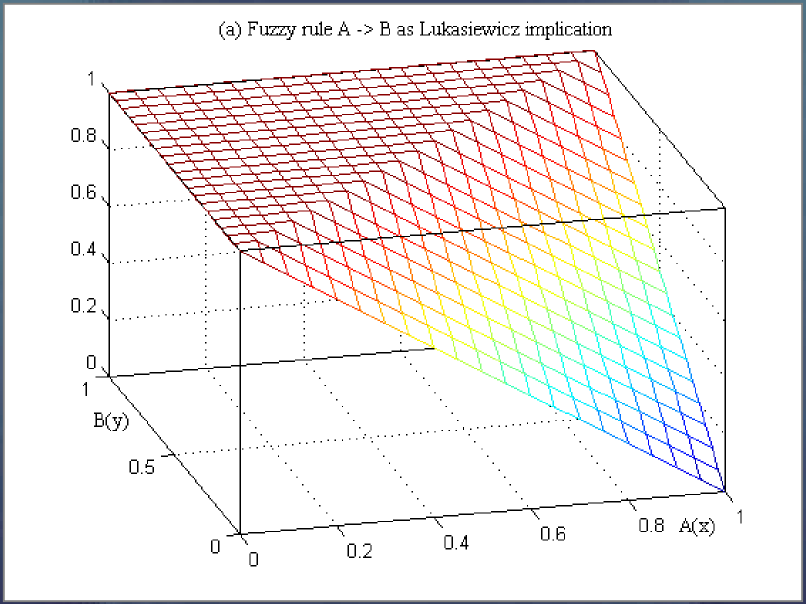
7. $f_i(A(x), B(y)) = 1 \Leftrightarrow A(x) \le B(y)$ — boundary condition

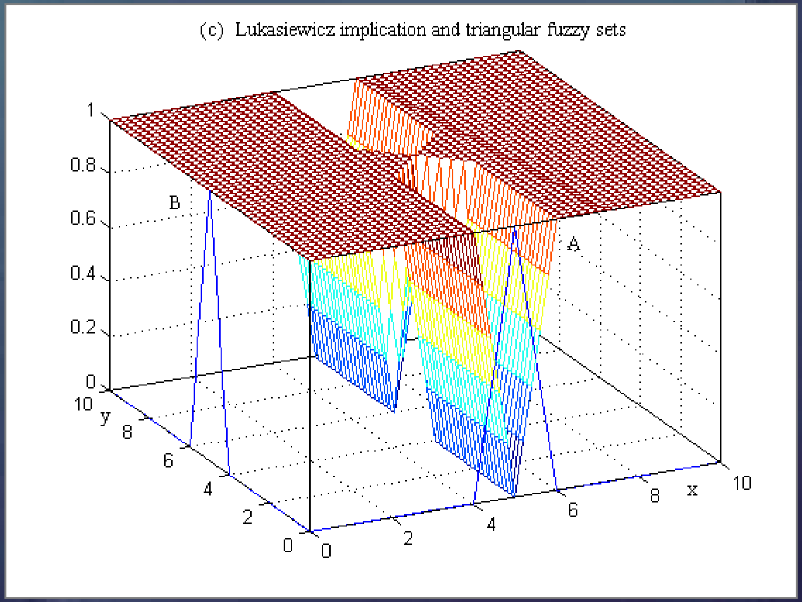
8. f_i is a continuous function — continuity

Examples of fuzzy implications

Name	Definition	Comment
Lukasiewicz	$f_\ell(A(x), B(y)) = \min\,[1, 1 - A(x) + B(y)]$	
Pseudo-Lukasiewicz	$f_\lambda(A(x), B(y)) = \min\left[1, \dfrac{1 - A(x) + (\lambda+1)B(y)}{1 + \lambda A(x)}\right]$	$\lambda > -1$
Pseudo-Lukasiewicz	$f_w(A(x), B(y)) = \min\,[1, (1 - A(x)^w + B(y)^w)^{1/w}]$	$w > 0$
Gaines	$f_a(A(x), B(y)) = \begin{cases} 1 & \text{if } A(x) \le B(y) \\ 0 & \text{otherwise} \end{cases}$	
Gödel	$f_g(A(x), B(y)) = \begin{cases} 1 & \text{if } A(x) \le B(y) \\ B(y) & \text{otherwise} \end{cases}$	
Goguen	$f_e(A(x), B(y)) = \begin{cases} 1 & \text{if } A(x) \le B(y) \\ \dfrac{B(y)}{A(x)} & \text{otherwise} \end{cases}$	
Kleene	$f_b(A(x), B(y)) = \max\,[1 - A(x), B(y)]$	
Reichenbach	$f_r(A(x), B(y)) = 1 - A(x) + A(x)B(y)]$	
Zadeh	$f_z(A(x), B(y)) = \max\,[1 - A(x), \min\,(A(x), B(y))]$	
Klir-Yuan	$f_k(A(x), B(y)) = 1 - A(x) + A(x)^2 B(y)$	

Example: f_ℓ = Lukasiewicz





$R_\ell(x,y) = \min\{1, 1 - A(x)+B(y)\}$

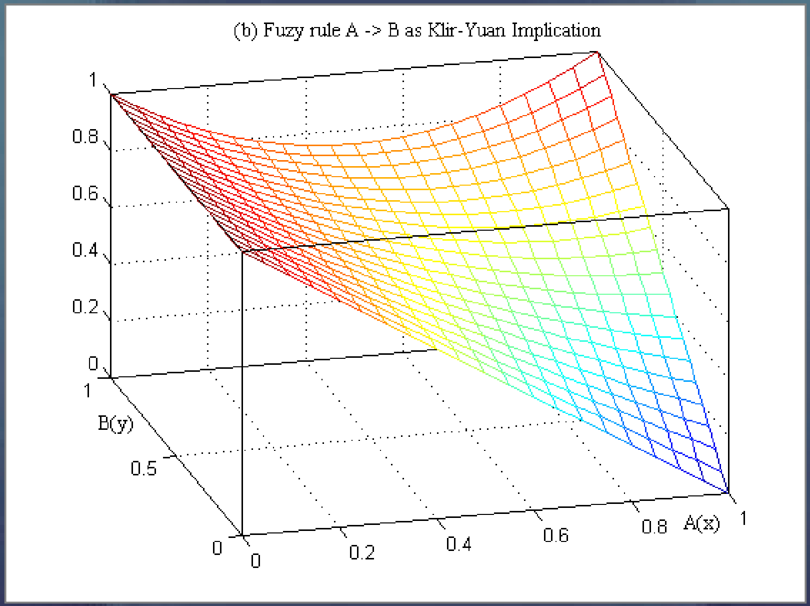
$\forall\ (A(x), B(y)) \in [0,1]^2$

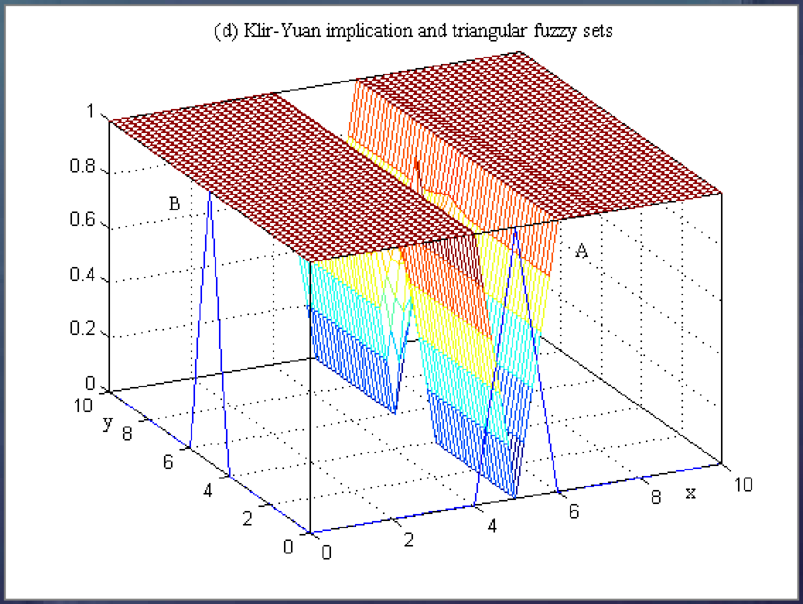
$R_\ell(x,y) = \min\{1, 1 - A(x)+B(y)\}$

$A(x) = A(x,4,5,6)$

$B(y) = B(y,4,5,6)$

Example: f_k = Klir–Yuan





$R_k(x,y) = 1 - A(x) + A(x)^2 B(y)$

$\forall\ (A(x), B(y)) \in [0,1]^2$

$R_k(x,y) = 1 - A(x) + A(x)^2 B(y)$

$A(x) = A(x,4,5,6)$

$B(y) = B(y,4,5,6)$

- Categories of fuzzy implications:

1. s-implications

$$f_{is}(A(x),B(y)) = \overline{A}(x)sB(y) \qquad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

$$f_b(A(x),B(y)) = \max[1-A(x),B(y)] \qquad \text{Kleene}$$

$$f_g(A(x),B(y)) = \min\{1, 1-A(x)+B(y)\} \qquad \text{Lukasiewicz}$$

2. r-implications

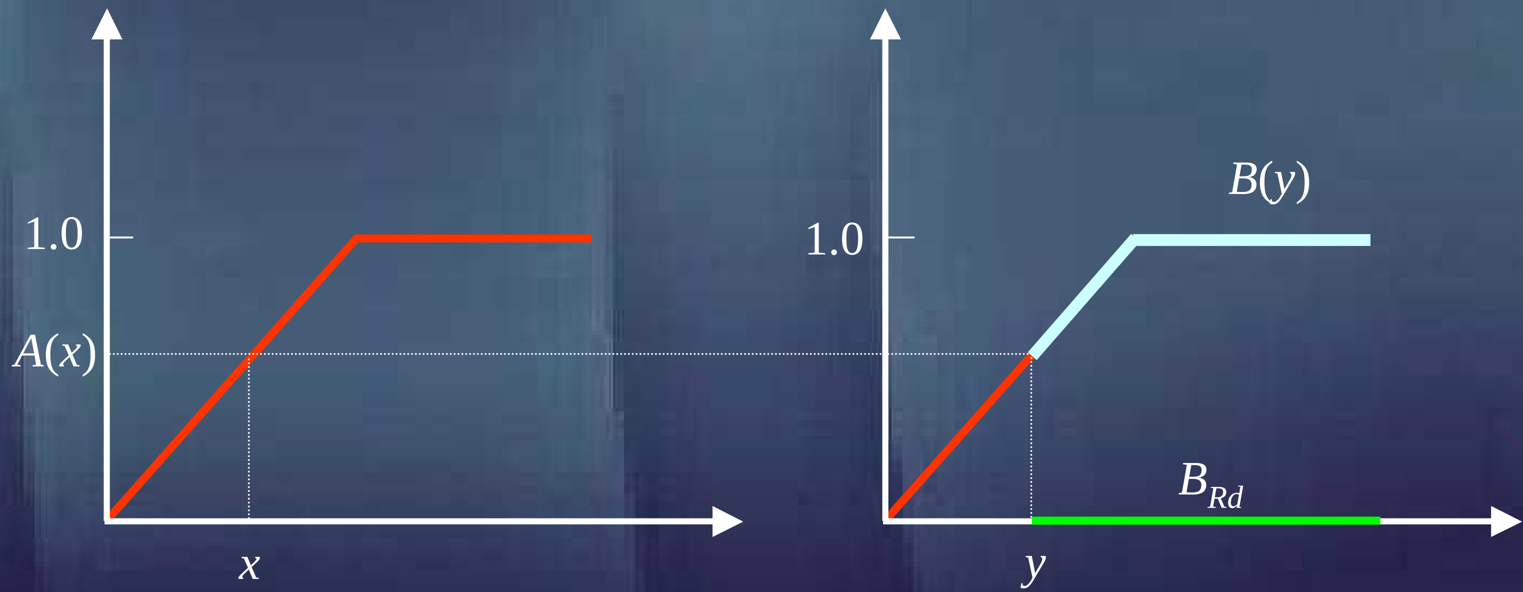
$$f_{ir}(A(x),B(y)) = \sup[c \in [0,1] \mid A(x)\,t\,c \le B(y)] \qquad \forall (x,y) \in \mathbf{X}\times\mathbf{Y}$$

$$t = \min$$

$$f_g(A(x),B(y)) = \begin{cases} 1 & \text{if } A(x) \le B(y) \\ B(y) & A(x) > B(y) \end{cases} \qquad \text{Gödel}$$

Semantics of gradual rules

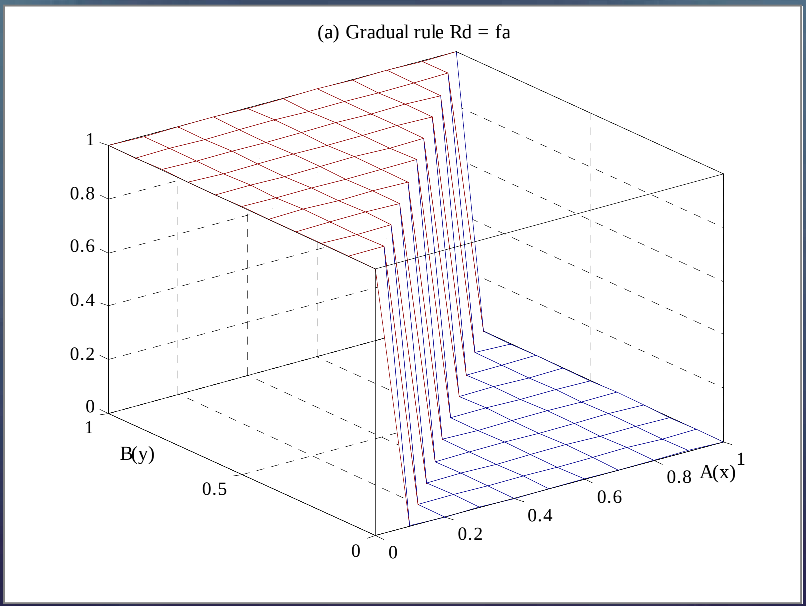
the *more* X is A, the *more* Y is B $\Rightarrow$ $B(y) \geq A(x)$ $\forall x \in \mathbf{X}$ and $\forall y \in \mathbf{Y}$

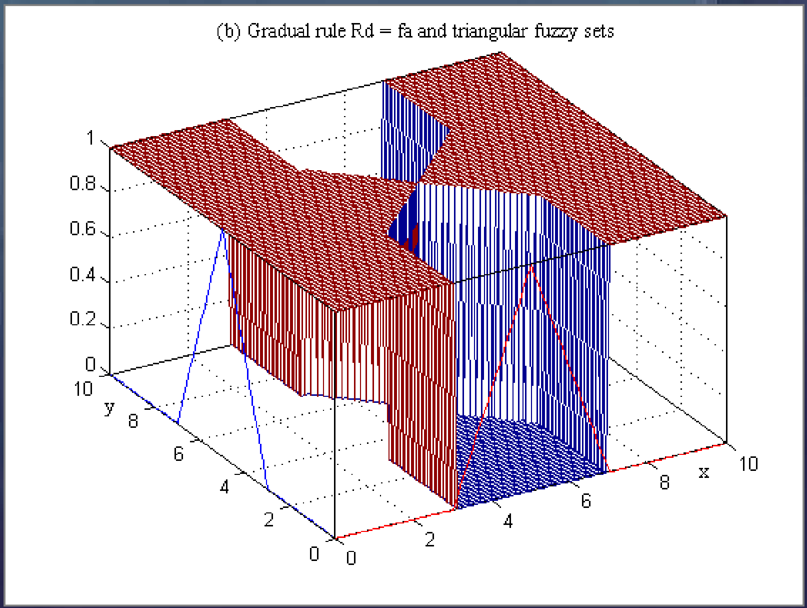


$$B_{Rd} = \{y \in \mathbf{Y} \mid B(y) \geq A(x)\} \text{ for each } x \in \mathbf{X}$$

Example: $R_d = f_a$ = Gaines

$$R_d(x, y) = \begin{cases} 1 & \text{if } B(y) \geq A(x) \\ 0 & \text{otherwise} \end{cases}$$





$R_d(x,y)$

$\forall\ (A(x), B(y)) \in [0,1]^2$

$R_d\,(x,y)$

$A(x) = A(x,3,5,7)$

$B(y) = B(y,3,5,7)$

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Main types of rule bases

- Fuzzy rule base ≡ $\{R_1, R_2,....,R_N\}$ ≡ finite family of fuzzy rules
- Fuzzy rule base can assume various formats:

1. fuzzy graph

R_i: **If** X is A_i **then** Y is B_i is a fuzzy granule in $\mathbf{X}\times\mathbf{Y}$, $i = 1,...,N$

2. fuzzy implication rule base

R_i: **If** X is A_i **then** Y is B_i is fuzzy implication, $i = 1,...,N$

3. functional fuzzy rule base

R_i: **If** X is A_i **then** $y = f_i(x)$ is a functional fuzzy rule, $i = 1,...,N$

Fuzzy graph

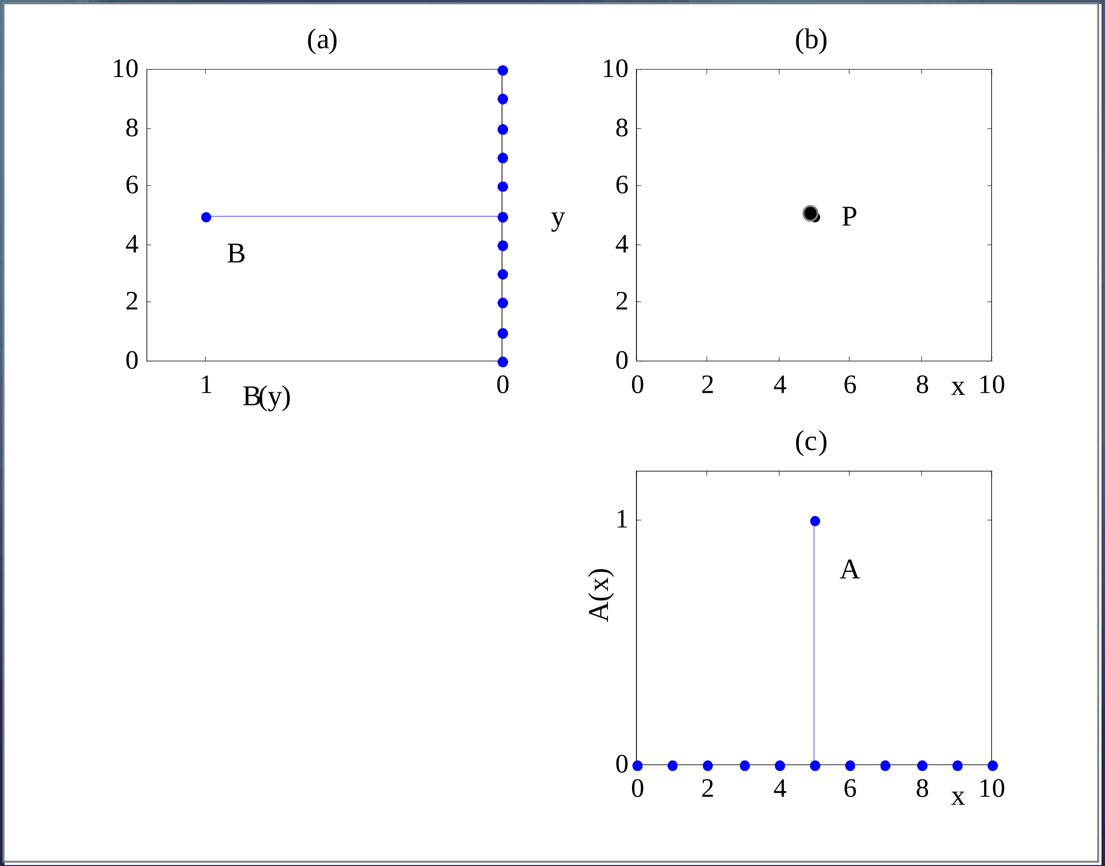
- Fuzzy rule base $R \equiv$ collection of rules $R_1, R_2,....,R_N$
- Each fuzzy rule R_i is a fuzzy granule (point)
- Fuzzy graph $\equiv R$ is a collection of fuzzy granules
  - granular approximation of a function

$$R = \bigcup_{i=1}^{N} R_i = \bigcup_{i=1}^{N} (A_i \times B_i)$$

  - $R = R_1 \text{ or } R_2 \text{ or....or } R_N$
  - general form

$$R(x, y) = \underset{i=1}{\overset{N}{S}} [A_i(x) t B_i(y)]$$

Point



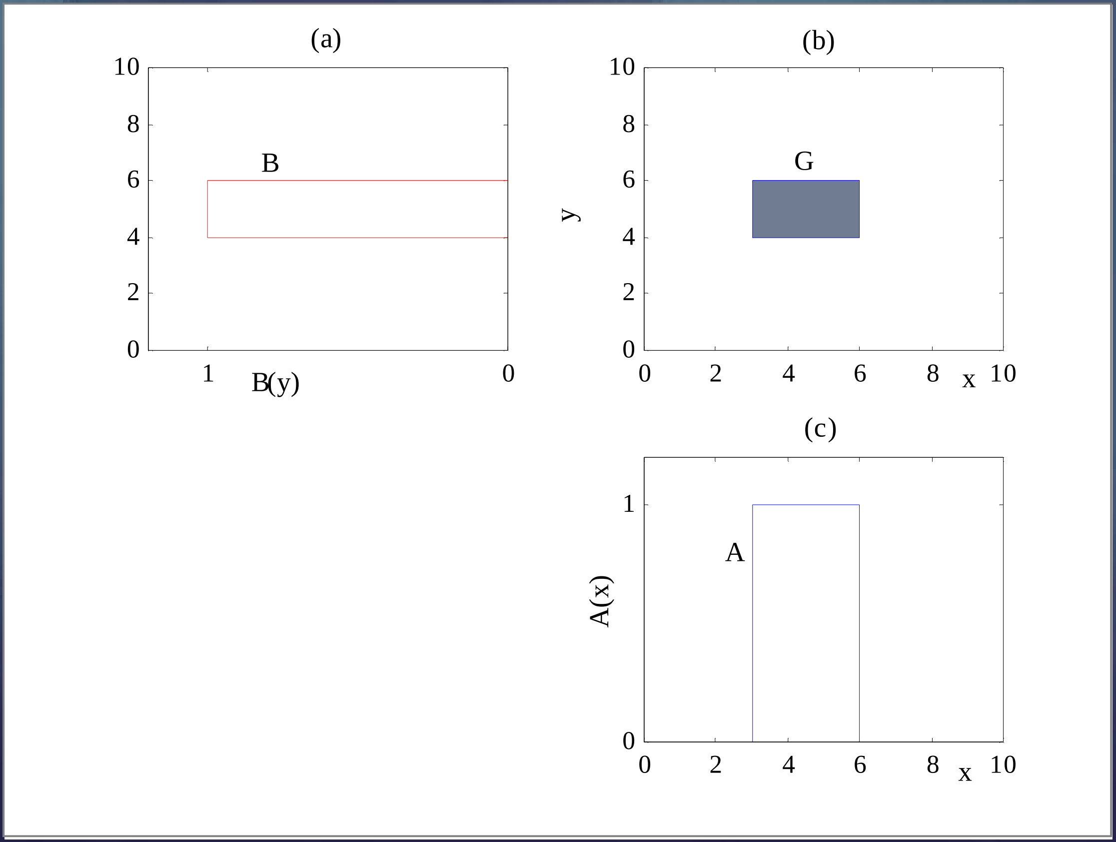
Point P in $\mathbf{X}\times\mathbf{Y}$

$P = A\times B$

A is a singleton in $\mathbf{X}$

B is a singleton in $\mathbf{Y}$

Granule



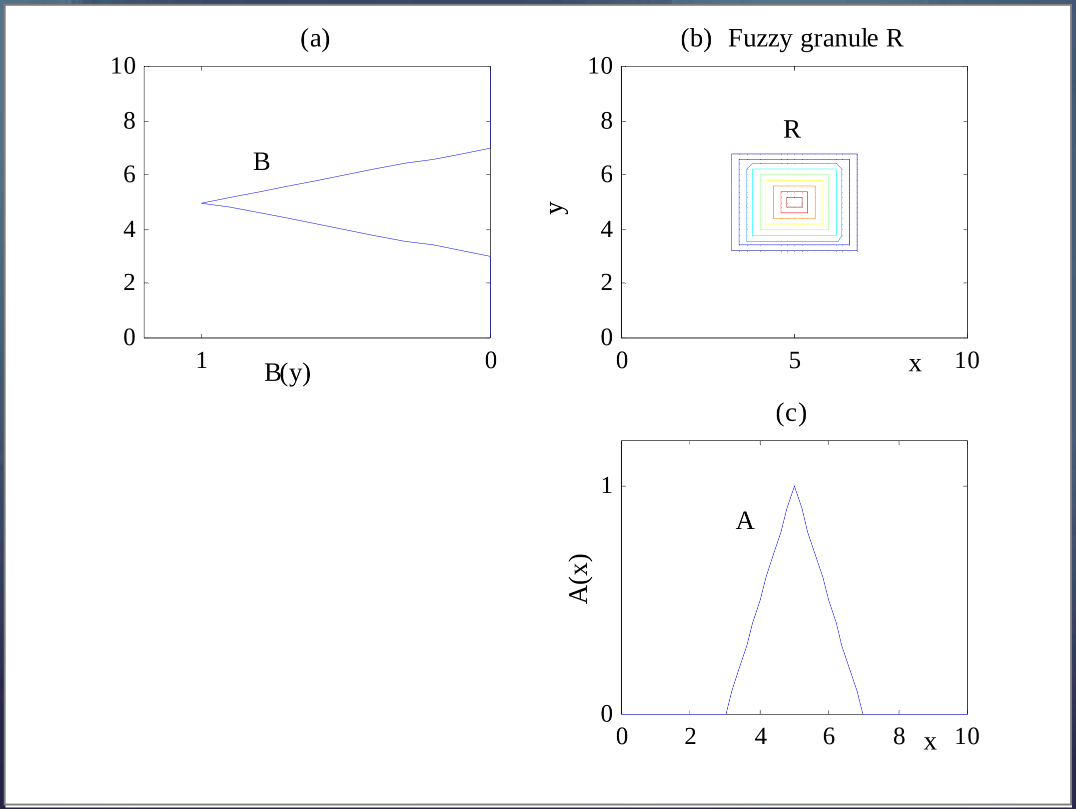
Granule G in $\mathbf{X}\times\mathbf{Y}$

$G = A\times B$

A is an interval in $\mathbf{X}$

B is an interval in $\mathbf{Y}$

Fuzzy granules ≡ fuzzy points



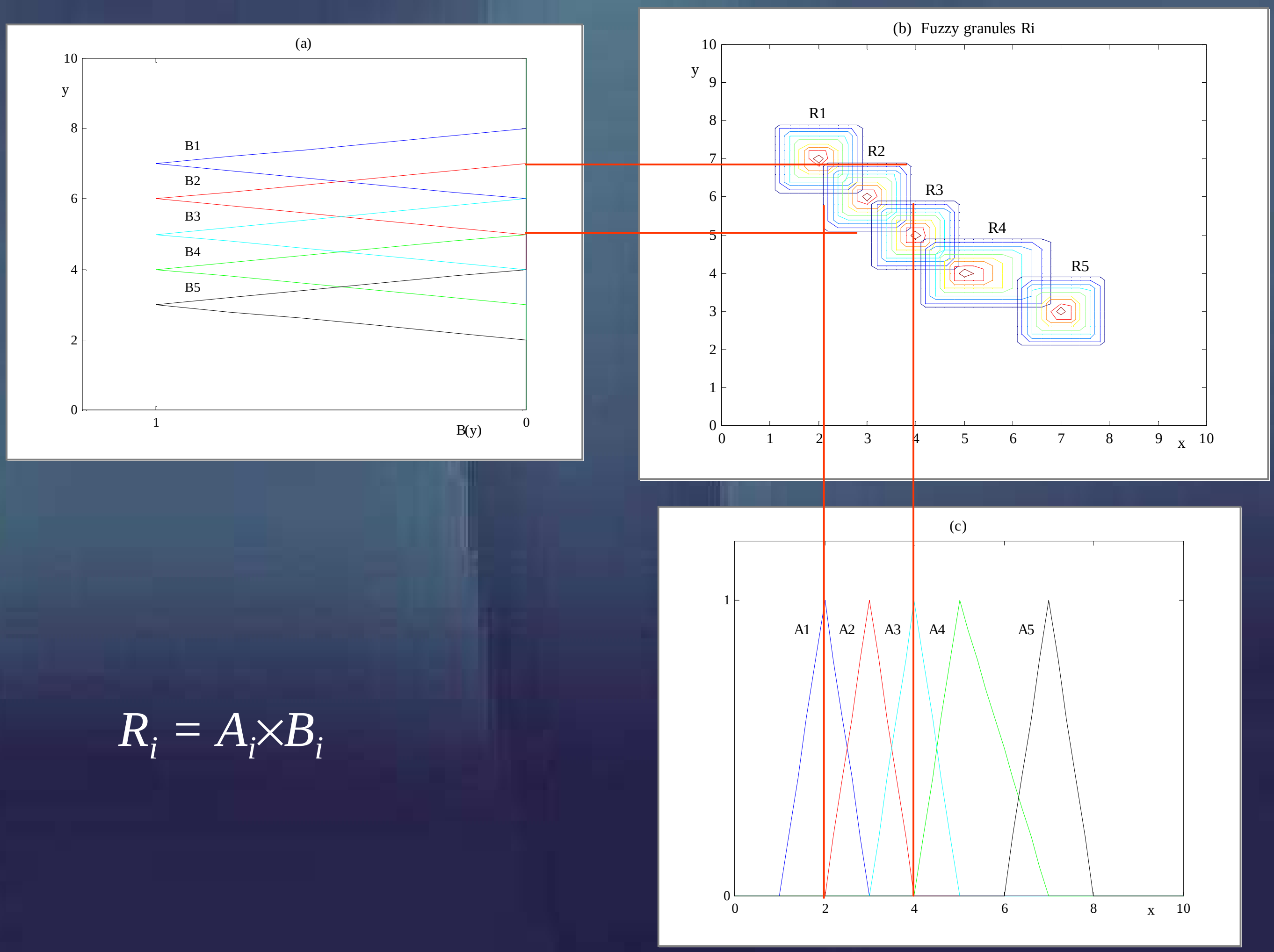
fuzzy granule R in $\mathbf{X}\times\mathbf{Y}$

$R = A\times B$

A is a fuzzy set on $\mathbf{X}$

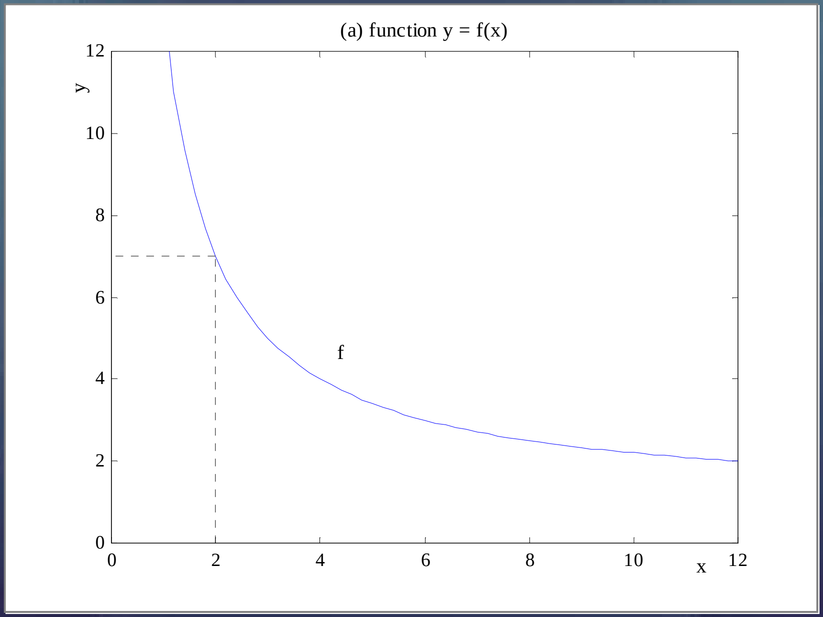
B is a fuzzy set on $\mathbf{Y}$

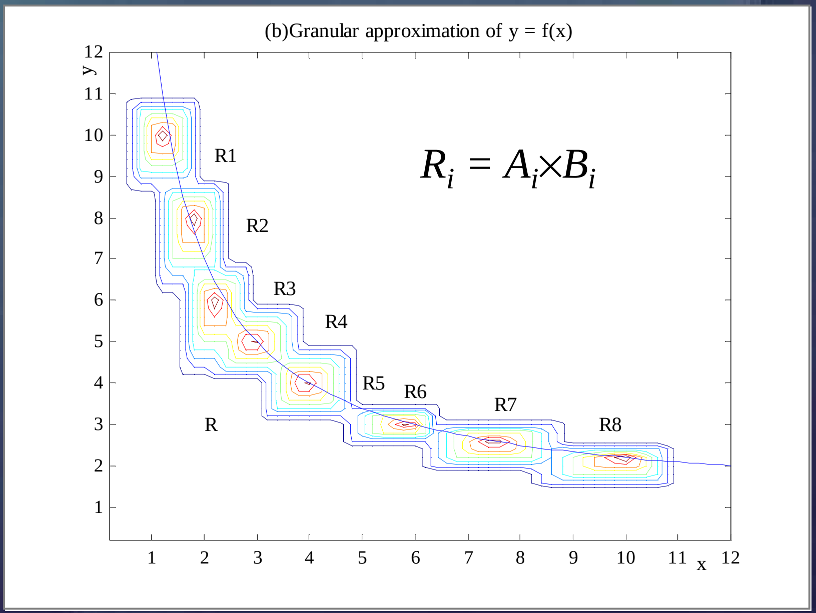
Fuzzy rule base as a set fuzzy granules



$$R_i = A_i \times B_i$$

Graph of a function *f* and its granular approximation *R*



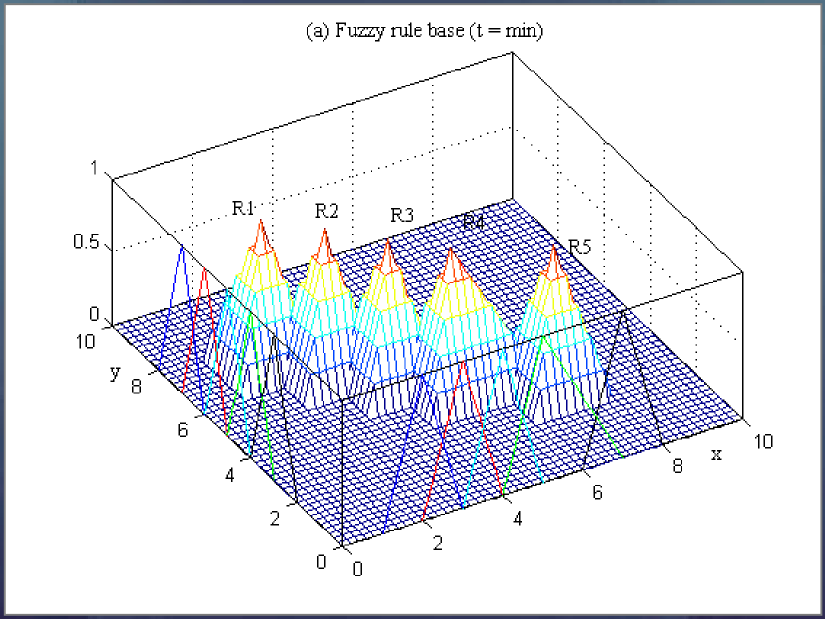


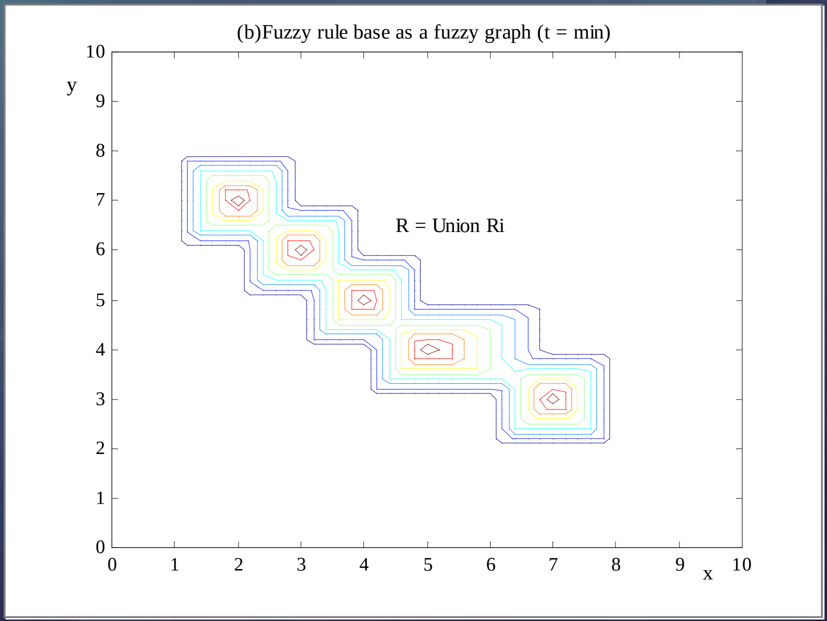
f

R

Fuzzy rule base and fuzzy graph

Example 1



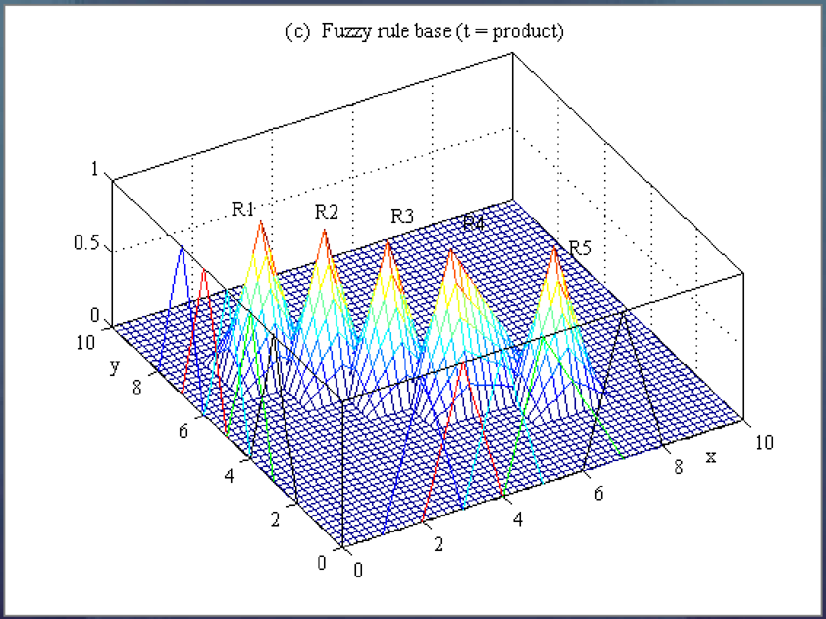


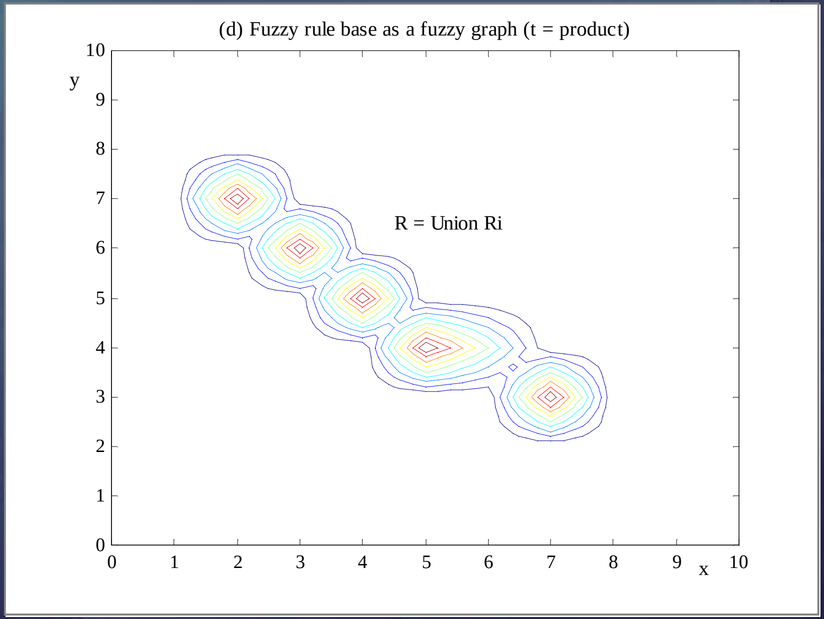
$R_i = A_i \times B_i \;\Rightarrow\; R_i(x,y) = \min\,[A_i(x), B_i(y)]$

$R = \;\cup\, R_i \;\Rightarrow\; R(x,y) = \max\,[R_i(\mathrm{x,y}), i = 1,\ldots, N\,]$

Fuzzy rule base and fuzzy graph

Example 2





$$R_i = A_i \, t \, B_i \;\Rightarrow R_{\mathrm{i}}(x,y) = A_i(x)\, B_i(y)$$

$$R = \;\cup R_i \;\Rightarrow\; R(x,y) = \max\,[R_i(x,y),\, i = 1,\ldots,\, N\,]$$

Fuzzy implication

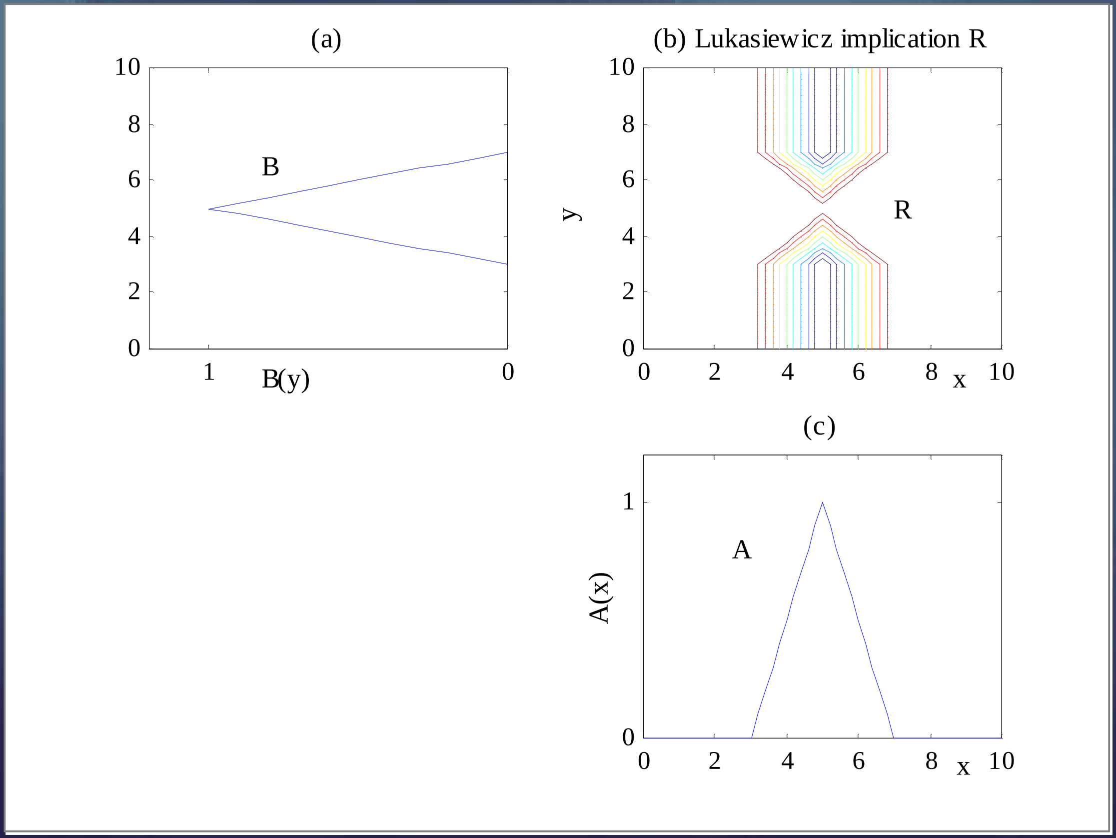
- Fuzzy rule base $R \equiv$ collection of rules $R_1, R_2, ..., R_N$
- Each fuzzy rule R_i is a fuzzy implication
- Fuzzy rule base R is a collection of fuzzy relations
  - relation R is obtained using intersection

$$R = \bigcap_{i=1}^{N} R_i = \bigcap_{i=1}^{N} f_i = \bigcap_{i=1}^{N} (A_i \Rightarrow B_i)$$

  - $R = R_1$ *and* R_2 *and....and* R_N
  - general form

$$R = \mathop{T}_{i=1}^{N} f_i(A_i(x), B_i(y))$$

Fuzzy rule as an implication



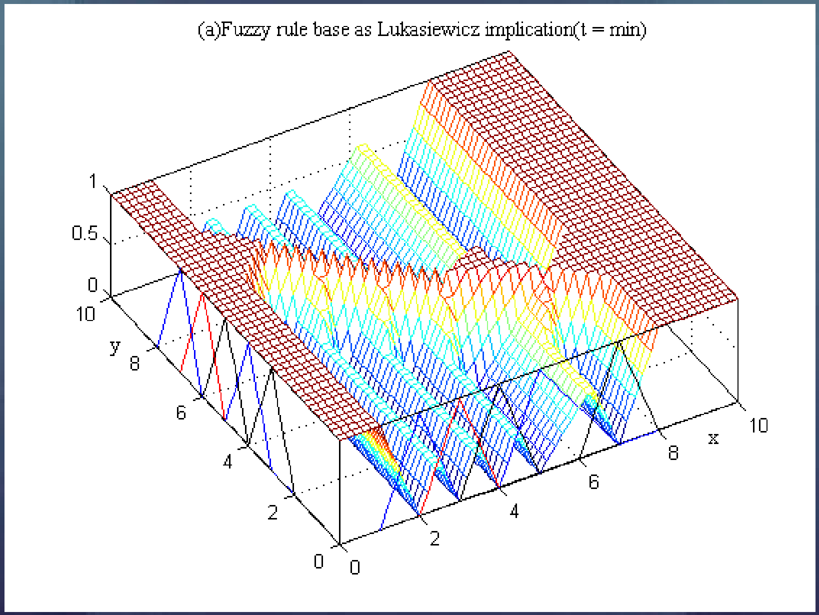
fuzzy rule R in $\mathbf{X}\times\mathbf{Y}$

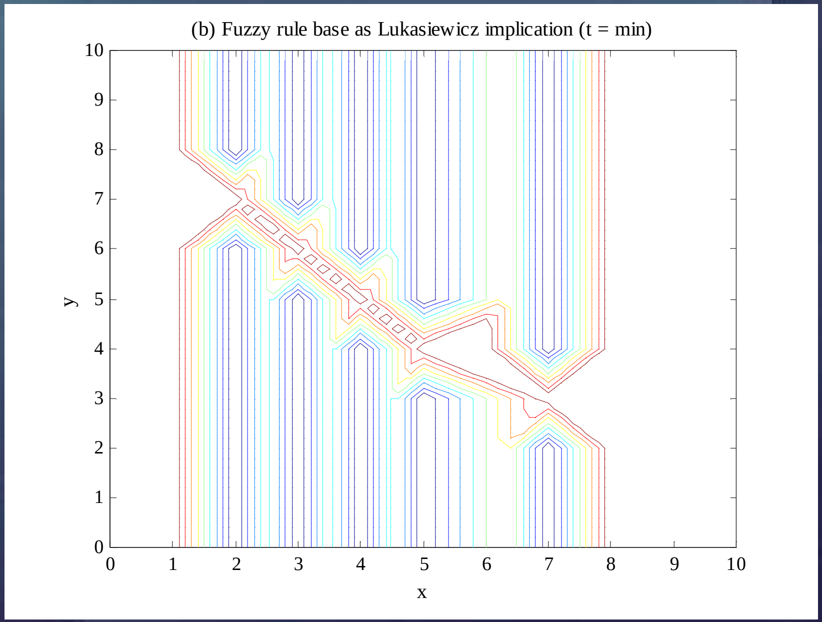
$R = f_{\ell}(A,B)$

Lukasiewicz implication

Fuzzy rule base and fuzzy implication

Example 1a



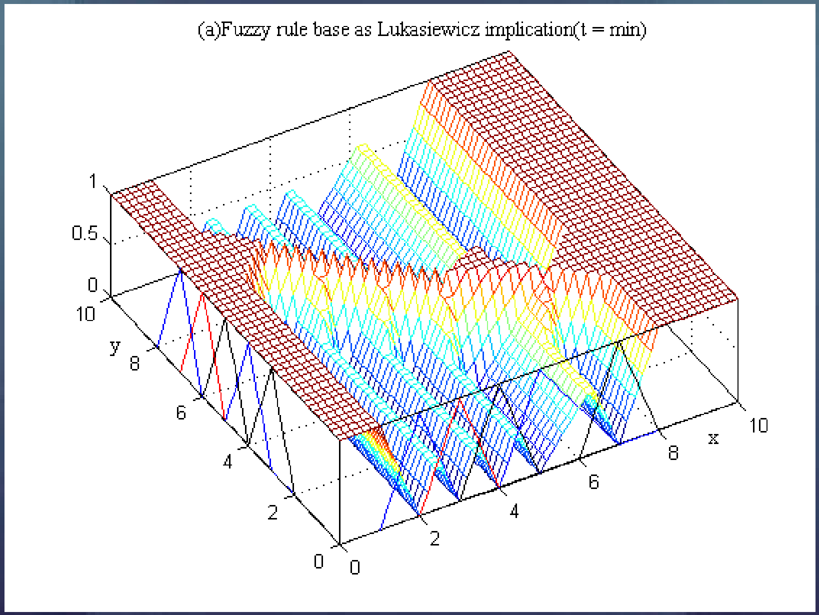


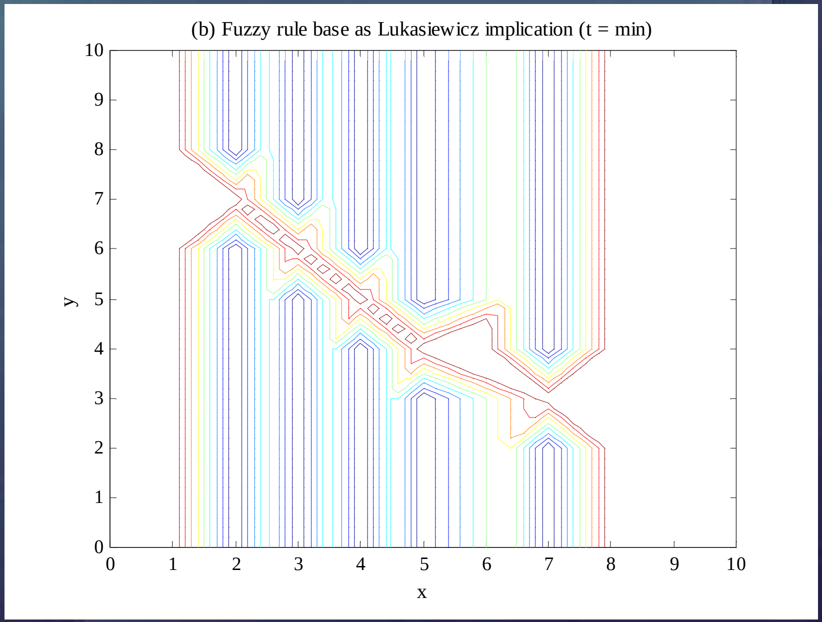
$R_i = f_\ell(A,B) \Rightarrow R_i(x,y) = \min\,[1, 1 - A_i(x) + B_i(y)]$ Lukasiewicz implication

$R = \cap R_i \Rightarrow R(x,y) = \min\,[R_i(x,y), i = 1,\ldots, 5]$ min t-norm

Fuzzy rule base and fuzzy implication

Example 1b



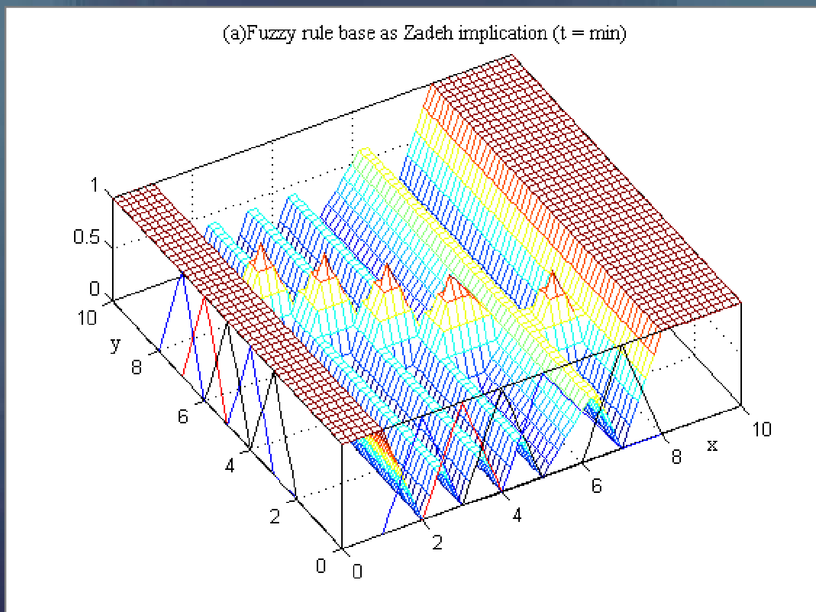


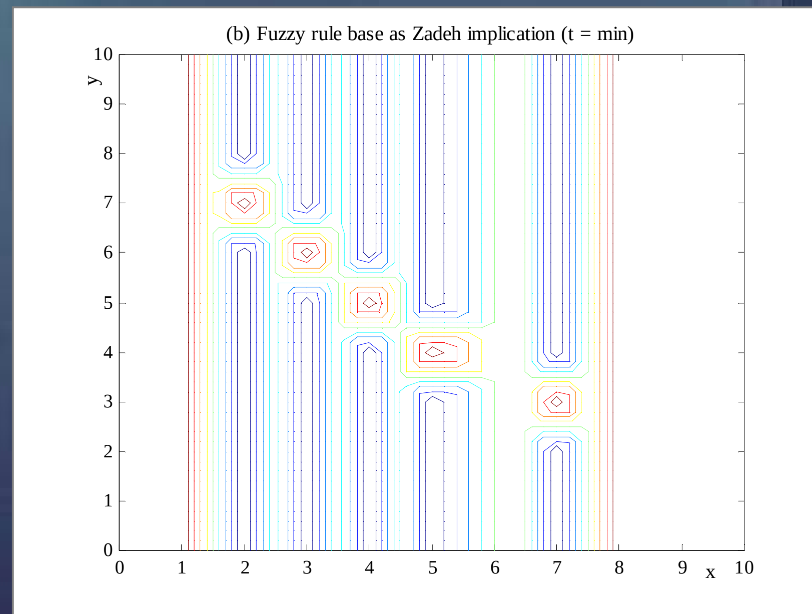
$R_i = f_\ell(A,B) \Rightarrow R_i(x,y) = \min\,[1, 1 - A_i(x) + B_i(y)]$ Lukasiewicz implication

$R = \cap R_i \Rightarrow R(x,y) = R_1(x,y)\ t_l\ R_2(x,y)\ t_l\ ....\ t_l\ R_i(x,y)$ Lukasiewicz t-norm

Fuzzy rule base and fuzzy implication

Example 2a



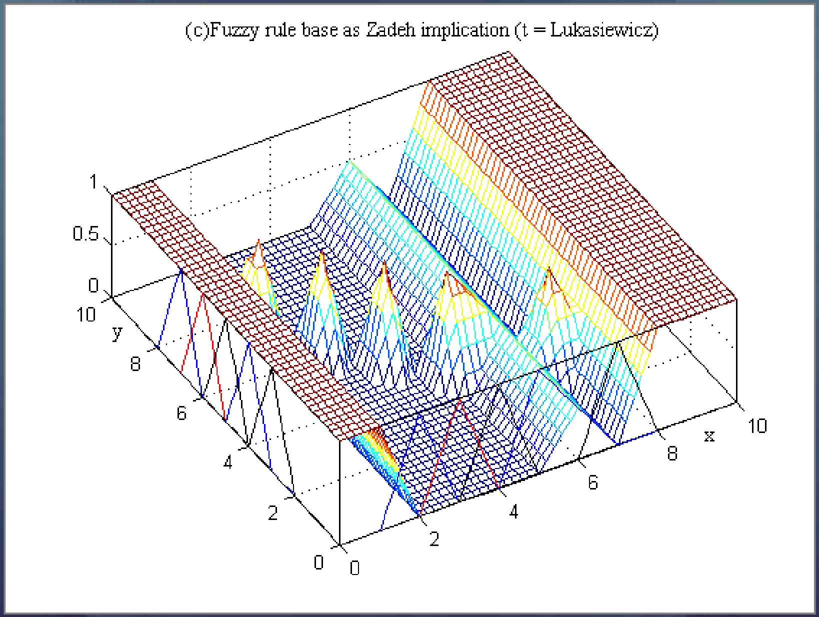


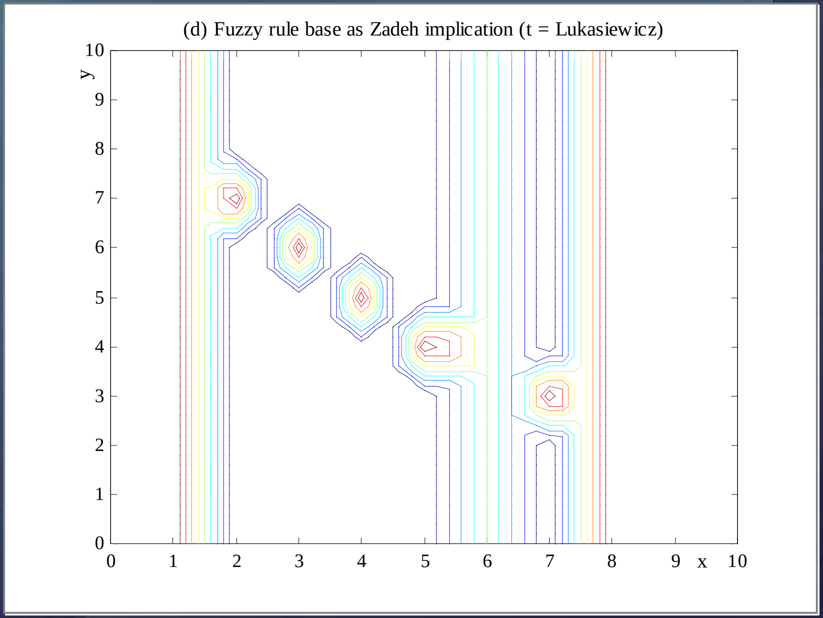
$R_i = f_z(A,B) \Rightarrow R_i(x,y) = \max\,[1 - A_i(x), \min(A_i(x), B_i(y)]$ Zadeh implication

$R = \cap R_i \Rightarrow R(x,y) = \min\,[R_i(x,y), i = 1,\ldots, 5]$ min t-norm

Fuzzy rule base and fuzzy implication

Example 2b





$R_i = f_z(A,B) \Rightarrow R_{\mathrm{i}}(x,y) = \max\,[1 - A_i(x), \min(A_{\mathrm{i}}(x), B_i(y)]$ Zadeh implication

$R = \cap R_i \;\Rightarrow\; R(x,y) = R_1(x,y)\; t_l\; R_2(x,y)\; t_l \;....\; t_l\; R_i(x,y)$ Lukasiewicz t-norm

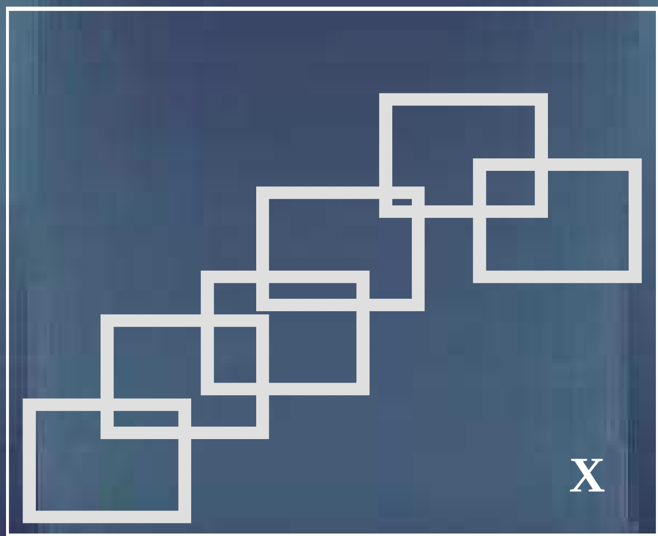
Data base

- Data base contains definitions of:
  - universes
  - scaling functions of input and output variables
  - granulation of the universes membership functions

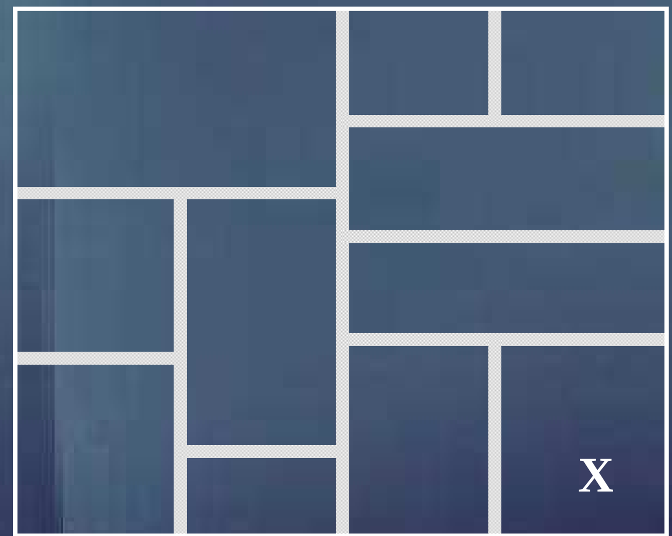
- Granulation
  - granular constructs in the form of fuzzy points
  - granules along different regions of the universes

- Construction of membership functions
  - expert knowledge
  - learning from data

Granulation



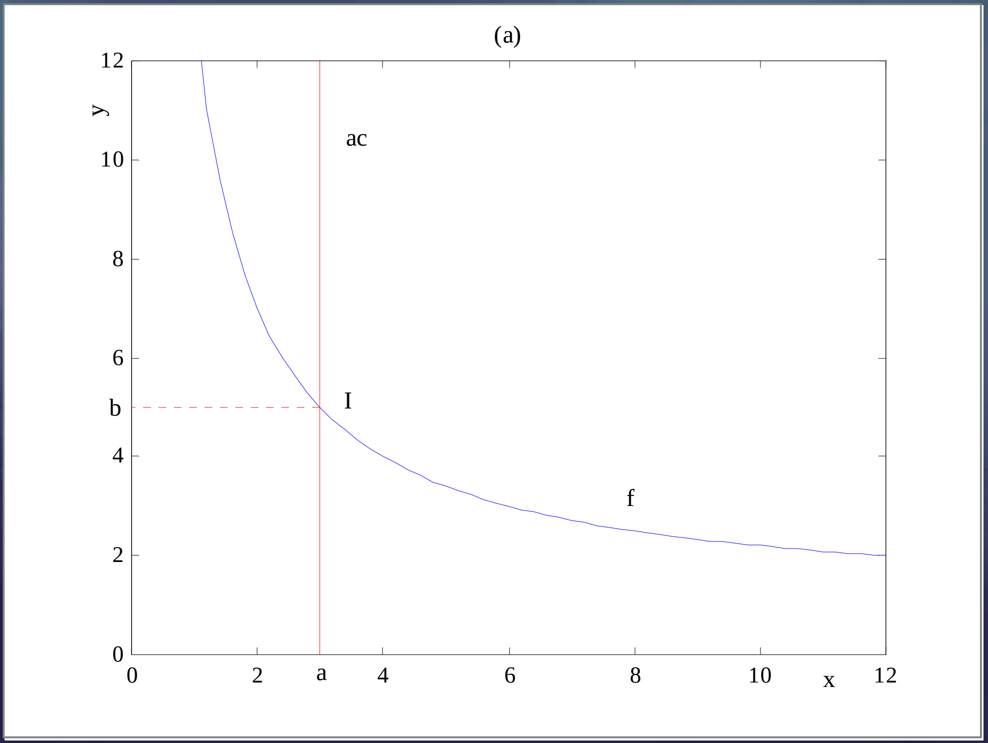
granular constructs in
the form of fuzzy points



granules along different
regions of the universes

Fuzzy inference

- Basic idea of inference



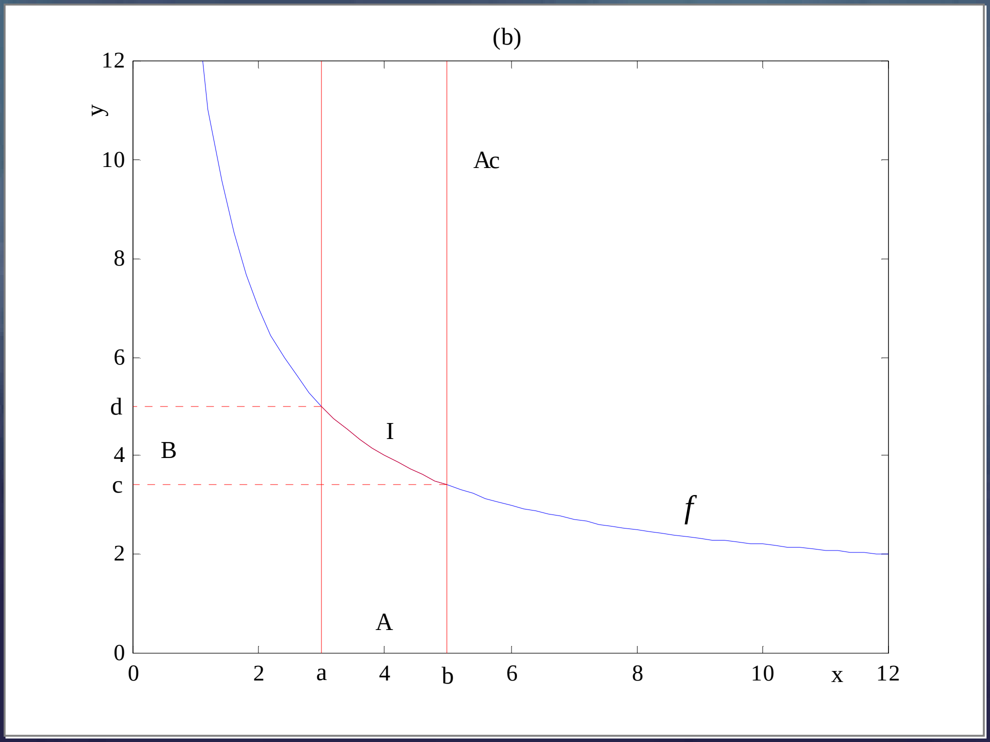
$x = \mathrm{a}$
$y = f(x)$
$y = b$

$$b = \mathrm{Proj}_{\mathbf{Y}}\,(a_c \cap f\,)$$

$$\Downarrow$$

$$b = \mathrm{Proj}_{\mathbf{Y}}\,(\,I\,)$$

Inference involves operations with sets



x = A

$y = f(x)$

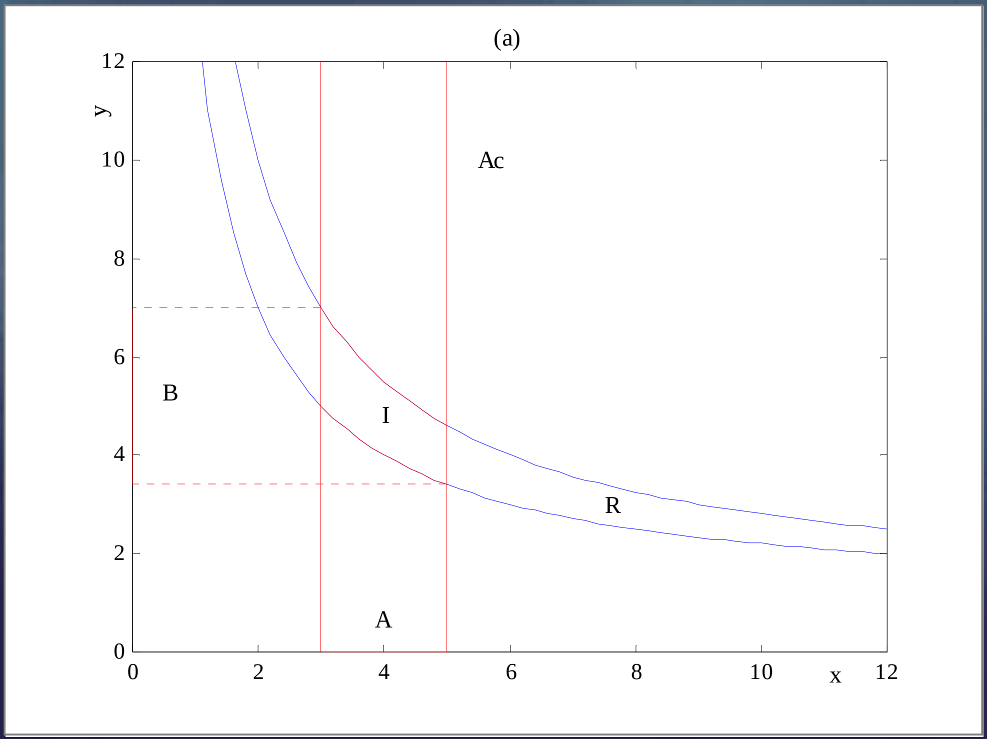
$B = f(A) = \{f(x), x \in A\}$

$B = \text{Proj}_{\mathbf{Y}}(A_c \cap f)$

$\Downarrow$

$B = \text{Proj}_{\mathbf{Y}}(I)$

Inference involving sets and relations



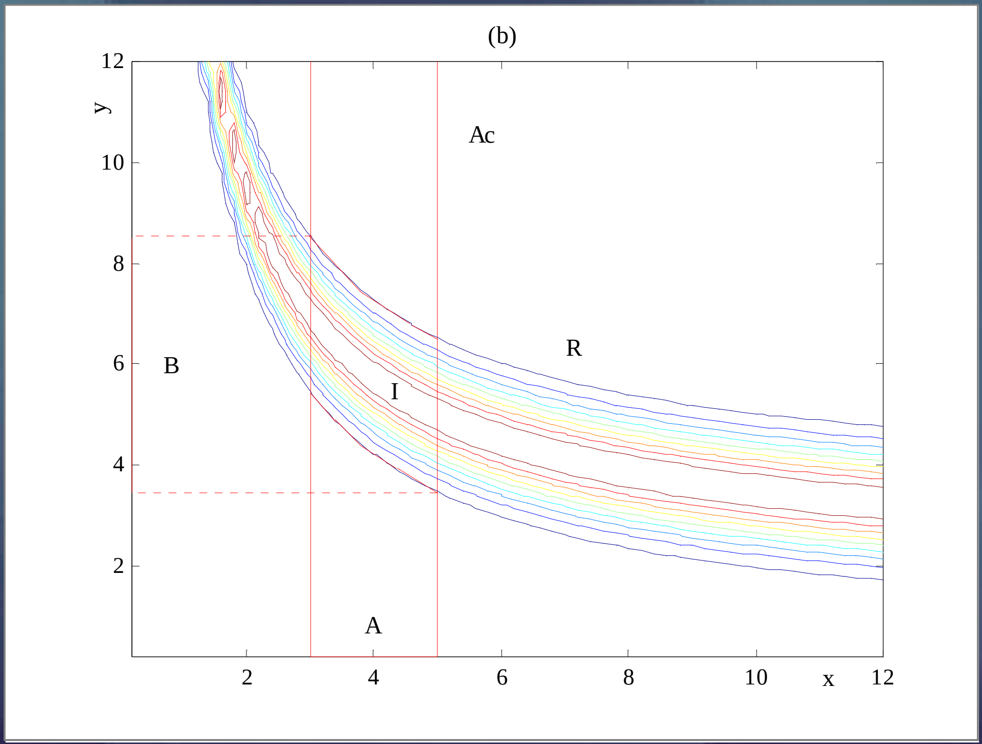
x is A
(x,y) is R
y is B

$$B = \text{Proj}_{\mathbf{Y}} (A_c \cap R)$$

$$\Downarrow$$

$$B = \text{Proj}_{\mathbf{Y}} (I)$$

Fuzzy inference ands operations with fuzzy sets and relations



X is A (fuzzy set on $\mathbf{X}$)
(X,Y) is R (fuzzy relation on $\mathbf{X}\times\mathbf{Y}$)
Y is B (fuzzy set on $\mathbf{Y}$)

$$B = \mathrm{Proj}_{\mathbf{Y}}\,(A_c \cap R)$$

$$\Downarrow$$

$$B = \mathrm{Proj}_{\mathbf{Y}}\,(I) \quad \Rightarrow \quad \boxed{B(y) = \sup_{x\in\mathbf{X}}\{A(x)tR(x,y)\}}$$

Fuzzy inference

- Compositional rule of inference

X is *A*
(*X*,*Y*) is *R*
Y is *B*

$$\boxed{B = A \circ R}$$

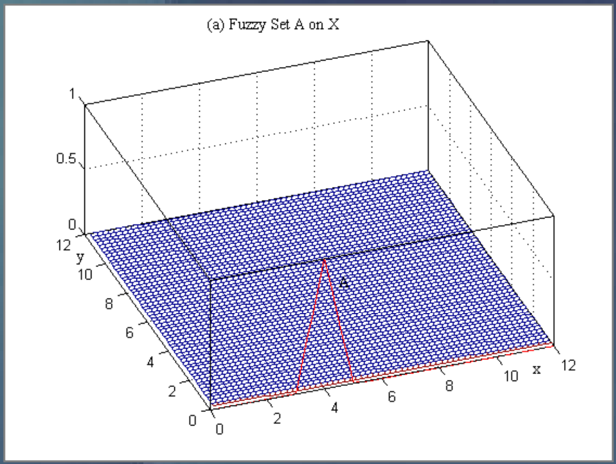
X is *A*
(*X*,*Y*) is *R*
Y is *A*o*R*

Fuzzy inference procedure

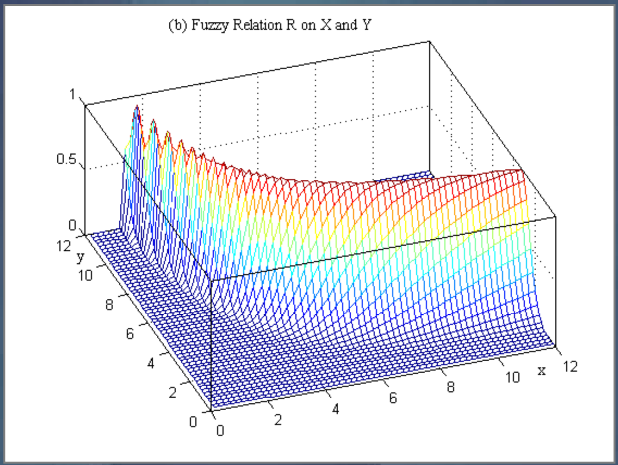
procedure FUZZY-INFERENCE (A, R) **returns** a fuzzy set
input : fuzzy relation: R
fuzzy set: A
local: x, y: elements of **X** and **Y**
t: t-norm

for all x and y **do**
$A_c(x,y) \leftarrow A(x)$
for all x and y **do**
$I(x,y) \leftarrow A_c(x,y)\ t\ R(x,y)$
$B(y) \leftarrow \sup_x I(x,y)$
return B

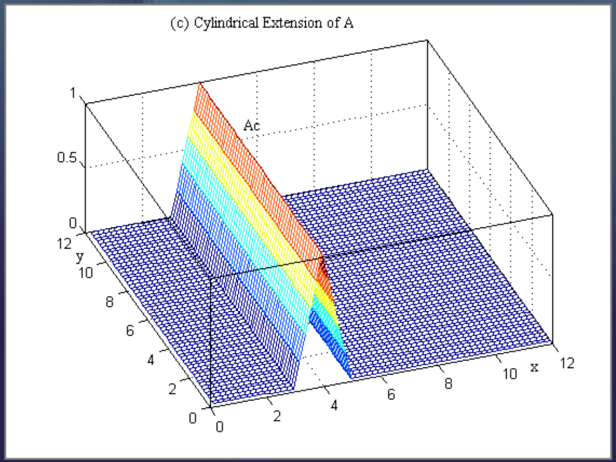
Example: compositional rule of inference



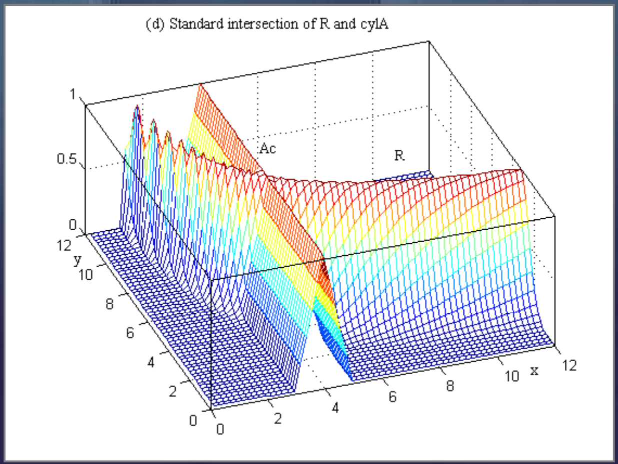
(a) Fuzzy Set A on X



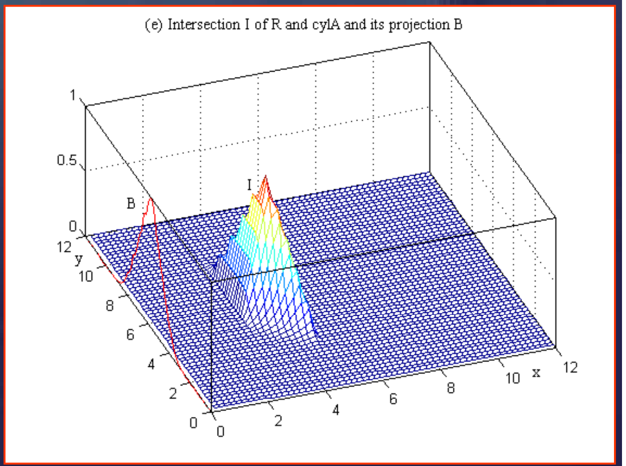
(b) Fuzzy Relation R on X and Y



(c) Cylindrical Extension of A



(d) Standard intersection of R and cylA



(e) Intersection I of R and cylA and its projection B

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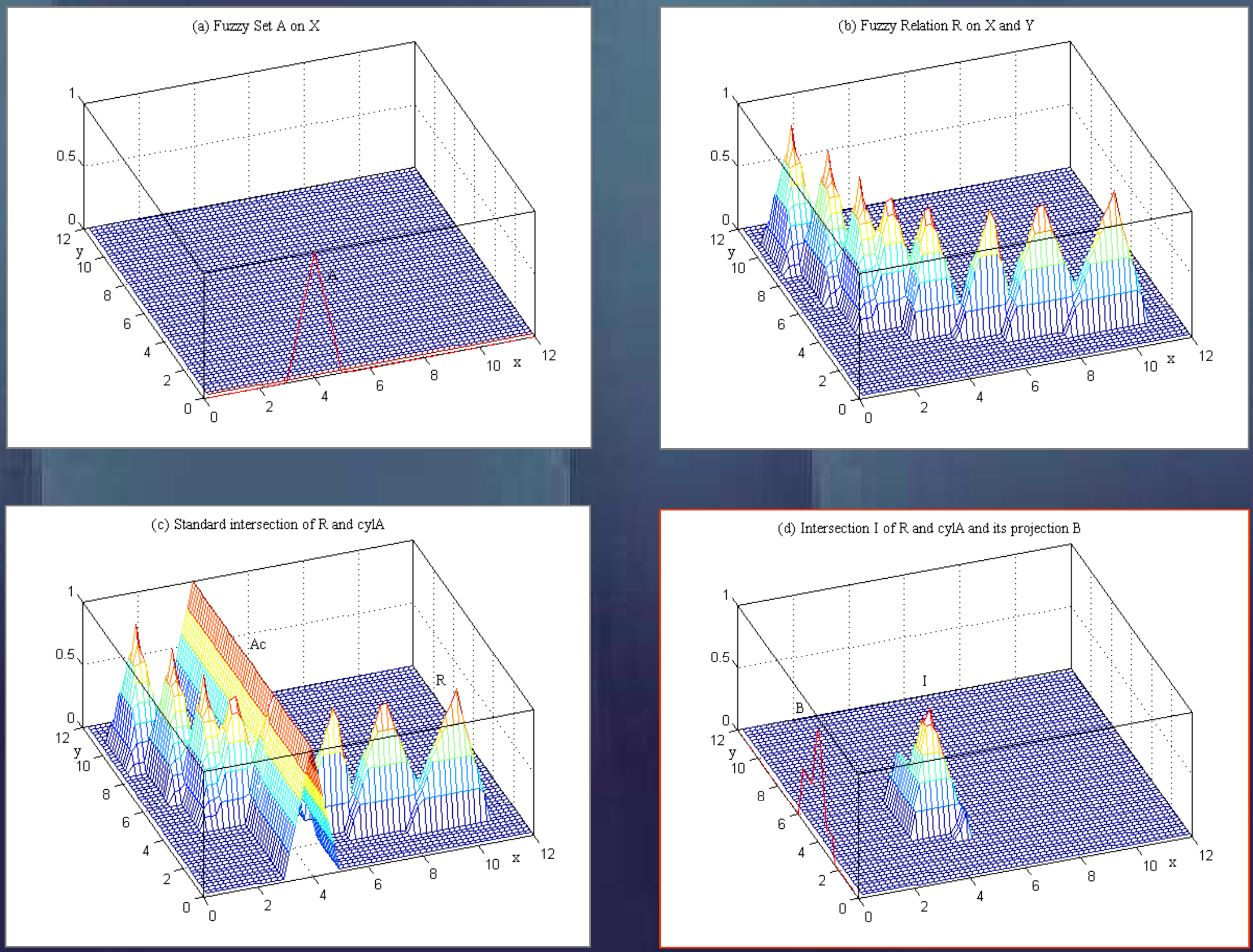
Example: fuzzy inference with fuzzy graph



11.5 Types of rule-based systems and architectures

Linguistic fuzzy models

P:	X is A ***and*** Y is B	input
R_1:	**If** X is A_1 ***and*** Y is B_1 **then** Z is C_1	
	....	
R_i:	**If** X is A_i ***and*** Y is B_i **then** Z is C_i	rule base
	....	
R_N:	**If** X is A_N ***and*** Y is B_N **then** Z is C_N	
Z:	Z is C	output

- all fuzzy sets A, B, A_i,s and B_i,s are given
- rule and connectives (***and***, ***or***) with known semantics
- membership function of fuzzy set C = ??

min-max fuzzy models

Assume

P: X is A ***and*** Y is B

$P(x,y) = \min\{A(x), B(y)\}$

R_i: **If** X is A_i ***and*** Y is B_i **then** Z is C_i

$R_i(x,y,z) = \min\{A_i(x), B_i(y), C_i(z)\}$

$i = 1,\ldots, N$

Using the compositional rule of inference (t = min)

$$C = P \circ R = P \circ \bigcup_{i=1}^{N} R_i$$

$$C(z) = \sup_{x,y}\{\min[P(x,y), \max(R_i(x,y,z), i = 1,\ldots,N)]\}$$

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$$C = P \circ R = P \circ \bigcup_{i=1}^{N} R_i = \bigcup_{i=1}^{N} (P \circ R_i) = \bigcup_{i=1}^{N} C'_i$$

$$C'_i = P \circ R_i$$

$$C'_i(z) = \sup_{x,y}\{\min[P(x,y), R_i(x,y,z)]\} = \sup_{x,y}\{A(x) \wedge B(y) \wedge A_i(x) \wedge B_i(y) \wedge C_i(z)]\}$$

$$\sup_{x}[A(x) \wedge A_i(x)] = \mathrm{Poss}(A, A_i) = m_i$$

$$\sup_{y}[B(y) \wedge B_i(y)] = \mathrm{Poss}(B, B_i) = n_i$$

$$C'_i(z) = m_i \wedge n_i \wedge C_i(z)$$

$$C(z) = \max\{(m_i \wedge n_i)C_i, i = 1,\ldots,N\} = \max\{\lambda_i \wedge C_i(z), i = 1,\ldots,N\}$$

λ_i is the degree of activation of i – th rule

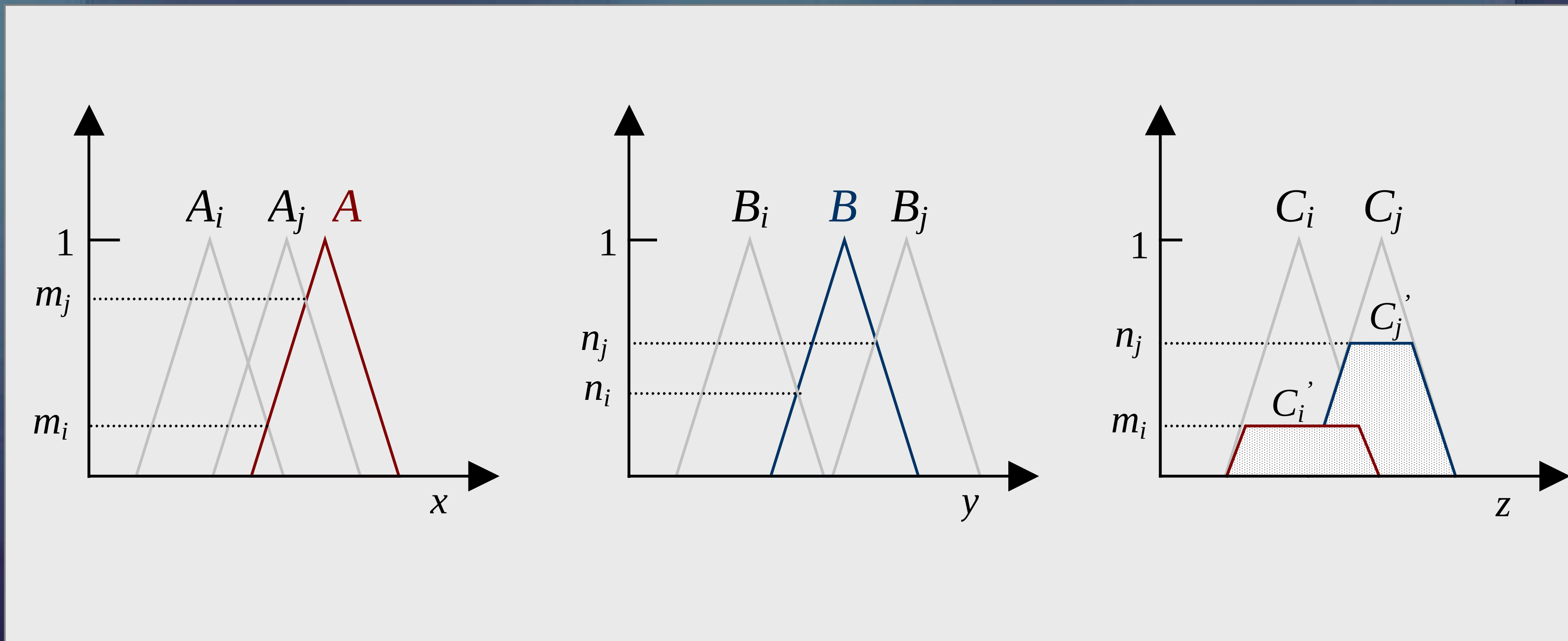
min-max fuzzy model processing

procedure MIN-MAX-MODEL (A,B) **returns** a fuzzy set
local: fuzzy sets: A_i, B_i, C_i, i =1,.., N
activation degrees: λ_i

Initialization $C = \varnothing$

for i = 1: N **do**
m_i = max (min (A, A_i))
n_i = max (min (B, B_i))
λ_i = min (m_i, n_i)
if $\lambda_i \neq 0$ **then** C_i' = min (λ_i , C_i) and C = max(C, C_i')
return C

Example: min-max fuzzy model processing



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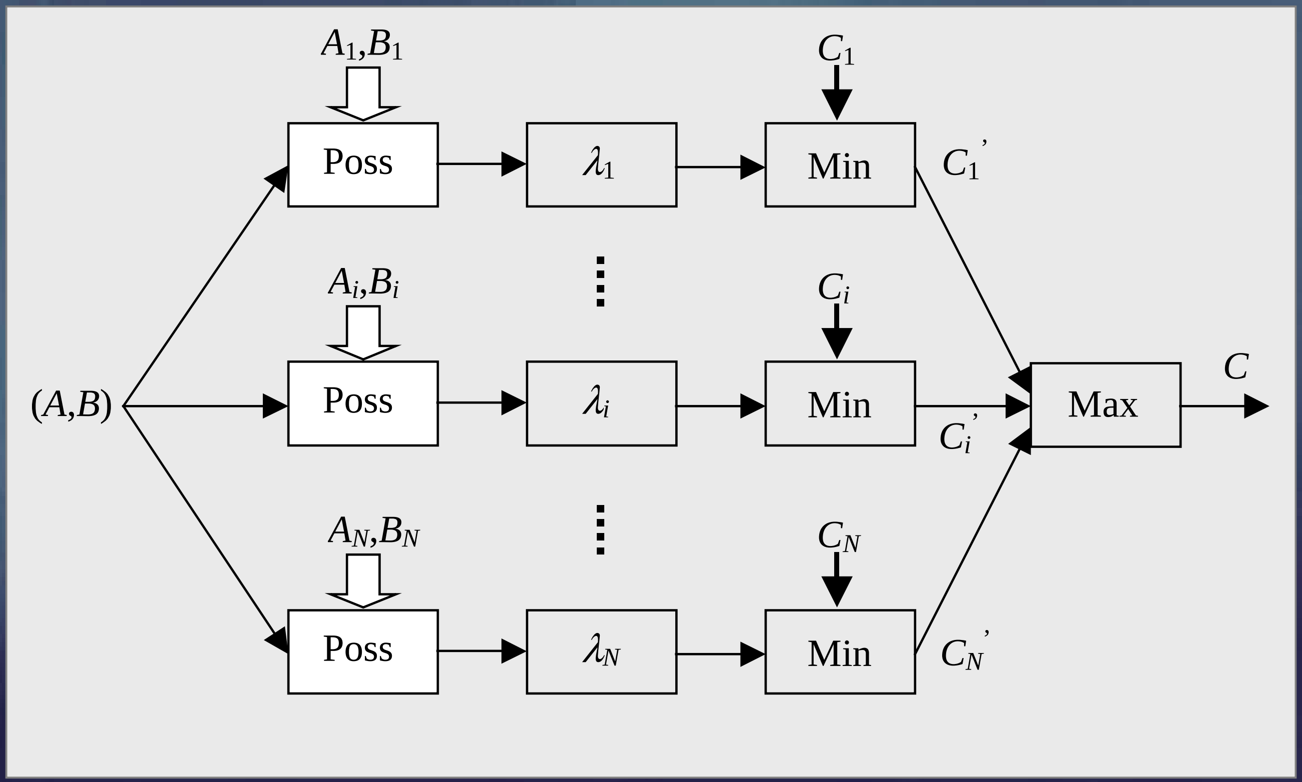
min-max fuzzy model architecture



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Special case: numeric inputs

$$A(x)=\begin{cases}1 & \text{if } x=x_o\\ 0 & \text{otherwise}\end{cases} \quad \textit{and} \quad B(y)=\begin{cases}1 & \text{if } y=y_o\\ 0 & \text{otherwise}\end{cases}$$

Numeric output

$$z=\frac{\int_{\mathbf{Z}} zC(z)dz}{\int_{\mathbf{Z}} C(z)dz} \qquad \text{centroid defuzzification}$$

$$z=\frac{\sum_{i=1}^{N}(m_i \wedge n_i)v_i}{\sum_{i=1}^{N}(m_i \wedge n_i)} \qquad \text{weighted average modal values } v_i$$

Example

P: X is x_o ***and*** Y is y_o inputs (x_o, y_o), $\forall x_o, y_o \in [-2, 2]$

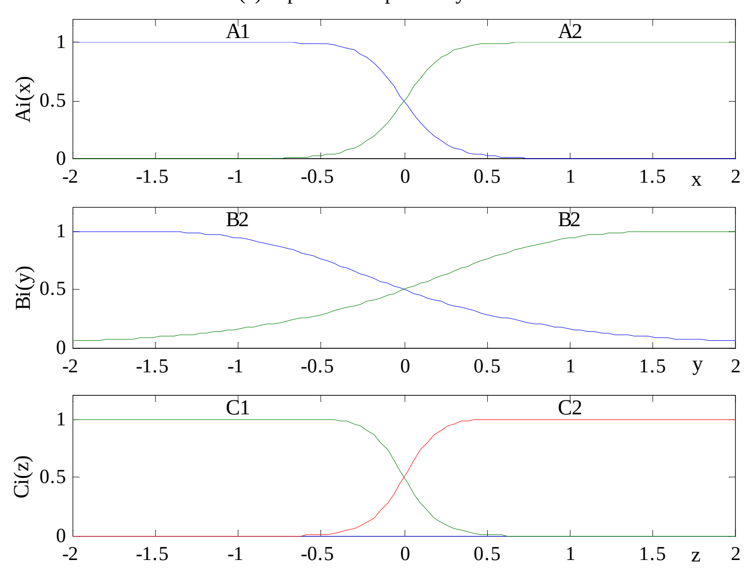
R_1: **If** X is A_1 ***and*** Y is B_1 **then** Z is C_1

R_2: **If** X is A_2 ***and*** Y is B_2 **then** Z is C_2

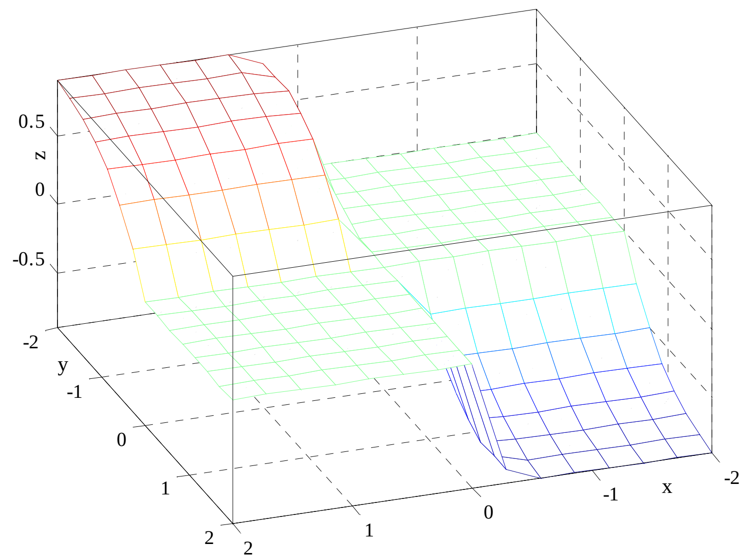
rules

$N = 2$, centroid defuzzification

(a) Input and output fuzzy sets



(b) Input-output mapping



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min-sum fuzzy models

Assume

P: X is A ***and*** Y is B $\qquad$ $P(x,y) = \min\{A(x), B(y)\}$

R_i: **If** X is A_i ***and*** Y is B_i **then** Z is C_i $\qquad$ $R_i(x,y,z) = \min\{A_i(x), B_i(y), C_i(z)\}$

$i = 1,\ldots, N$

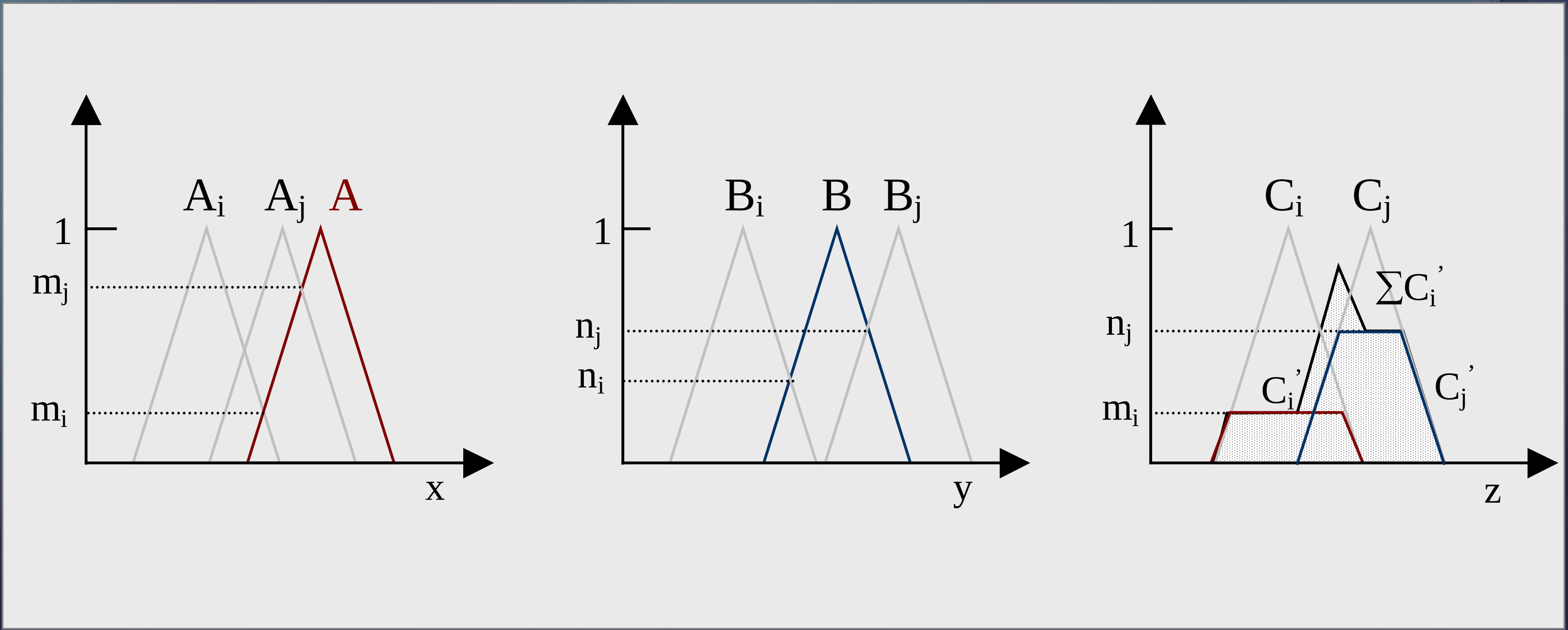
Using the compositional rule of inference (t = min)

$$C_i'(z) = \sup_{x,y}[A(x) \wedge B(y) \wedge A_i(x) \wedge B_i(y) \wedge C_i(z)]$$

$$C(z) = \sum_{i=1}^{N} w_i C_i'$$

Additive fuzzy models
(Kosko, 1992)

Example: min-sum fuzzy model processing



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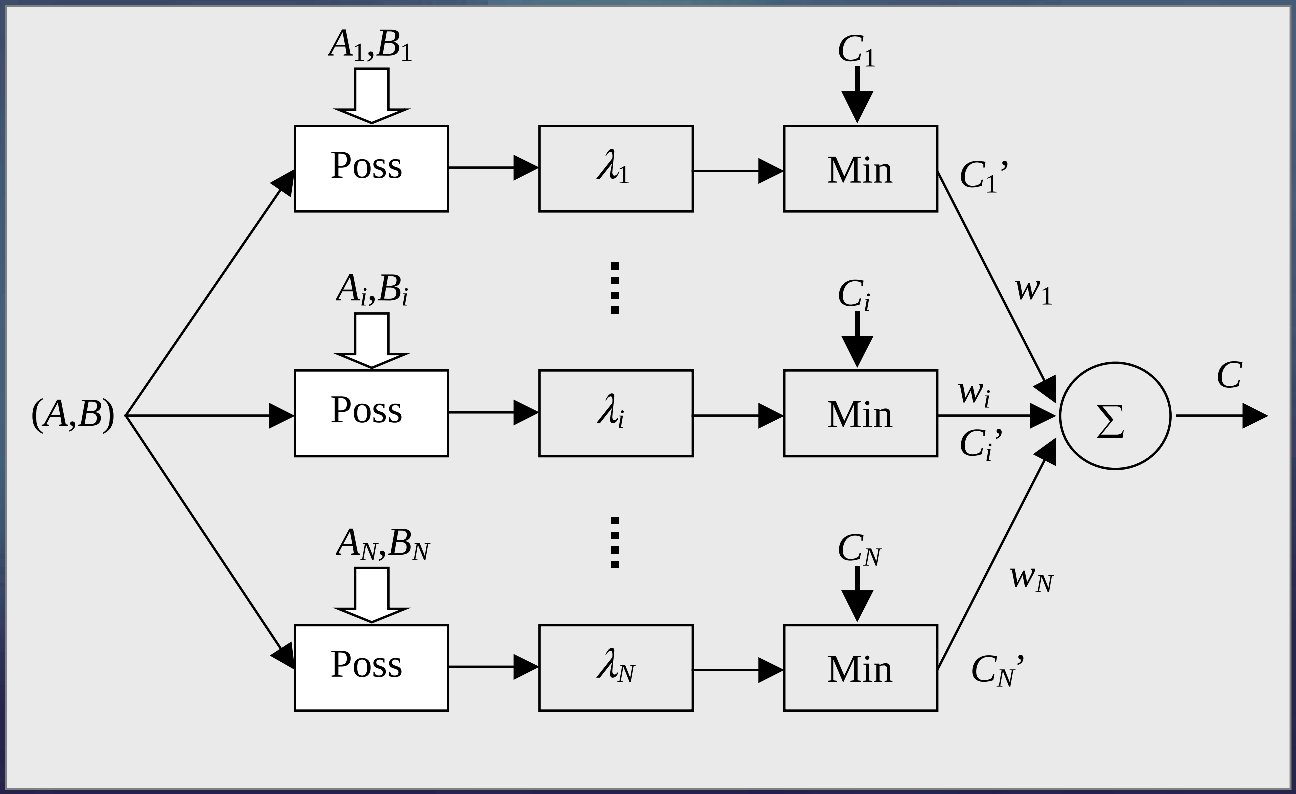
min-sum fuzzy model architecture



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Example

P: X is x_o ***and*** Y is y_o

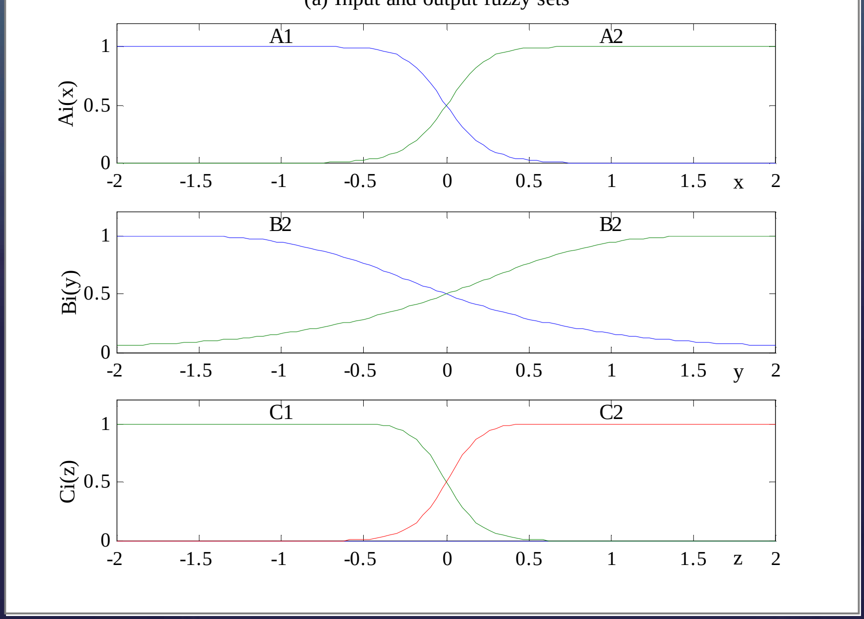
inputs (x_o, y_o), $\forall x_o, y_o \in [-2, 2]$

R_1: **If** X is A_1 ***and*** Y is B_1 **then** Z is C_1

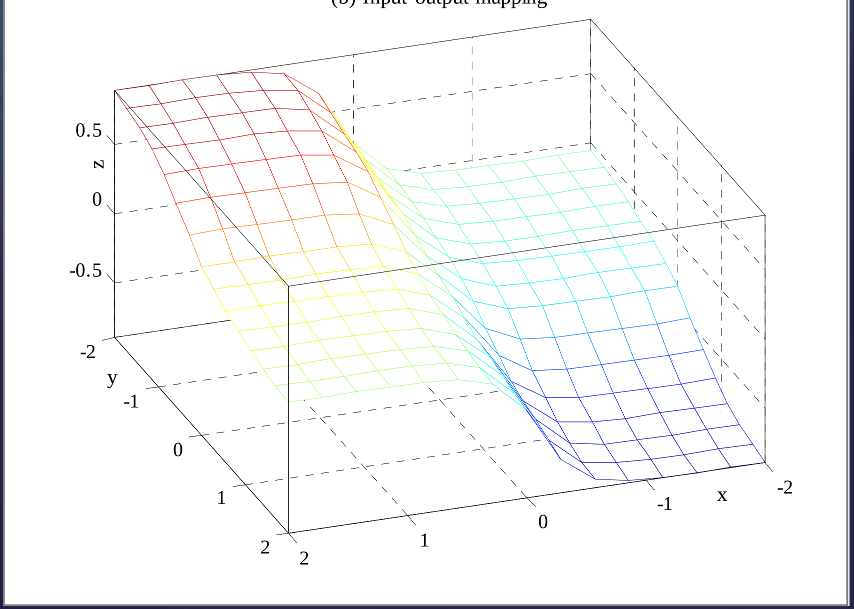
R_2: **If** X is A_2 ***and*** Y is B_2 **then** Z is C_2

rules

$N = 2$ $w_1 = w_2 = 1$, centroid defuzzification



(a) Input and output fuzzy sets



(b) Input-output mapping

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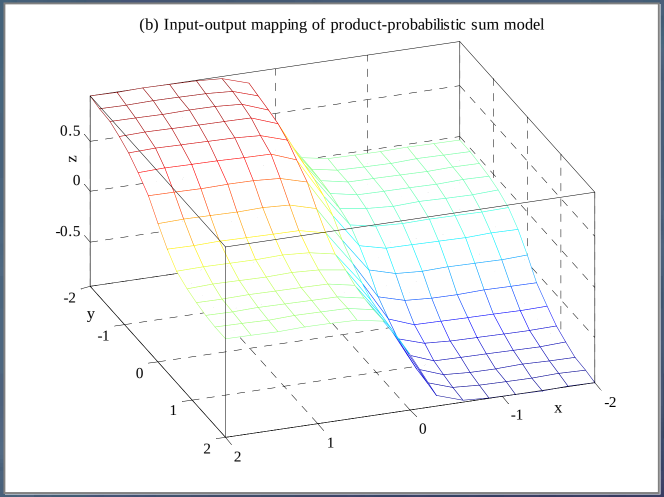
product-sum fuzzy models

1- Product–probabilistic sum

$$C_i'(z) = m_i n_i C_i(z)$$

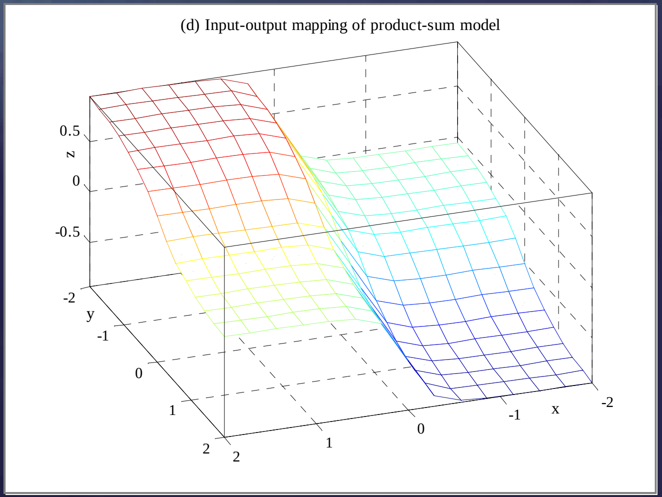
$$C(z) = \underset{i=1}{\overset{N}{S_p}}\, C_i'(z)$$



2- Product–sum

$$C_i'(z) = m_i n_i C_i(z)$$

$$C(z) = \sum_{i=1}^{N} C_i'(z)$$



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3 - Bounded product-bounded sum

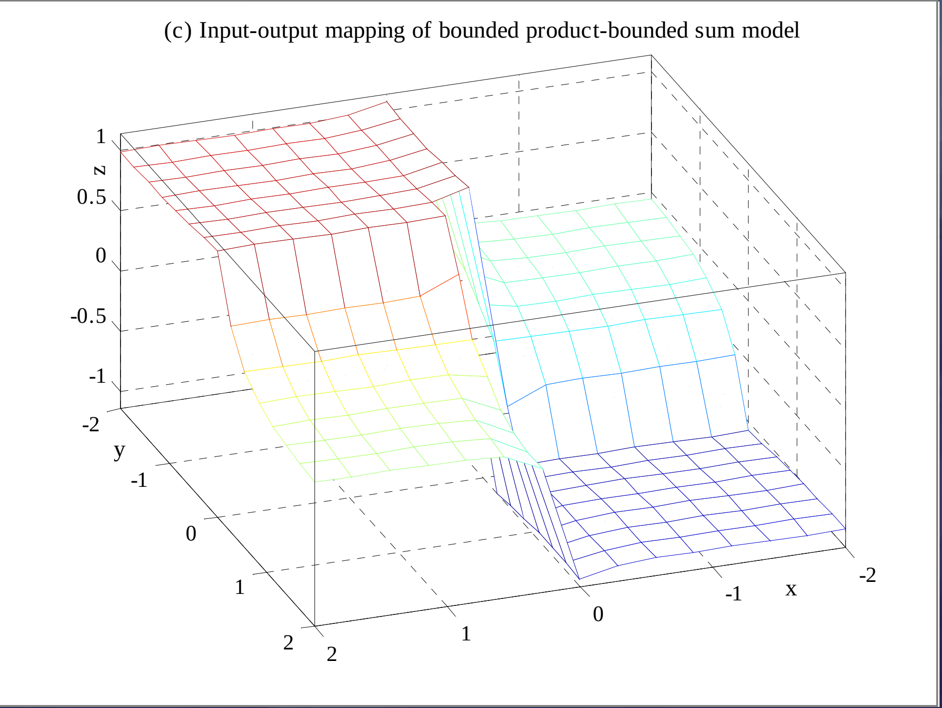
$$C_i'(z) = m_i \otimes n_i \otimes C_i(z)$$

$$C(z) = \bigoplus_{i=1}^{N} C_i'(z)$$

$$a \otimes b = \max\{0, a + b - 1\}$$

$$a \oplus b = \min\{1, a + b\}$$

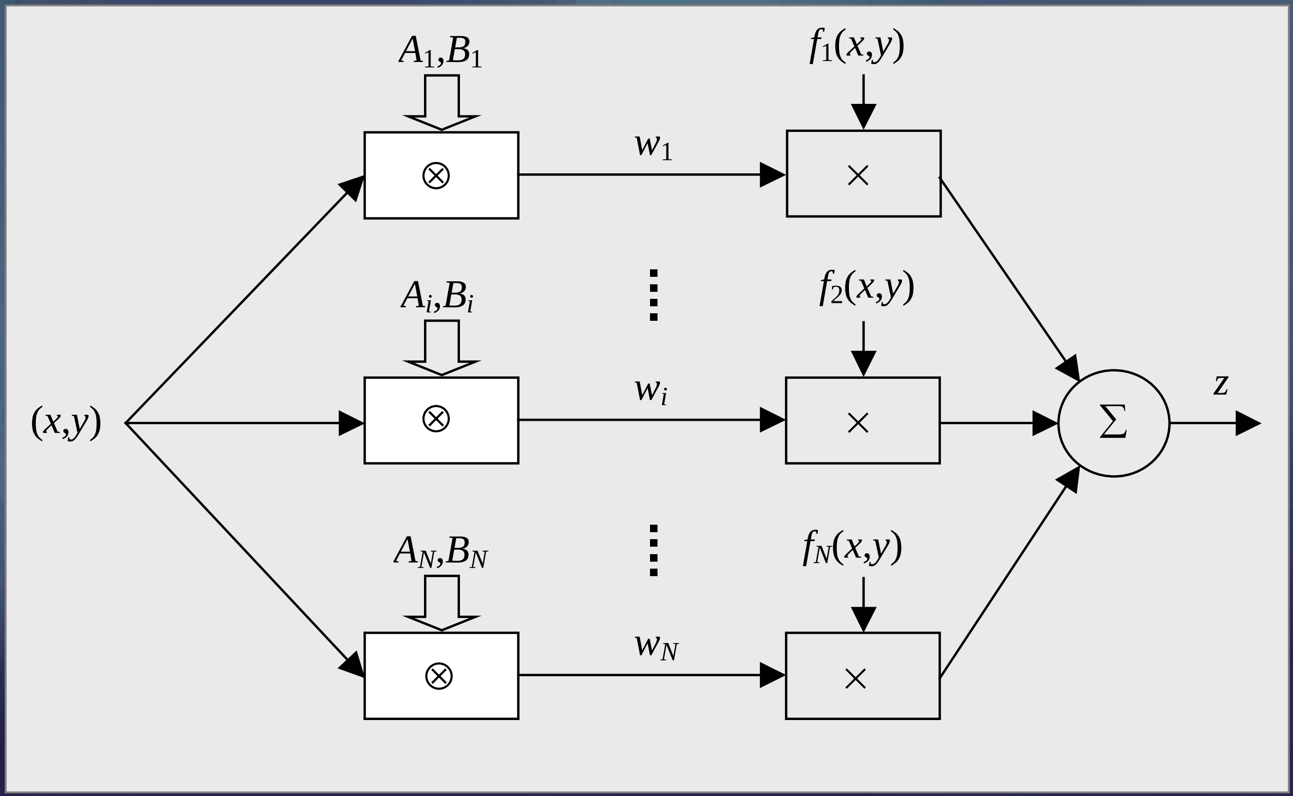
$$a,b \in [0,1]$$



Functional fuzzy models

P:	X is x ***and*** Y is y	input
R_1:	**If** X is A_1 ***and*** Y is B_1 **then** $z = f_1(x,y)$	
	....	
R_i:	**If** X is A_i ***and*** Y is B_i **then** $z = f_i(x,y)$	rule base
	....	
R_N:	**If** X is A_N ***and*** Y is B_N **then** $z = f_N(x,y)$	
	$\lambda_i(x,y) = A_i(x)\ t\ B_i(y)$ t = t-norm	degree of activation
	$z = \sum_{i=1}^{N} w_i(x,y)\, f_i(x,y)\,, \qquad w_i = \dfrac{\lambda_i}{\sum_{i=1}^{N} \lambda_i(x,y)}$	output

Functional fuzzy model architecture



Example 1

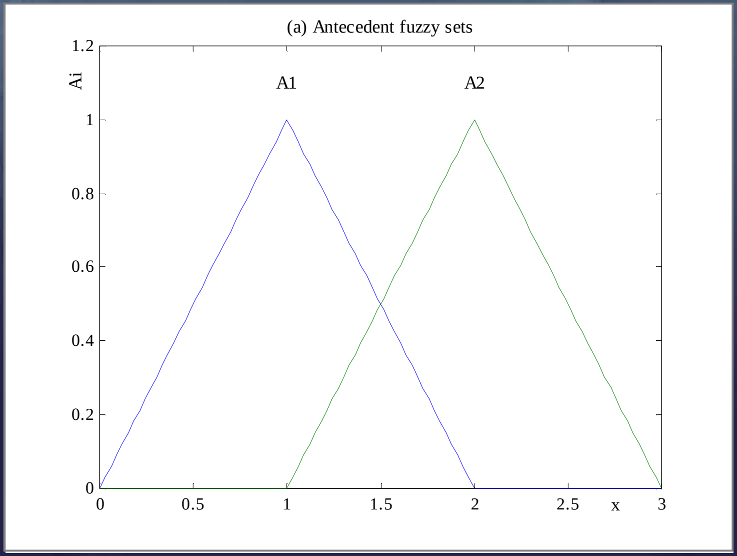
P: X is x

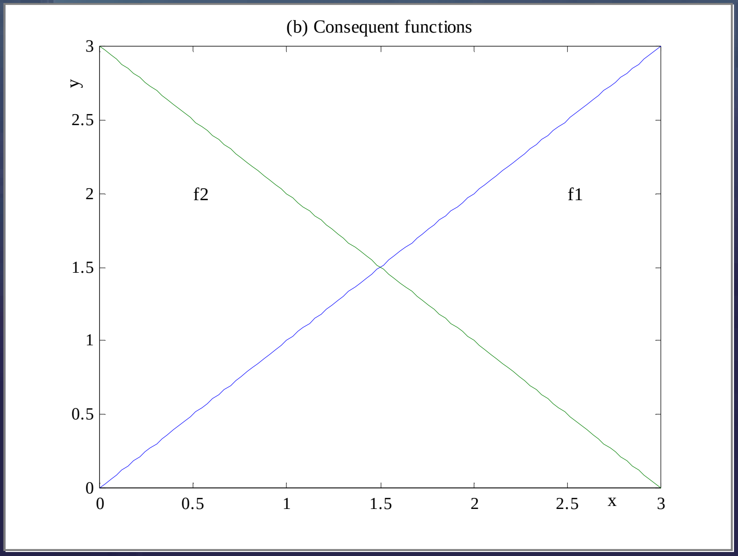
R_1: **If** X is A_1 **then** $z = x$

R_2: **If** X is A_2 **then** $z = -x + 3$

inputs $x \in [0, 3]$

rules





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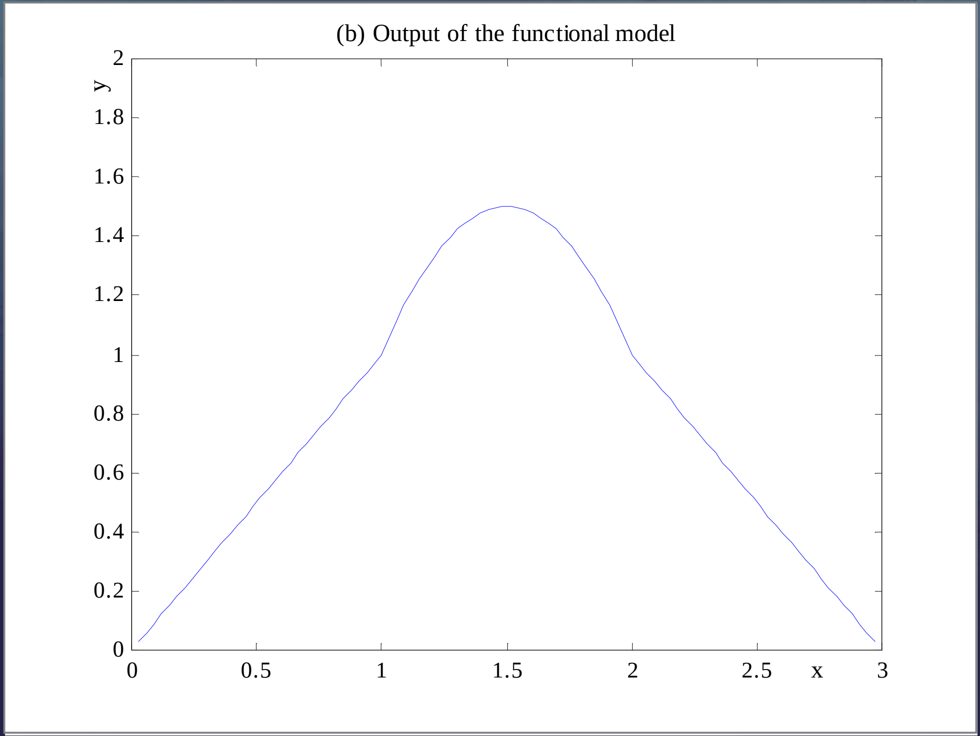
$$z = \begin{cases} x & \text{if } x \in (0,1] \\ A_1(x)x + A_2(x)(-x+3) & \text{if } x \in [1,2] \\ -x+3 & \text{if } x \in [2,3) \end{cases}$$



output

Example 2

P: X is x

inputs $x \in [0, 3]$

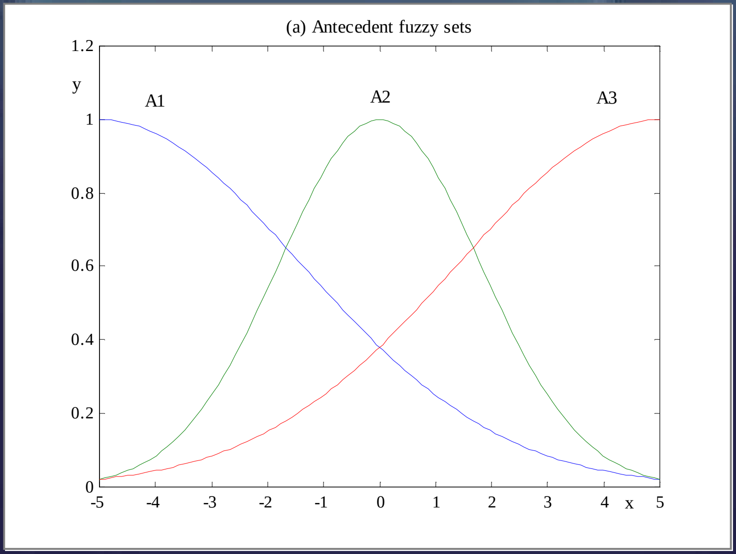
R_1: **If** X is A_1 **then** $y = -\sin(2x)$

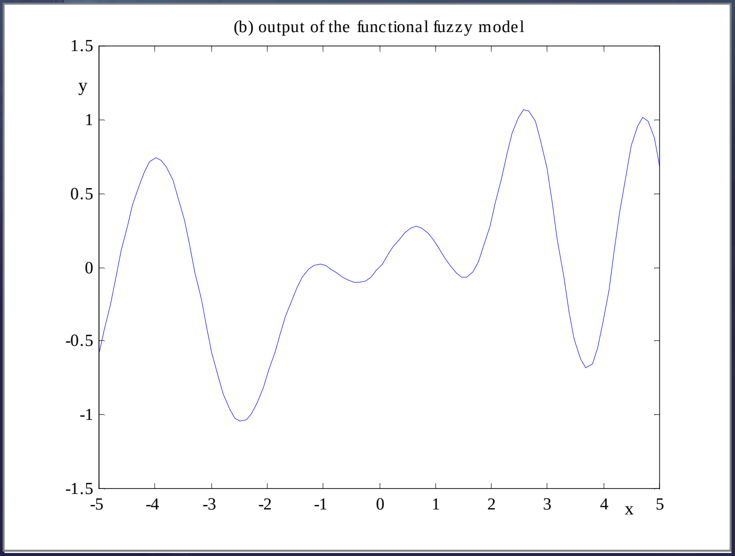
R_2: **If** X is A_2 **then** $y = -0.5x$

R_3: **If** X is A_3 **then** $y = \sin(3x)$

rules

output





Example 2

P: X is x

R_1: **If** X is A_1 **then** $y = -1$

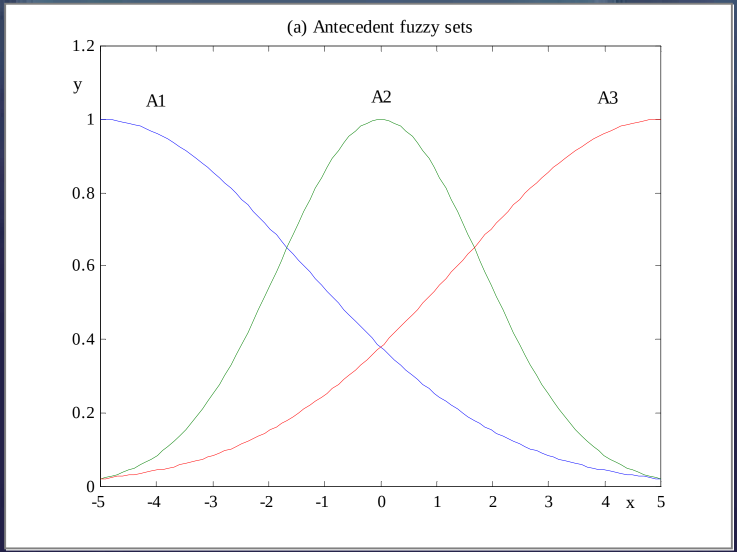
R_2: **If** X is A_2 **then** $y = x$

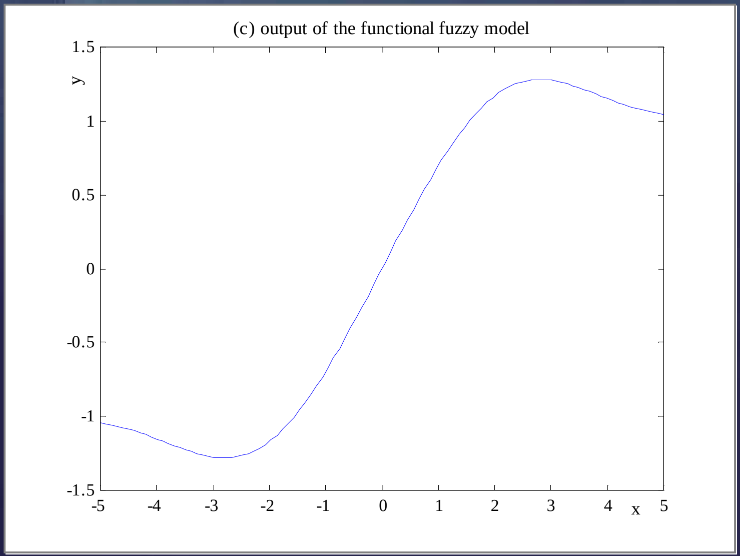
R_3: **If** X is A_3 **then** $y = 1$

inputs $x \in [0, 3]$

rules

output





Gradual fuzzy models

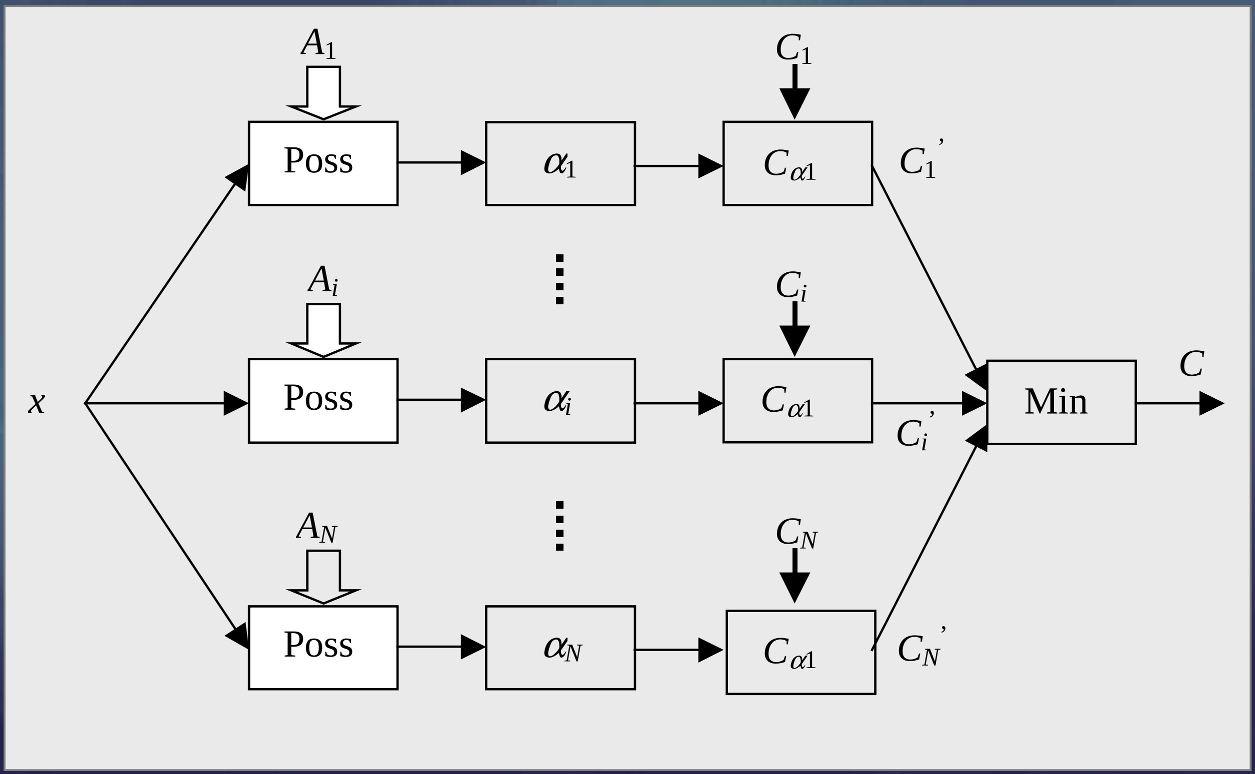
R_i: The *more* X is A_i the *more* Z is C_i

$i = 1,..., N$

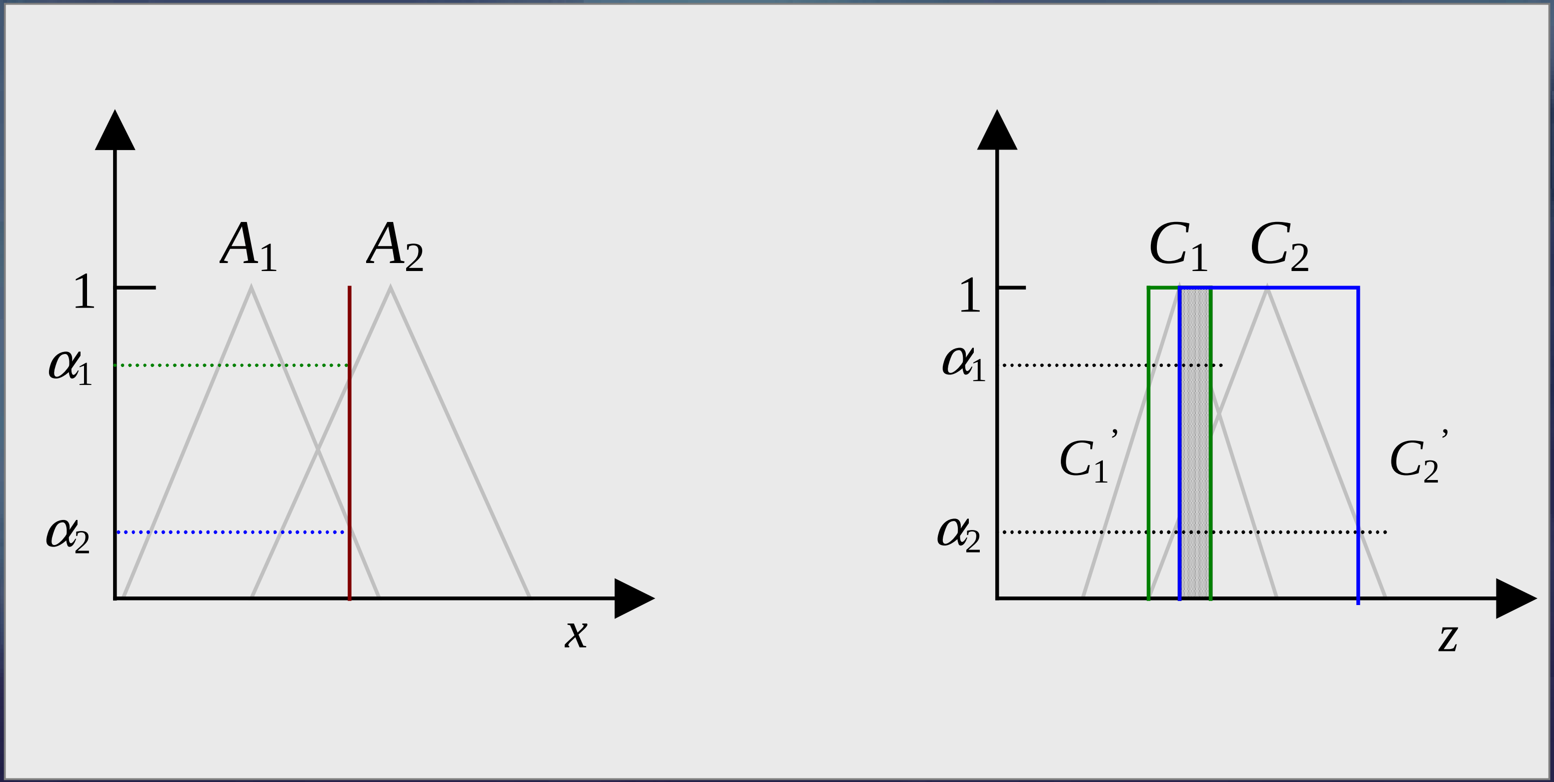
$$R_i(x, y) = \begin{cases} 1 & \text{if } C_i(z) \geq A_i(x) \\ 0 & \text{otherwise} \end{cases}$$

$$C = \bigcap_{i=1}^{N} (C_i')_{\alpha_i} = \bigcap_{i=1}^{N} C_{\alpha_i}$$

Gradual fuzzy model architecture



Example: gradual fuzzy model processing



Example

P: X is x

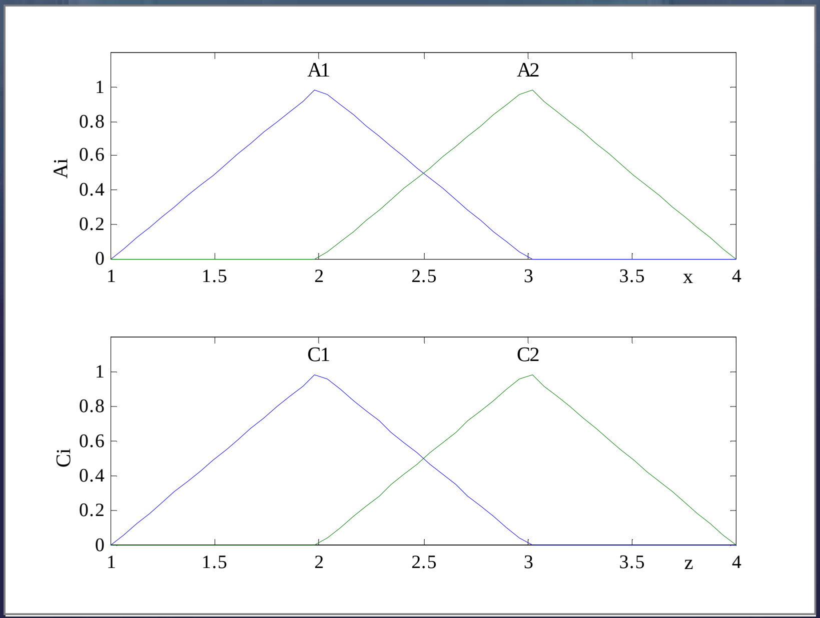
inputs $x \in [0, 3]$

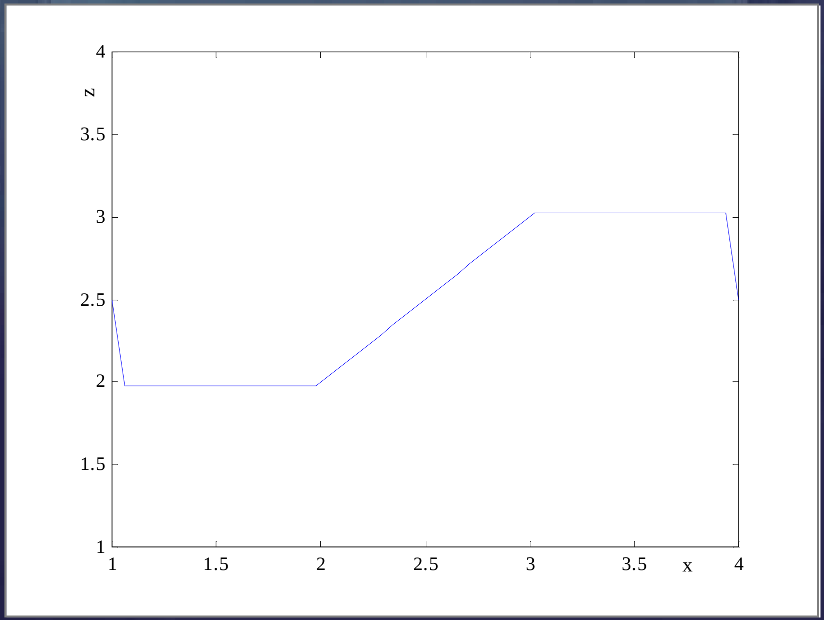
R_1: The *more* X is A_1 the *more* Z is C_1

R_2: The *more* X is A_1 the *more* Z is C_1

rules

output





Pedrycz and Gomide, FSE 2007

11.6 Approximation properties of fuzzy rule-based models

- FRBS uniformly approximates continuous functions
  - any degree of accuracy
  - closed and bounded sets

- Universal approximation with (Wang & Mendel, 1992):
  - algebraic product t-norm in antecedent
  - rule semantics via algebraic product
  - rule aggregation via ordinary sum
  - Gaussian membership functions
  - sup-min compositional rule of inference
  - pointwise inputs
  - centroid defuzzification

- Universal approximation when (Kosko, 1992):
  - min t-norm in antecedent
  - rule aggregation via ordinary sum
  - symmetric consequent membership functions
  - sup-min compositional rule of inference
  - pointwise inputs
  - centroid defuzzification

(additive models)

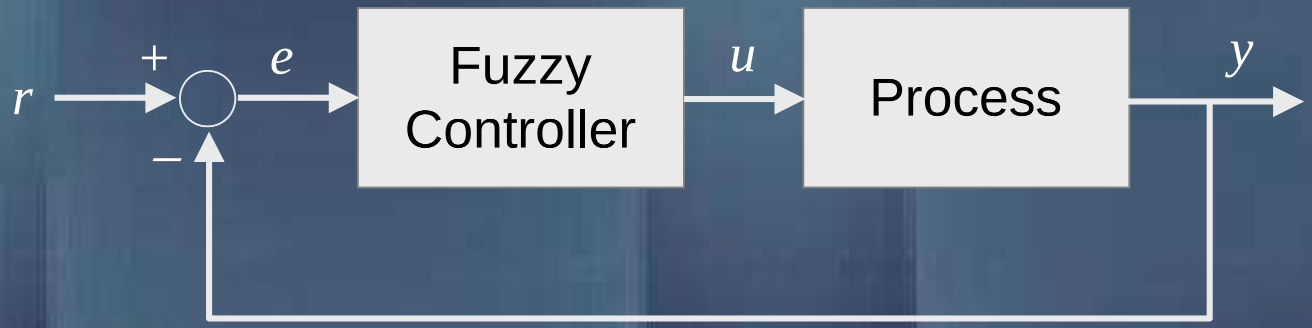
- Universal approximation with (Castro, 1995):
  - arbitrary t-norm in antecedent
  - rule semantics: r-implications or conjunctions
  - triangular or trapezoidal membership functions
  - sup-min compositional rule of inference
  - pointwise inputs
  - centroid defuzzification

11.7 Development of rule-based systems

Expert-based development

- Knowledge provided by domain experts
  - basic concepts and variables
  - links between concepts and variables to form rules
- Reflects existing knowledge
  - can be readily quantified
  - short development time

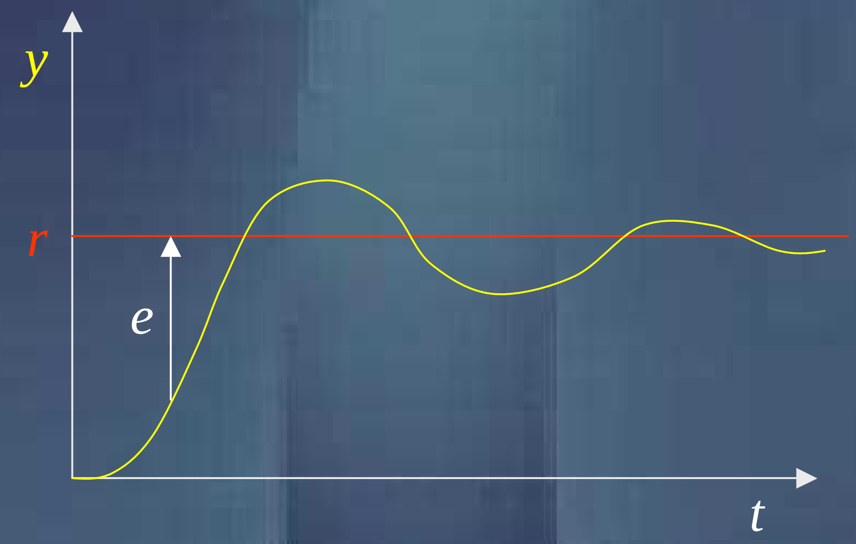
Example: fuzzy control



R_i: **If** Error is A_i ***and*** Change of Error is B_i **then** Control is C_i

R_i: **If** e is A_i ***and*** de is B_i **then** u is C_i

R_i: **If** e is A_i ***and*** de is B_i **then** u is C_i



Change of Error (*de*) / Error (*e*)	NM	NS	ZE	PS	PM
NB	PM	NB	NB	NB	NM
NM	PM	NB	NS	NM	NM
NS	PM	NS	Z	NS	NM
Z	PM	NS	Z	NS	NM
PS	PM	PS	Z	NS	NM
PM	PM	PM	PS	PM	NM
PB	PM	PM	PM	PM	NM

Data-driven development

- Given a finite set of input/output pairs

$\{(\mathbf{x}_k, y_k),\ k = 1,\ldots, M\}$

$\mathbf{x}_k = [x_{1k}, x_{2k},\ldots, x_{nk}] \in \mathrm{R}^n$

$\mathbf{z}_k = [\mathbf{x}_k, y_k] \in \mathrm{R}^{n+1},\ k = 1,\ldots, M$

- Clustering $\mathbf{z}_k = [\mathbf{x}_k, y_k] \in \mathrm{R}^{n+1},\ k = 1,\ldots, M$ (e.g. using FCM)

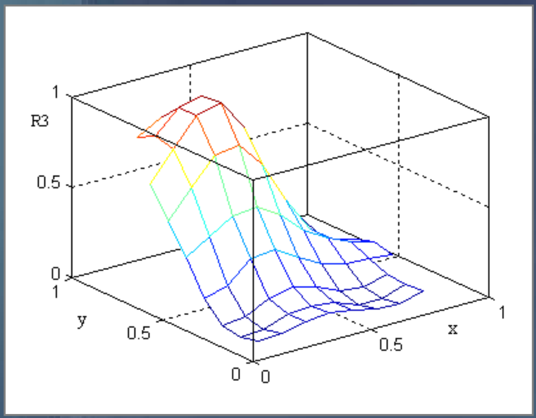
$\mathbf{v}_1, \mathbf{v}_2,\ldots,\mathbf{v}_N$ prototypes/cluster centers

$\mathbf{v}_i \in \mathrm{R}^{n+1}, i = 1,\ldots, N$

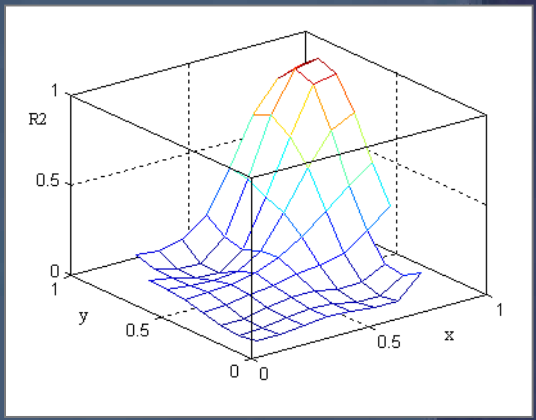
- Idea: fuzzy clusters ≡ fuzzy rules

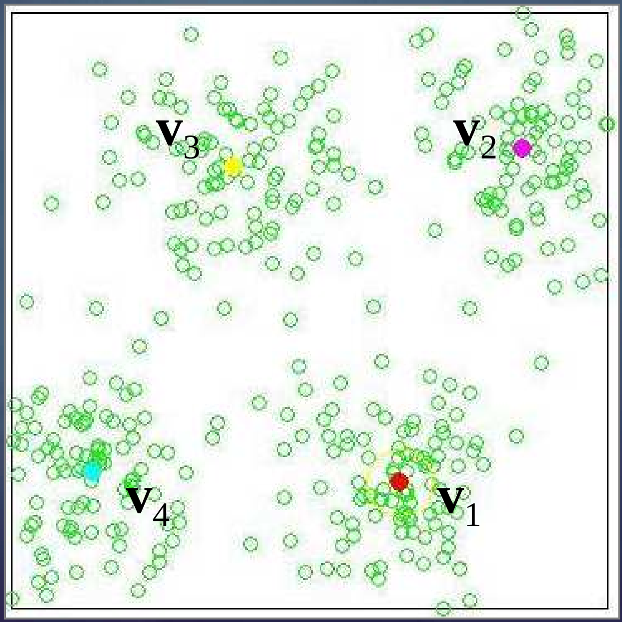
Example

R_3

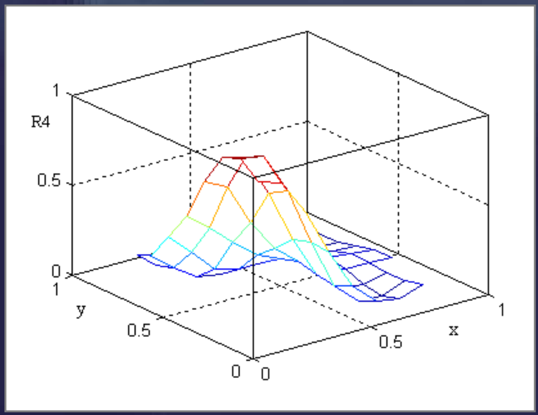


R_2

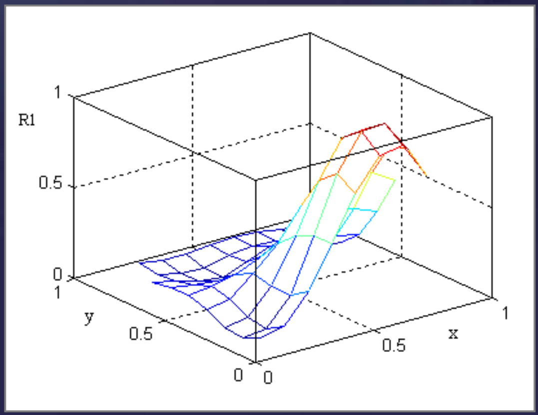




R_4



R_1



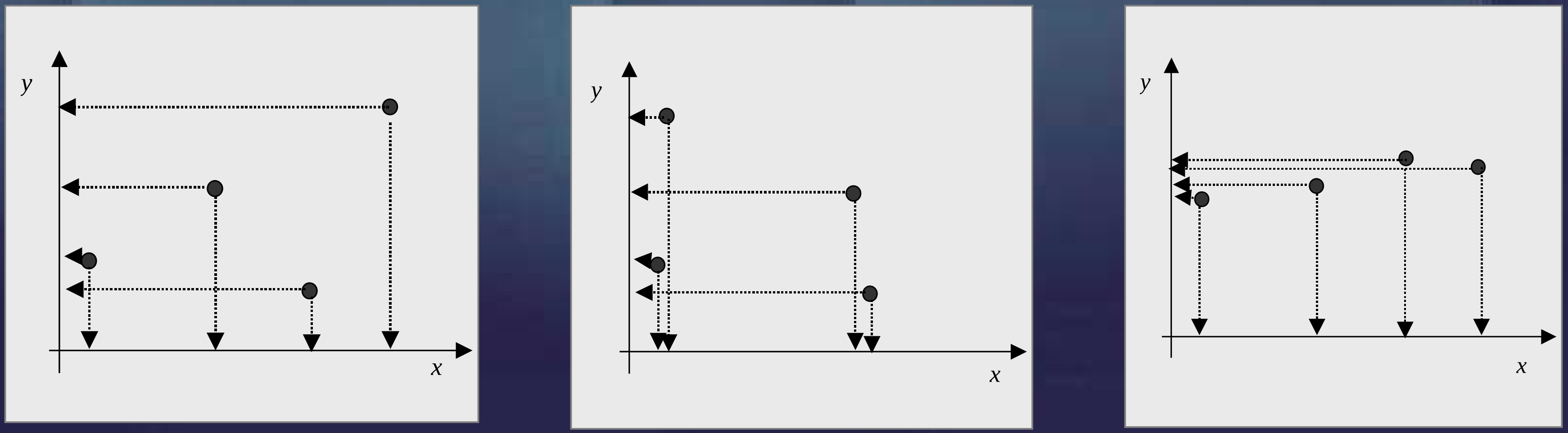
Pedrycz and Gomide, FSE 2007

- Projecting the prototypes in the input and output spaces

  $\mathbf{v}_1[y]$, $\mathbf{v}_2[y]$,....,$\mathbf{v}_N[y]$ projections of prototypes in $\mathbf{Y}$

  $\mathbf{v}_1[\mathbf{x}]$, $\mathbf{v}_2[\mathbf{x}]$,....,$\mathbf{v}_N[\mathbf{x}]$ projections of prototypes in $\mathbf{X}$

- R_i: **If** X is A_i **then** Y is C_i, $i = 1,..., N$



11.8 Parameter estimation for functional rule-based systems

- Functional fuzzy rules

- R_i: **If** X_{i1} is A_{i1} ***and*** ... ***and*** X_{in} is A_{in} **then** $z = a_{io} + a_{i1}x_1 + \ldots + a_{in}x_n$

  $i = 1,\ldots, N$

- Given input/output data: $\{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2),\ldots,(\mathbf{x}_M, y_M)\}$

- Let $\mathbf{a}_i = [a_{io}, a_{i1}, a_{i2},\ldots, a_{in}]^{\mathrm{T}}$

- Output of functional models

$$\hat{y}_k = \sum_{i=1}^{N} w_{ik} f_i(\mathbf{x}_k, \mathbf{a}_i), \qquad w_{ik} = \frac{\lambda_i(x_k)}{\sum_{i=1}^{N} \lambda_i(x_k)}$$

- Output for linear consequents

$$\hat{y}_k = \sum_{i=1}^{N} \mathbf{z}_{ik}^{\mathrm{T}} \mathbf{a}_i, \qquad \mathbf{z}_{ik} = [1, w_{ik}\mathbf{x}_k^{\mathrm{T}}]^{\mathrm{T}}$$

Let

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_N \end{bmatrix} \qquad \hat{y}_k = \begin{bmatrix} \mathbf{z}_{1k}^{\mathrm{T}} & \mathbf{z}_{2k}^{\mathrm{T}} & \cdots & \mathbf{z}_{Nk}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_N \end{bmatrix}$$

and

$$\mathbf{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_M \end{bmatrix} \qquad Z = \begin{bmatrix} \mathbf{z}_{11}^{\mathrm{T}} & \mathbf{z}_{12}^{\mathrm{T}} & \cdots & \mathbf{z}_{N1}^{\mathrm{T}} \\ \mathbf{z}_{12}^{\mathrm{T}} & \mathbf{z}_{22}^{\mathrm{T}} & \cdots & \mathbf{z}_{N2}^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{z}_{1M}^{\mathrm{T}} & \mathbf{z}_{2M}^{\mathrm{T}} & \cdots & \mathbf{z}_{NM}^{\mathrm{T}} \end{bmatrix}$$

then $\mathbf{y} = Z\mathbf{a}$

Global least squares approach

$$\mathrm{Min}_{\mathbf{a}}\ J_G(\mathbf{a}) = \|\ \mathbf{y} - \mathrm{Z}\mathbf{a}\|^2$$

$$\|\ \mathbf{y} - \mathrm{Z}\mathbf{a}\|^2 = (\mathbf{y} - \mathrm{Z}\mathbf{a})^{\mathrm{T}}\ (\mathbf{y} - \mathrm{Z}\mathbf{a})$$

Solution

$$\mathbf{a}_{\mathrm{opt}} = \mathrm{Z}^{\#}\,\mathbf{y}$$

$$\mathrm{Z}^{\#} = (\mathrm{Z}^{\mathrm{T}})^{-1}\mathrm{Z}^{\mathrm{T}}$$

Local least squares approach

$$\mathrm{Min}_{\mathbf{a}} J_L(\mathbf{a}) = \sum_{i=1}^{N} \left\| \mathbf{y} - Z_i \mathbf{a}_i \right\|^2$$

$$Z_i = \begin{bmatrix} \mathbf{z}_{i1}^{\mathrm{T}} \\ \mathbf{z}_{i2}^{\mathrm{T}} \\ \vdots \\ \mathbf{z}_{iM}^{\mathrm{T}} \end{bmatrix}$$

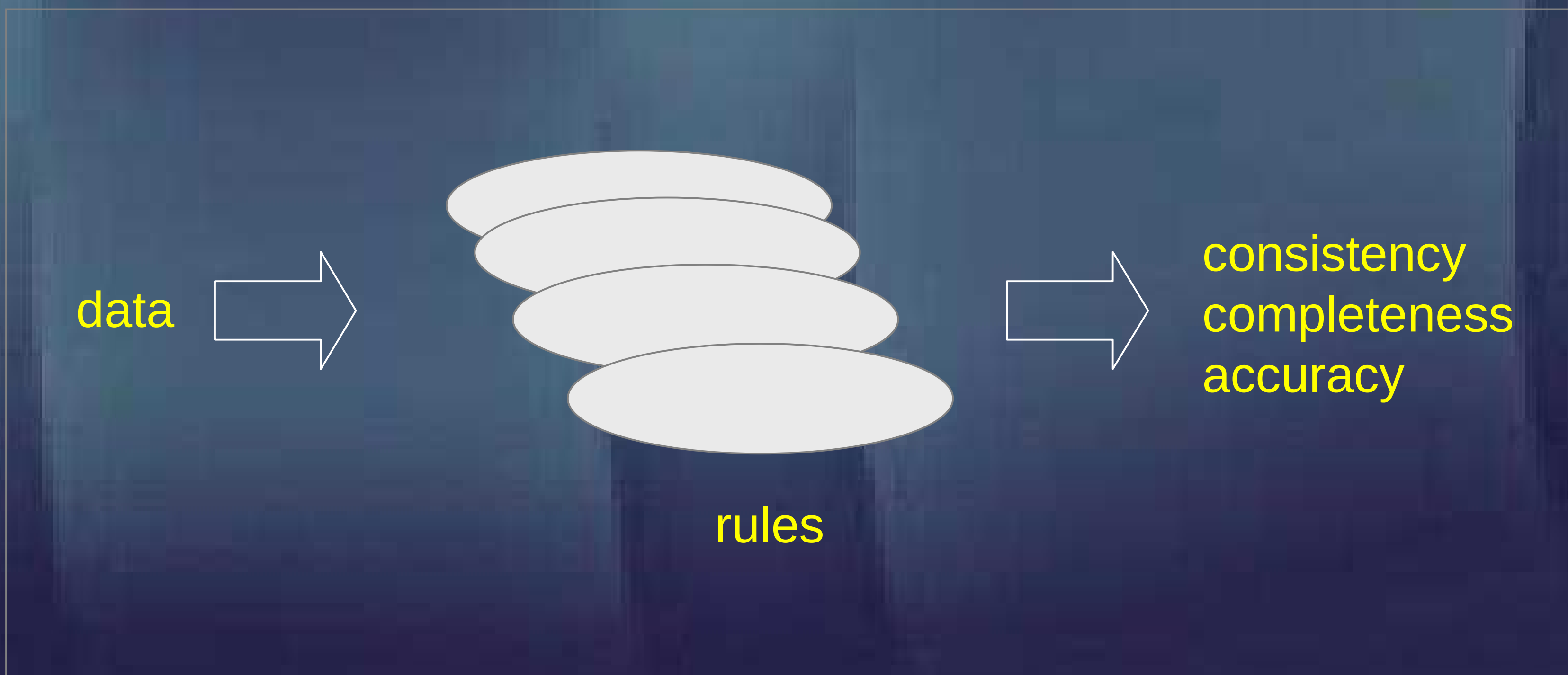
Solution

$$\mathbf{a}_{iopt} = \mathrm{Z}_i^{\#}\,\mathbf{y}$$

$$\mathrm{Z}_i^{\#} = (\mathrm{Z}_i^{\mathrm{T}})^{-1}\mathrm{Z}_i^{\mathrm{T}}$$

11.9 Design issues of FRBS: Consistency and completeness

Given input/output data: $\{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2),\ldots,(\mathbf{x}_M, y_M)\}$



Issue: quality of the rules

Completeness of rules

- All data points represented through some fuzzy set

$$\max_{i=1,\ldots,M} A_i(\mathbf{x}_k) > 0 \text{ for all } k = 1,2,\ldots, M$$

- Input space completely covered by fuzzy sets

$$\max_{i=1,\ldots,M} A_i(\mathbf{x}_k) > \delta \text{ for all } k = 1,2,\ldots, M$$

Consistency of rules

- Rules in conflict
  - similar or same conditions
  - completely different conclusions

Conditions and Conclusions	*Similar Conclusions*	*Different Conclusions*
Similar Conditions	rules are redundant	**rules are in conflict**
Different Conditions	different rules; could be eventually merged	different rules

R_i: **If** X is A_i **then** Y is B_i

R_j: **If** X is A_j **then** Y is B_j

$$\mathrm{cons}(i,j) = \sum_{i=1}^{M} \{ | B_i(y_k) - B_j(y_k) | \Rightarrow | A_i(x_k) - A_j(x_k) | \}$$

Alternatively

$$\mathrm{cons}(i,j) = \sum_{i=1}^{M} \{ \mathrm{Poss}(A_i(x_k), A_j(x_k)) \Rightarrow \mathrm{Poss}(B_i(y_k), B_j(y_k)) \}$$

$\Rightarrow$ is an implication induced by some t-norm (r-implication)

$$\mathrm{cons}(i) = \frac{1}{M} \sum_{j=1}^{M} \mathrm{cons}(i,j)$$

11.10 The curse of dimensionality in rule-based systems

- Curse of dimensionality
  - number of variables increase
  - exponential growth of the number of rules

- Example
  - n variables
  - each granulated using p fuzzy sets
  - number of different rules = p^n

- Scalability challenges

11.11 Development scheme of fuzzy rule-based models

- Spiral model of development
  - – incremental design, implementation and testing
  - – multidimensional space of fundamental characteristics