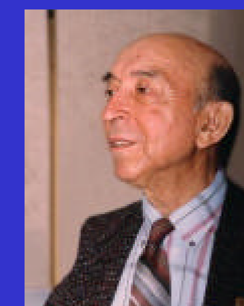
INFORMATION AND CONTROL 8, 338–353 (1965)

1965: Fuzzy Sets* Appear – A Contribution to the 40th Anniversary

L. A. Zadeh

Department of Electrical Engineering and Electronics Research Laboratory, University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.



Rudolf Seising

Medical Statistics and Informatics

Medical University of Vienna

Vienna - Austria

What is a little conference?

Criteria A:	few parallel sessions.
Criteria B:	few plenary talks.
Criteria C:	cheap registration fees.

……

What is a big conference?

Criteria A:	many participants.
Criteria B:	important speakers.
Criteria C:	long and great tradition.

………

Criterion *A*:

- Conference C_1 might be better than C_2, and
- Conference C_2 might be better than C_3.

Criterion *B*:

- Conference C_2 might be better than C_3, and
- Conference C_3 might be better than C_1.

Criterion C:

- Conference C_3 might be better than C_1, and
- Conference C_1 might be better than C_2.

1958 IRE TRANSACTIONS ON INFORMATION THEORY

What Is Optimal?

LOTFI A. ZADEH

How reasonable is our insistence on optimal solutions? Not too long ago we were content with designing systems which merely met given specifications. It was primarily Wiener's work on optimal filtering and prediction that changed profoundly this attitude toward the design of systems and their components. Today we tend, perhaps, to make a fetish of optimality. If a system is not "best" in one sense or another, we do not feel satisfied. Indeed, we are apt to place too much confidence in a system that is, in effect, optimal by definition.

To find an optimal system we first choose a criterion of performance. Then we specify a class of acceptable systems in terms of various constraints on the design, cost, etc. Finally, we determine a system within the specified class which is "best" in terms of the criterion adopted. Is this procedure more rational than the relatively unsophisticated approach of the pre-Wiener era?

Criterion *A*:

- Design D_1 might be better than D_2, and
- Design D_2 might be better than D_3.

Criterion *B*:

- Design D_2 might be better than D_3, and
- Design D_3 might be better than D_1.

Criterion *C*:

- Design D_3 might be better than D_1, and
- Design D_1 might be better than D_2.

History of the Theory of Fuzzy Sets

- Prehistory of the Theory of Fuzzy Sets
  1920s-1960s

- Genesis of the Theory of Fuzzy Sets
  1960s

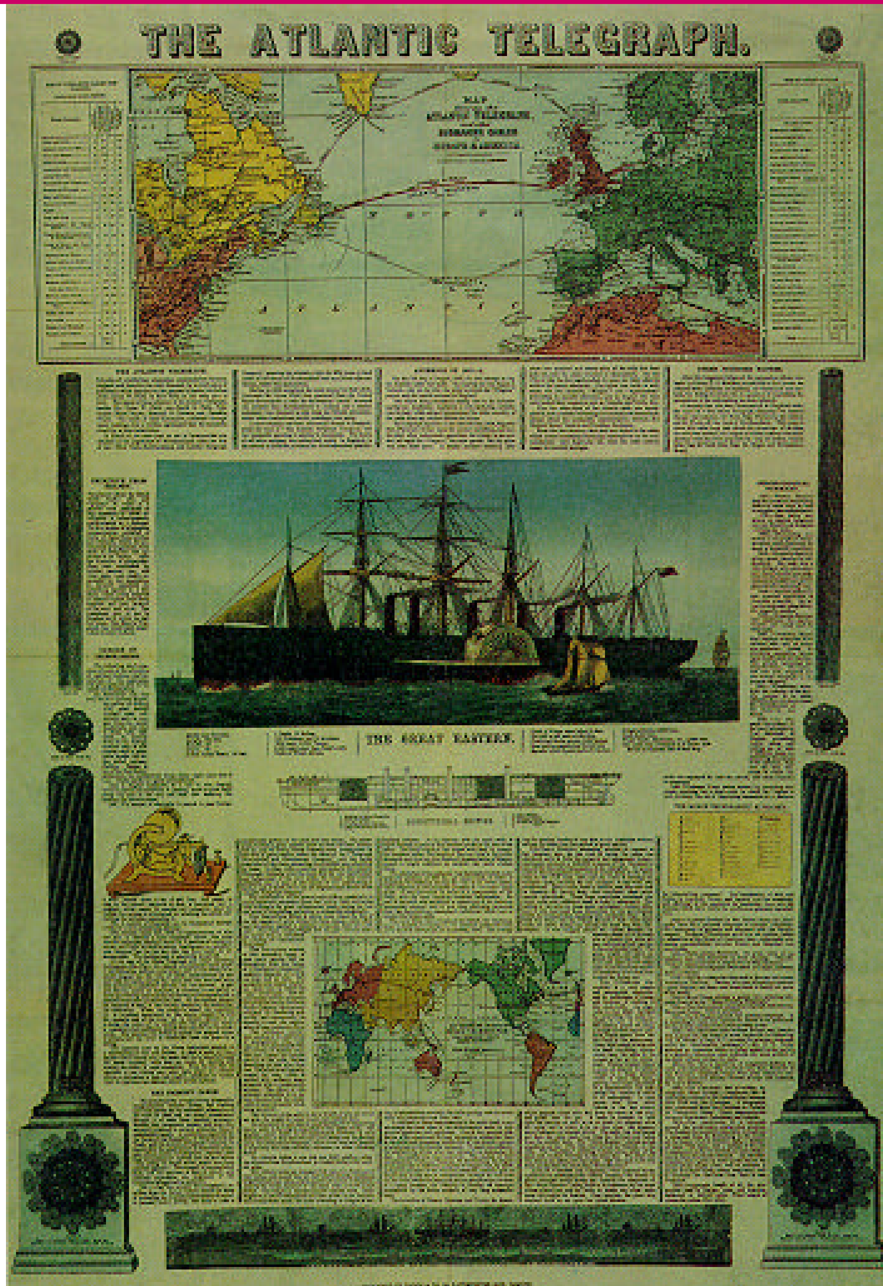
- Applications of the Theory of Fuzzy Sets
  1970s

- Enforcement of the Theory of Fuzzy Sets as a scientific paradigm
  1980s - 1990s

History of the Theory of Fuzzy Sets

History of the Theory of Fuzzy Sets

- From Circuit Theory to System Theory
  1940s-1960s
  - *From Signals to Filters*
  - *From Filters to Systems*
- From System Theory to Fuzzy Systems
  1960 - 1964
  - *The State Space Approach*
  - *A New View on System Theory*
- The Appearance of „Fuzzy Sets“
  1964 and 1965


THE ATLANTIC TELEGRAPH.
THE GREAT EASTERN.


Long Distance for Business
IT'S QUICKER AND COSTS LESS
PHONE-DON'T WRITE

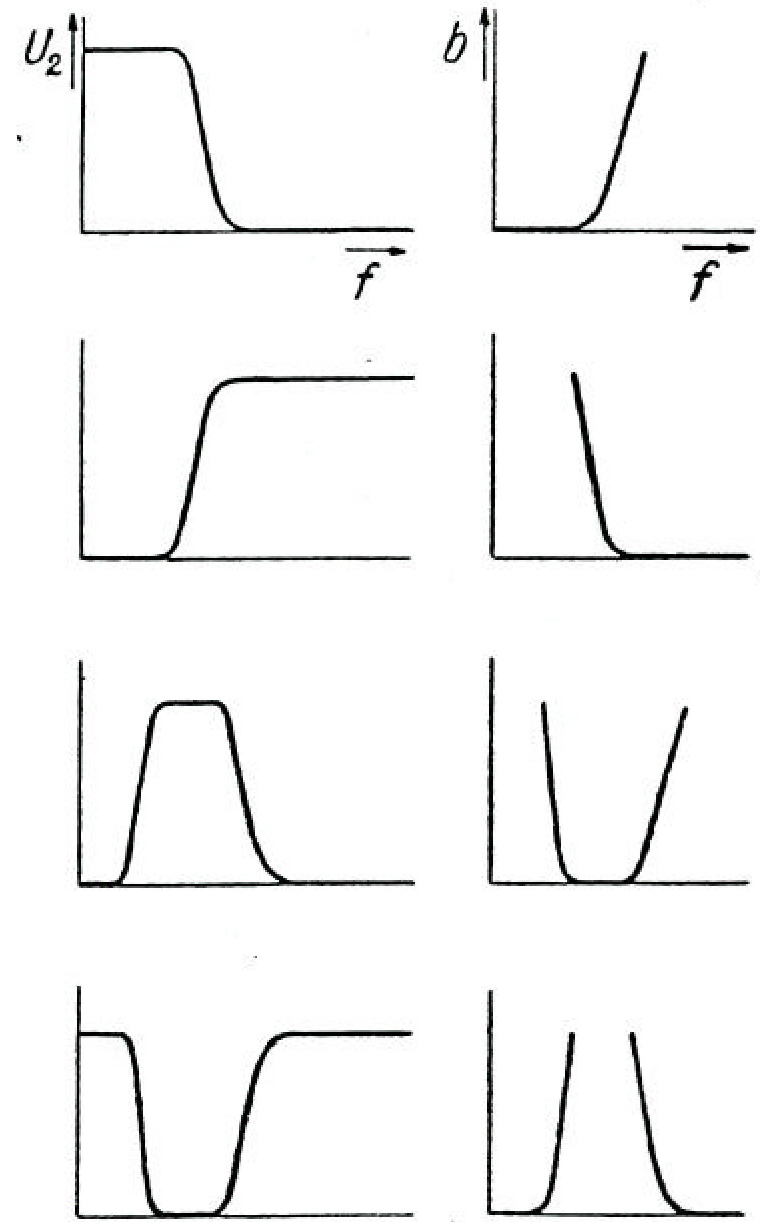
Pioneers of mathematical Electrical Engineering





Hendrik Bode, Otto Brune, Sidney Darlington, Wilhelm Cauer, Ronald Foster

Electrical Filters, Sieves

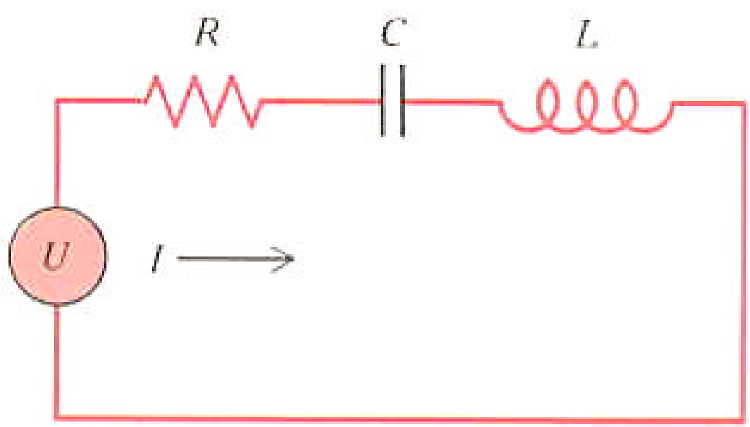


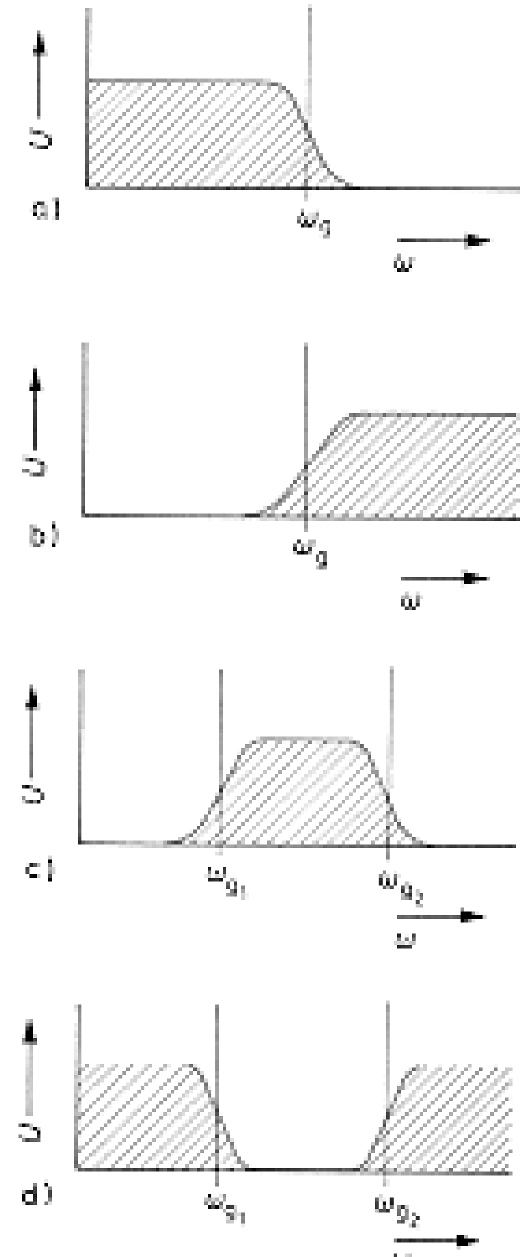
Electrical Filters, Sieves

Karl Ferdinand Braun (1850-1918)

George Ashley Campbell (1870-1954)

Wilhelm Cauer (1900-1945)





Ernst Adolphe Guillemin (1898-1970)

- Studies in *Electrical Engineering,* University of Wisconsin - Madison
- 1922 - 1926 Ph. D. Studies in Munich (Saltonstall Traveling Fellowship), Ph. D. Thesis Supervisor: Prof. Arnold Sommerfeld
- 07.07.1926 *Zur Theorie der Frequenzverviel-fachung durch Eisenkernkopplung*
- 1928 Assistant professor
- 1936 Associate professor
- 1944 Full professor
- 1931 *Communication Networks I*
- 1935 *Communication Networks II*
- 1953 *Introductory Circuit Theory*
- 1957 *Synthesis of Passive Networks*

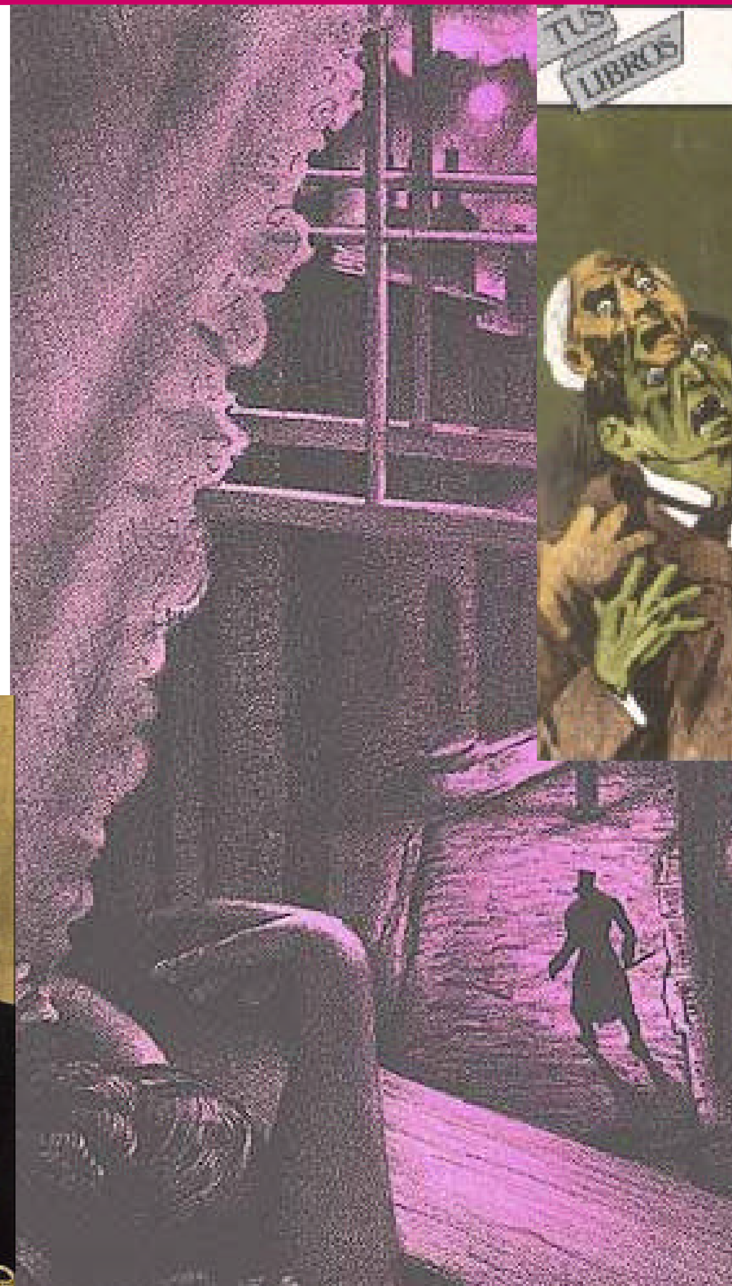
Burnell Dedicates Guillemin Laboratory

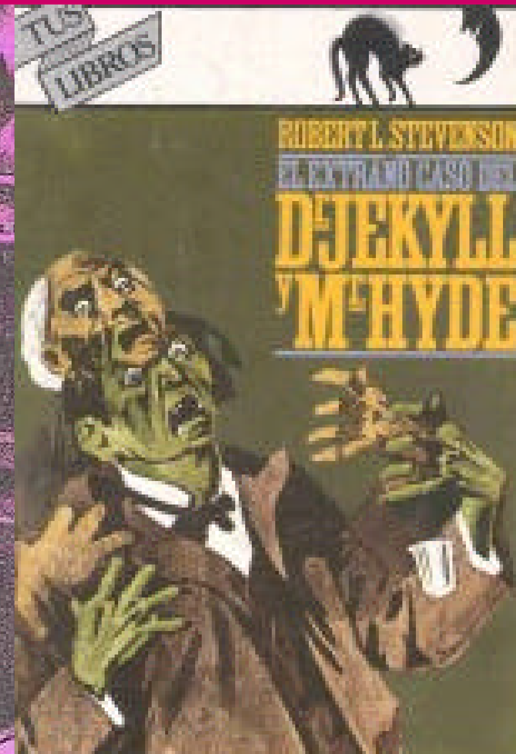
Science and industry joined in the dedication (Oct. 26, 1961) of **Burnell & Co., Inc.'s,** new Guillemin Research Laboratory in Cambridge, Mass. Honoring Dr. Ernst A. Guillemin, eminent M.I.T. scientist who is also vice president in charge of research of Burnell & Co., the lab is believed to be the first facility of its kind devoted exclusively to research in electronic filters and networks.

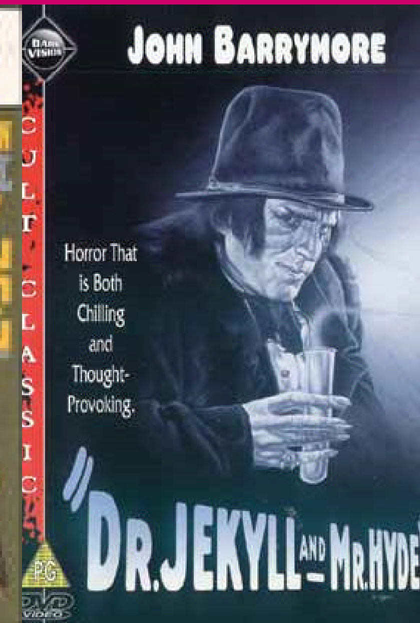
Above, Dr. Guillemin (center) is shown in a discussion with Dr. Lan J. Chu (right), professor of electrical engineering at M.I.T., and Lewis G. Burnell, executive vice president and director of engineering of Burnell & Co., who were among the guests at the dedication. The firm's main plant is located at 10 Pelham Parkway, Pelham, N. Y.

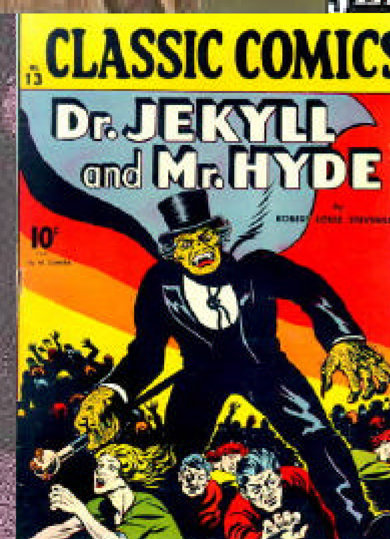
The Strange Case of Dr. Jekyll and Mr. Hyde

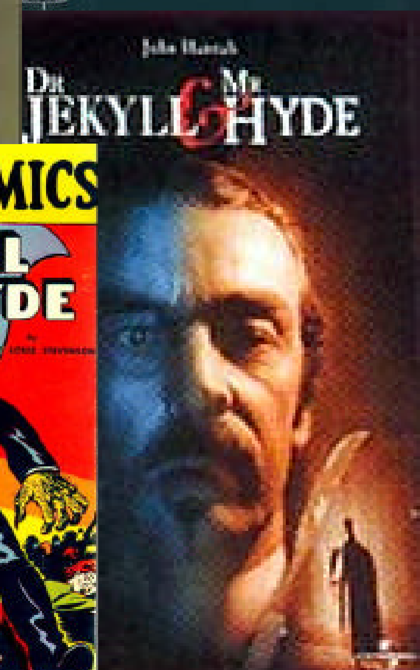
Robert Louis Stevenson
(1850-1894)







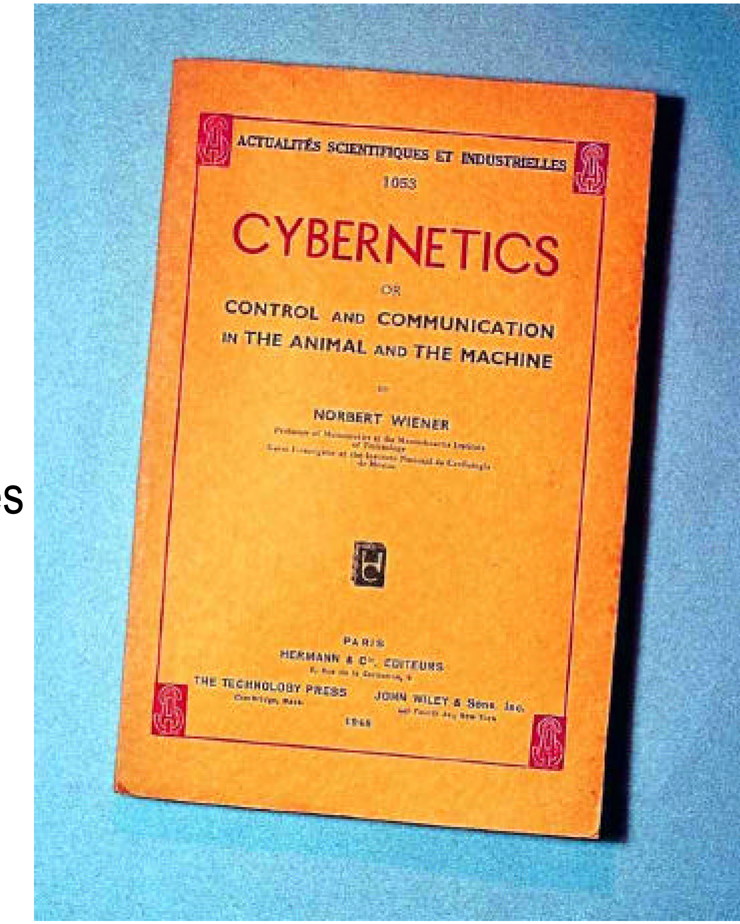




Norbert Wiener, 1948: Cybernetics

Norbert Wiener
(1894-1964)

- a new theory of information
- a new theory of prediction
- connections of both new theories
- a new way to communication techniques
- analogies between
  human nervous system and
  computing and control systems



Claude Elwood Shannon (1916-2001)

1938: A Symbolic Analysis of Relay and Switching Circuits, *Transactions of the AIEE*.

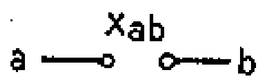


Figure 1 (left). Symbol for hindrance function

Figure 2 (middle). Interpretation of addition

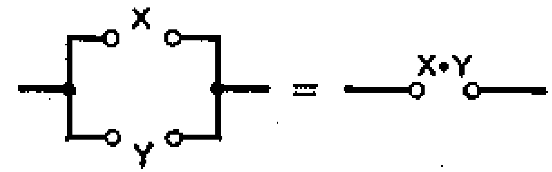


Figure 3 (right). Interpretation of multiplication

Table I. Analogue Between the Calculus of Propositions and the Symbolic Relay Analysis

Symbol	Interpretation in Relay Circuits	Interpretation in the Calculus of Propositions
X	The circuit X	The proposition X
0	The circuit is closed	The proposition is false
1	The circuit is open	The proposition is true
$X + Y$	The series connection of circuits X Y	The proposition which is true if either X or Y is true
XY	The parallel connection of circuits X and Y	The proposition which is true if both X and Y are true
X'	The circuit which is open when X is closed, and closed when X is open	The contradictory of proposition X
=	The circuits open and close simultaneously	Each proposition implies the other

Lotfi Aliasker Zadeh

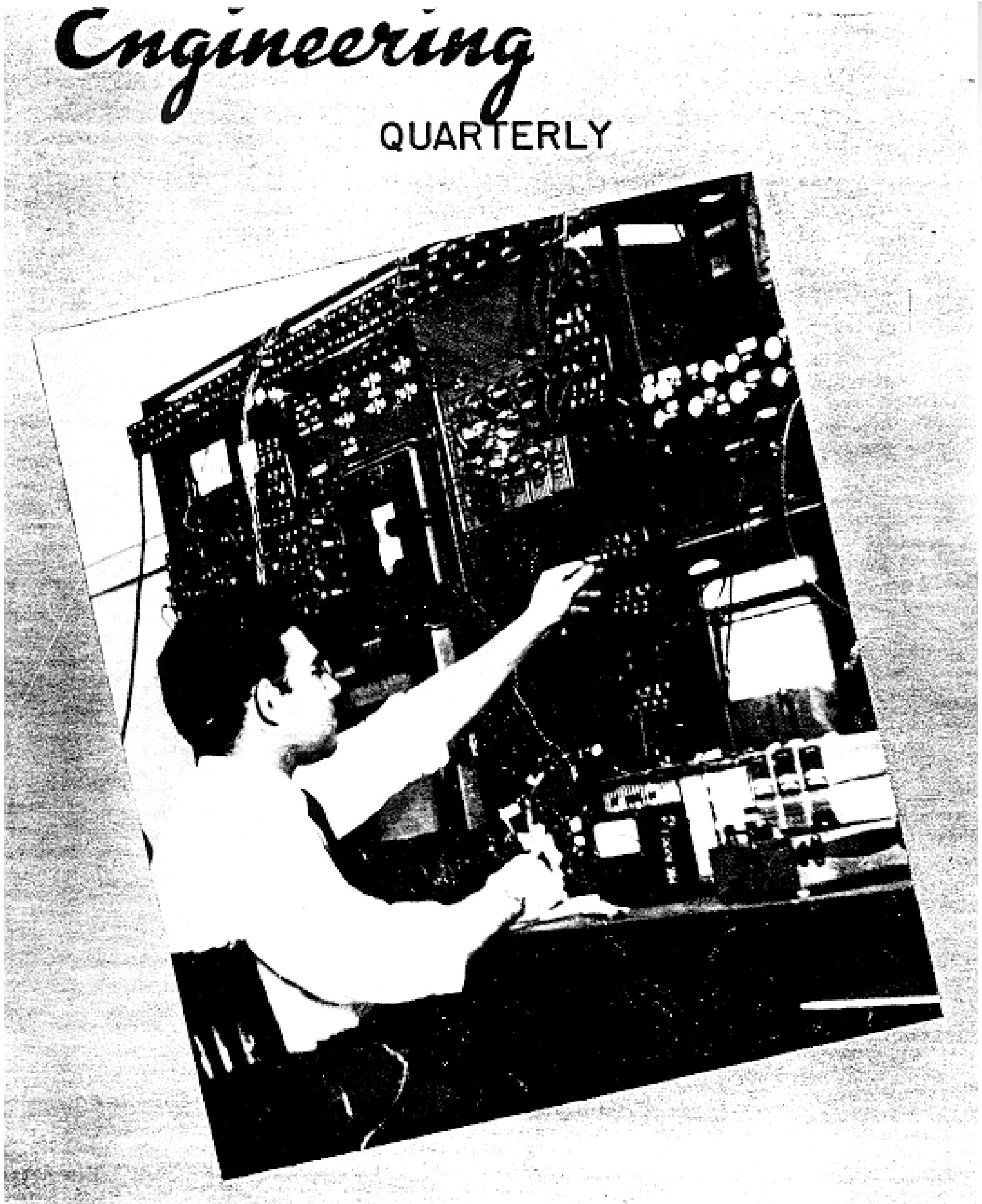
L. A. Zadeh

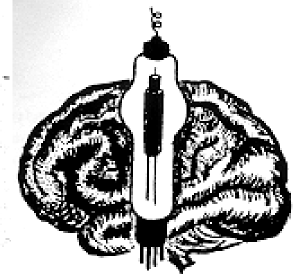
Robert Fano

J. R.Ragazzini

- born 1921 in Buku, Azerbaijan
- since 1942: Electrical Engineering, University Tehran
- then: Technical Associate of the US Army Forces in Iran
- 1944: Emigration into the USA,
  International Electronic Laboratories, New York
  Studies of Electrical Engineering at the MIT
- 1946: Master of Science, Supervisor: Robert Fano,
  Then: Columbia University, New York
- 1949: Ph. D. Thesis: *Frequency Analysis of Variable Networks*
  Supervisor: John Ralph Ragazzini
- 1950: (with Ragazzini) *An Extension of Wiener's Theory of Prediction*
- since 1952: Scientific Work: *Information Theory* and *System Theory*
- since 1964: Fuzzy Sets

Lotfi A. Zadeh, 1950: Thinking Machines, *Columbia Engineering Quarterly*, Jan. 1950.

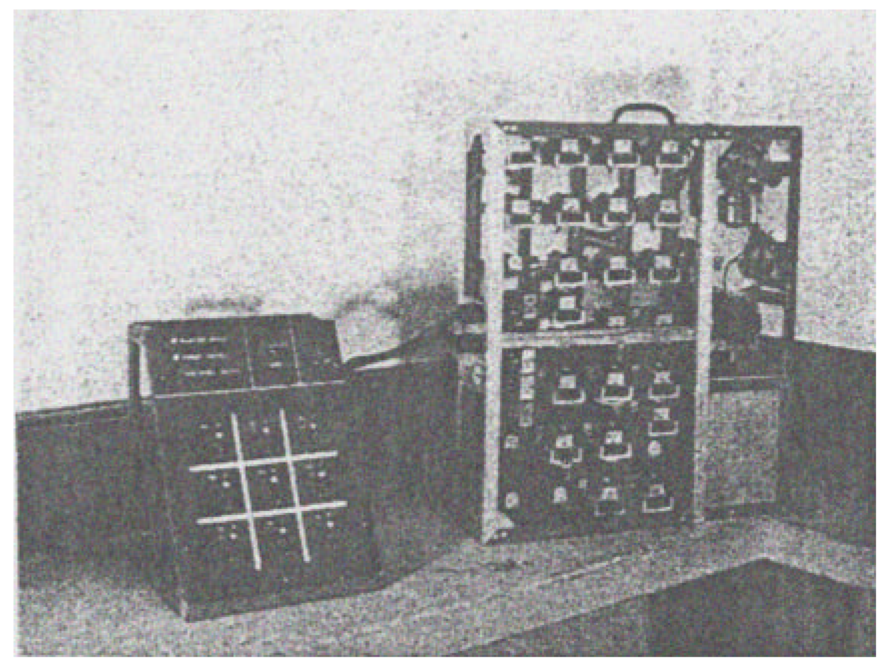




THINKING MACHINES

A New Field in Electrical Engineering

DR. LOFTI A. ZADEH
ELECTRICAL ENGINEERING DEPT.



The two units of R. Haufes Tit-Tat-Toe machine.

Lotfi A. Zadeh, 1950: Thinking Machines, *Columbia Engineering Quarterly*, Jan. 1950.

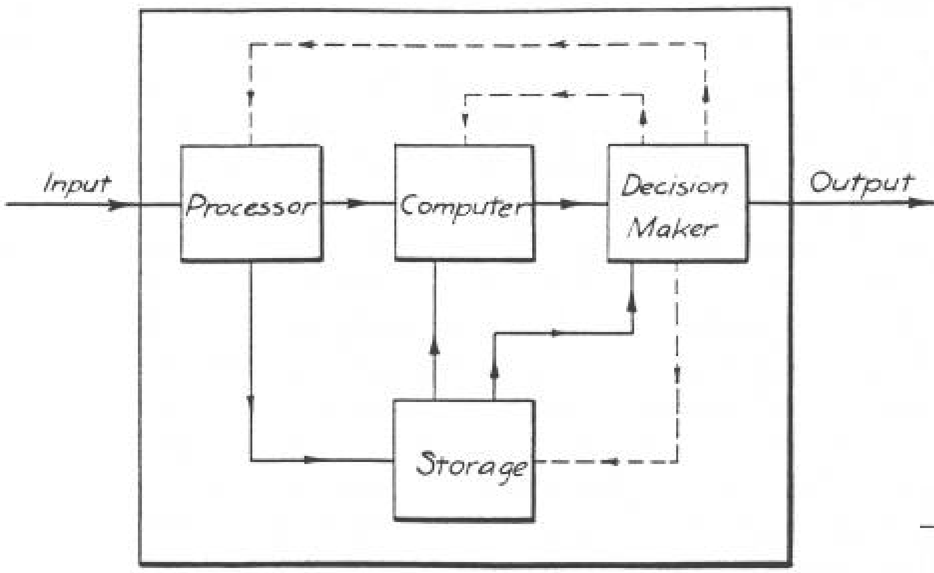


Figure 1—A schematic diagram illustrating how the basic elements of a thinking machine are arranged.

Lotfi A. Zadeh

Relay Circuit Element	Symbolic Logic Interpretation
Circuit A	Statement A
Closed circuit	A is false
Open circuit	A is true
Series connection of A and B	A and/or B $(A \vee B)$
Parallel connection of A and B	A and B $(A \cdot B)$

Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)
Series II, Vol. 14, No. 5, pp. 201-204.

Problem:

Let $X = \{x(t)\}$ be a set of signals.
An arbitrarily selected member of this set, say $x(t)$,
is transmitted through a noisy channel G
and is received as $y(t)$.

As a result of the noise and distortion introduced by G, the received signal $y(t)$ is, in general, quite different from $x(t)$.

Nevertheless, under certain conditions it is possible to recover $x(t)$ – or rather a time-delayed replica of it – from the received signal $y(t)$.

$$y = G\ x \qquad \textit{resp.} \qquad x = G^{-1} y$$

Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)
Series II, Vol. 14, No. 5, pp. 201-204.

Special case: *reception process*:

Let $X = \{x(t)\}$ consist of a finite number of discrete signals $x_1(t)$, $x_2(t)$, …, $x_n(t)$, which play the roles of symbols or sequences of symbols.

The replicas of all these signals are assumed to be available at the receiving end of the system. Suppose that a transmitted signal x_k is received as y.

To recover the transmitted signal from y, the receiver evaluates the 'distance' between y and all possible transmitted signals x_1, x_2, …, x_n, by the use of a suitable distance function $d(x, y)$, and then selects that signal which is 'nearest' to y in terms of this distance function.

Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)
Series II, Vol. 14, No. 5, pp. 201-204.

Distance functions:

- $d(x, y) = \text{l.u.b.}\ |x(t) - y(t)|$
- $d(x, y) = \{1/T \int_0^T [x(t) - y(t)]^2\, dt\}^{1/2}$
- $d(x, y) = \text{l.u.b.}\ \{1/T_0 \int_0^{t+T} [x(t) - y(t)]^2\, dt\}^{1/2}$
- $d(x, y) = 1/T \int_0^T |x(t) - y(t)|\, dt$

Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)
Series II, Vol. 14, No. 5, pp. 201-204.

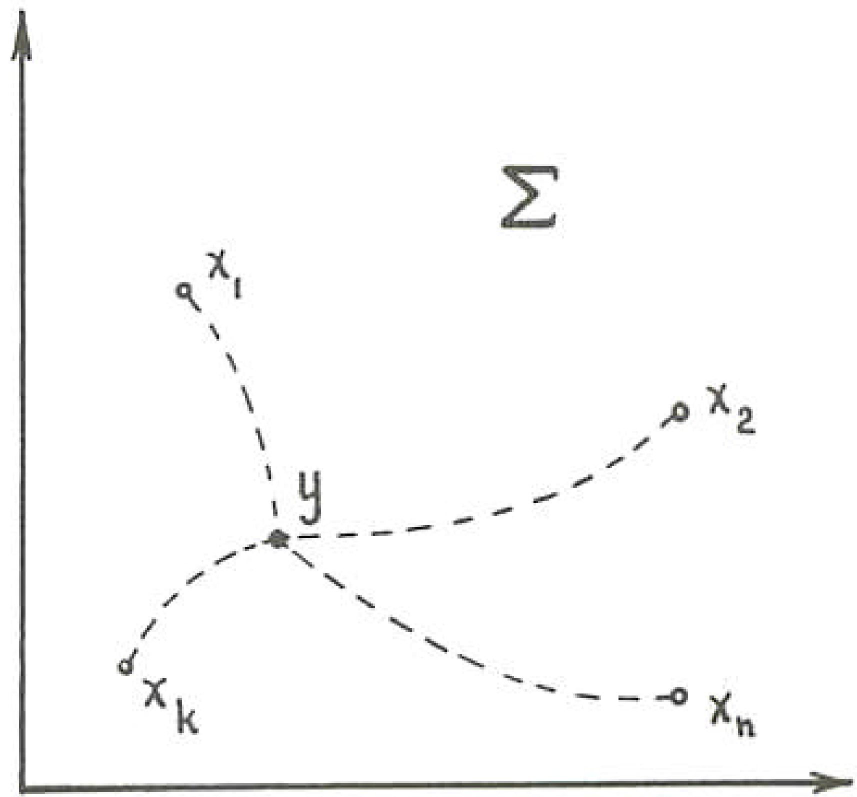


FIGURE 1. Recovery of the input signal by means of a comparison of the distances between the received signal y and all possible transmitted signals.

$d(x_k, y) < d(x_i, y)$ $i \neq k$, for all k and i.

‘In many practical situations it is inconvenient, or even impossible, to define a quantitative measure, such as a distance function, of the disparity between two signals.

In such cases we may use instead the concept of neighorhood, which is basic to the theory of topological spaces.’

Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)
Series II, Vol. 14, No. 5, pp. 201-204.

Problem: multiplex transmission of two or more signals; the system has two channels.

$X = \{x(t)\}$ and $Y = \{y(t)\}$: sets af signals assigned to their respective channels.

At the receiving end: sum signal: $u(t) = x(t) + y(t)$.

Extract $x(t)$ and $y(t)$ from $u(t)$!

That means Find two filters N_1 and N_2 such, that, for any x in X and any y in Y,

$$N_1(x + y) = x \quad \text{and} \quad N_2(x + y) = y$$

Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)
Series II, Vol. 14, No. 5, pp. 201-204.

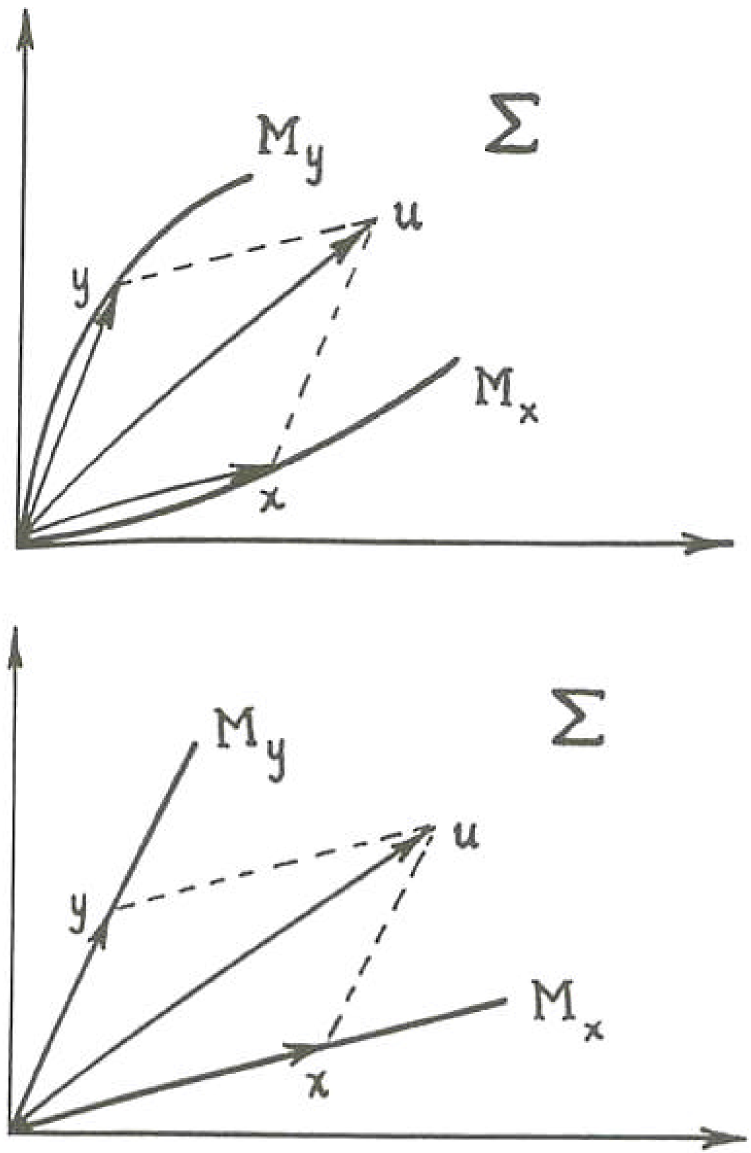


FIGURE 2. (a) Geometrical representation of nonlinear filtering. (b) Geometrical representation of linear filtering.

Geometrical representation of

nonlinear filtering

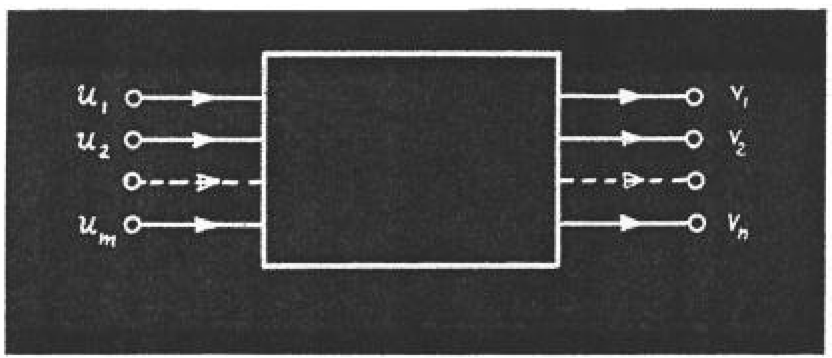
and

linear filtering

in terms of two-dimensional signal spaces.

Lotfi A. Zadeh, 1954: System Theory, *Columbia Engineering Quarterly*, Nov. 1954.

System Theory



L. A. Zadeh
Associate Professor
Electrical Engineering

System:
„an aggregation or assemblage of objects united by some form of interaction or interdependence“

(*Webster's dictionary*)

Block diagram

A: z = f(x, y, w)
B: u = g(z, v)
C: v = h(u)
w = k(u)

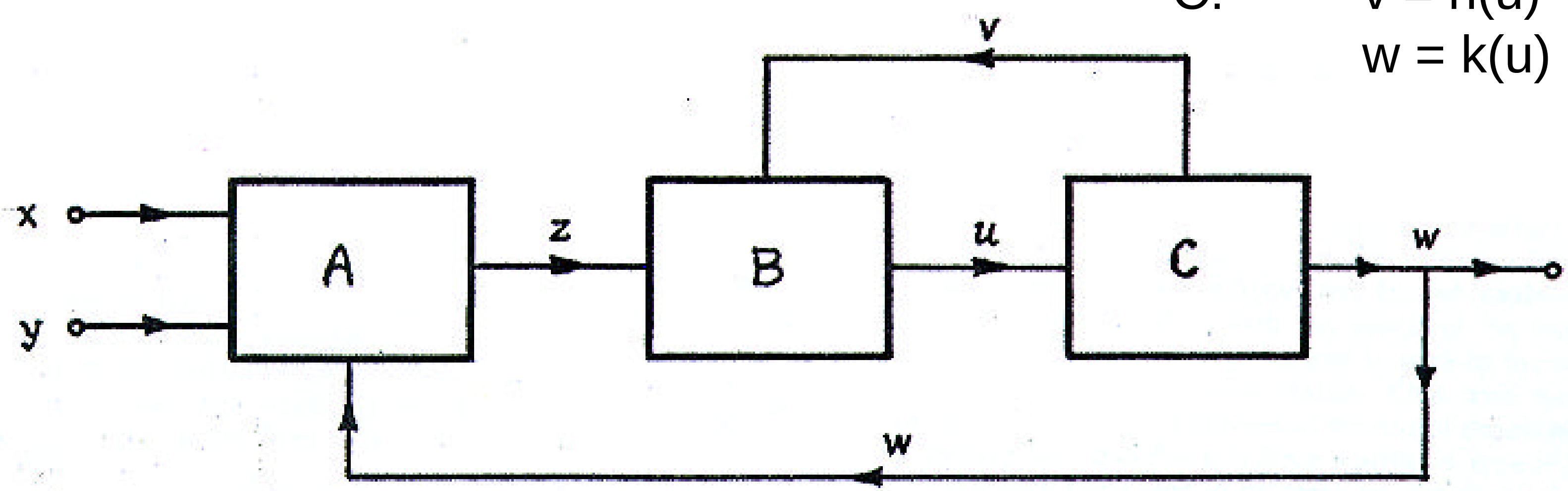
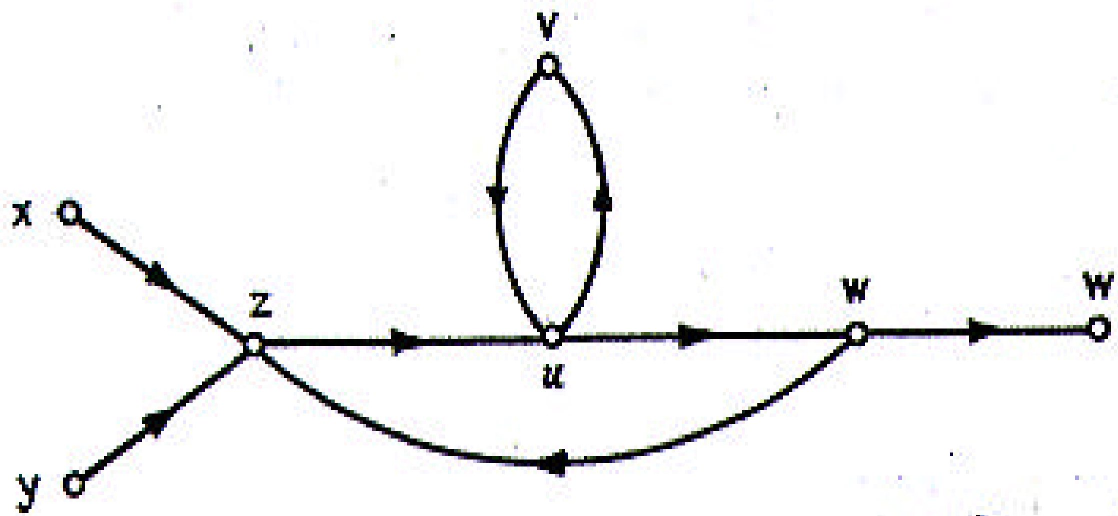


Fig. 2a

Lotfi A. Zadeh, 1954: System Theory, *Columbia Engineering Quarterly*, Nov. 1954.



(b)
Fig. 2b

Linear graph

Matrix

	u	v	w	x	y	z
u	0	1	1	0	0	0
v	1	0	0	0	0	0
w	0	0	0	0	0	1
x	0	0	0	0	0	1
y	0	0	0	0	0	1
z	1	0	0	0	0	0

Lotfi A. Zadeh, 1954: System Theory, *Columbia Engineering Quarterly*, Nov. 1954.

Fig. 1.2.1 Diagrammatic representation of a system $\mathfrak{A}$ with input u and output y.

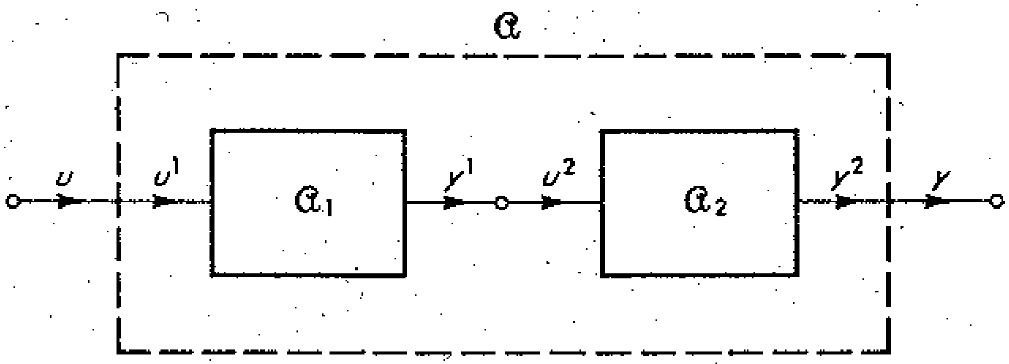


Fig. 1.4.1 Tandem combination of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

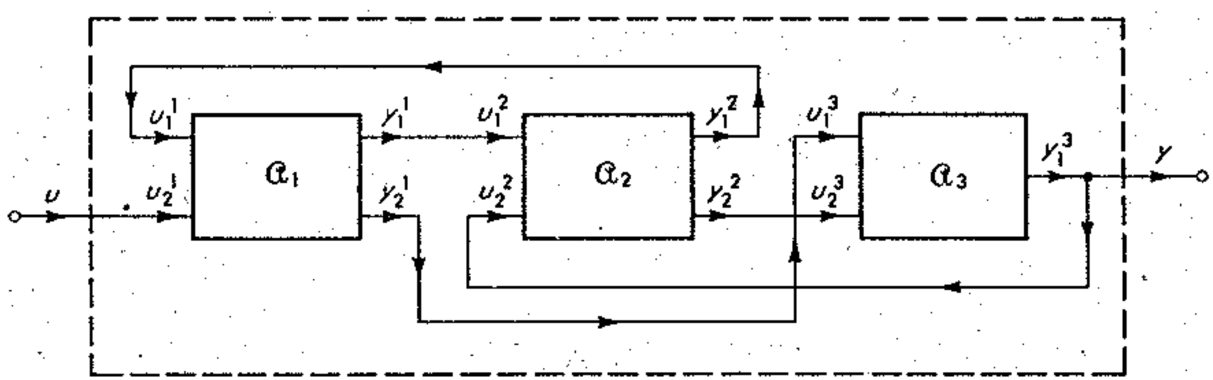


Fig. 1.4.2 Example of a system which is a combination of three component systems $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$.

Input-output-relationship:

$y = f(u)$

Lotfi A. Zadeh: 1963, Linear System Theory

System with two variables v_1 and v_2;

$$\frac{dv_2}{dt^2} = \frac{d^2 v_1}{dt^2} + v_1$$

This system can be realized in different forms.

Lotfi A. Zadeh: 1963, Linear System Theory

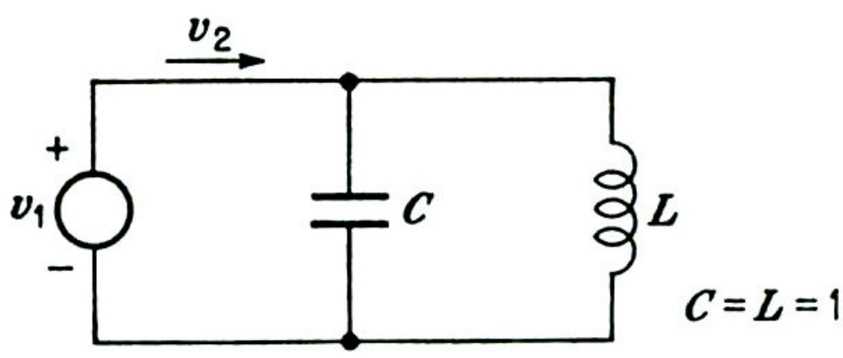


Fig. 1.4.1 A network realization of the object of Example *1.4.14*.

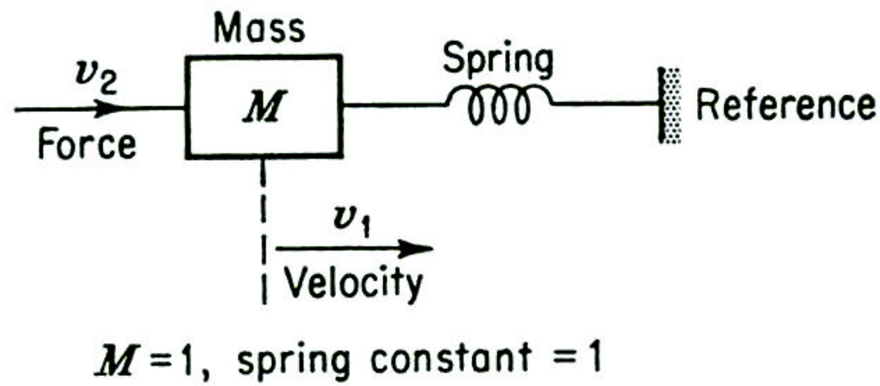


Fig. 1.4.2 A mechanical realization of the object of Example *1.4.14*.

Physical Realization 1:

elektrical network.

v_1: voltage

v_2: current .

Physical Realization 2:

mechanical system.

v_2: force at particle

v_1: velocity of the particle

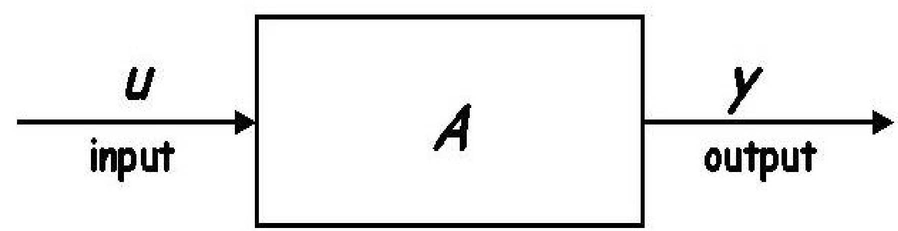
Lotfi A. Zadeh, 1963: Views on General Systems Theory

Proceedings of The Second Systems Symposium
at Case Institute of Technology, April 1963, Cleveland, Ohio

A System is a big black box
Of which we can't unlock the locks,
And all we can find out about
Is what goes in and what goes out.
Perceiving input-output pairs,
Related by parameters,
Permits us, sometimes, to relate
An input, output, and a state.
If this relation's good and stable
Then to predict we may be able,
But if this fails us – heaven forbid!
We'll be compelled to force the lid!

Kenneth E. Boulding

u : input y : output s : state

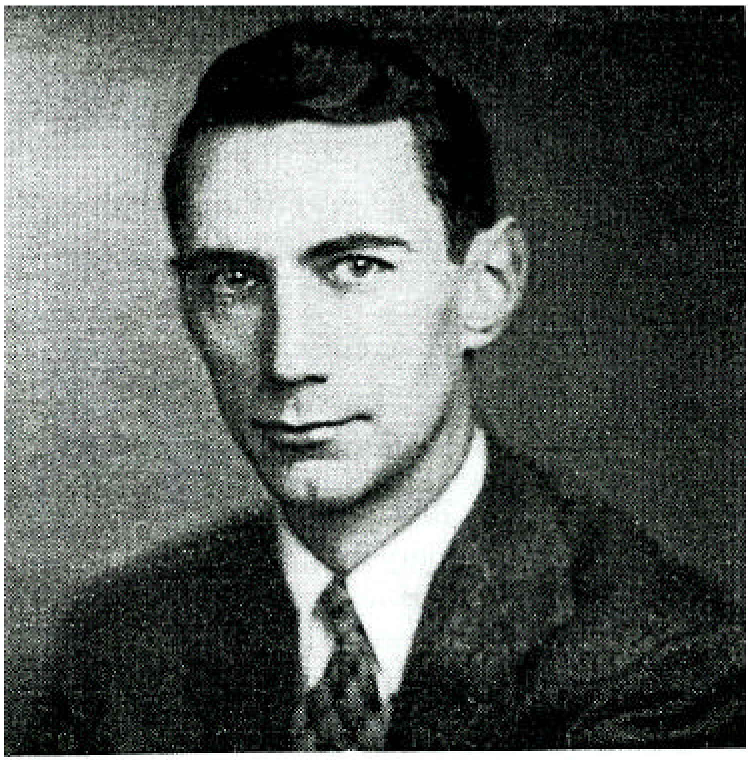


$s_{t+1} = f(s_t, u_t)$, t = 0, 1, 2, ...

$y_t = g(s_t, u_t)$

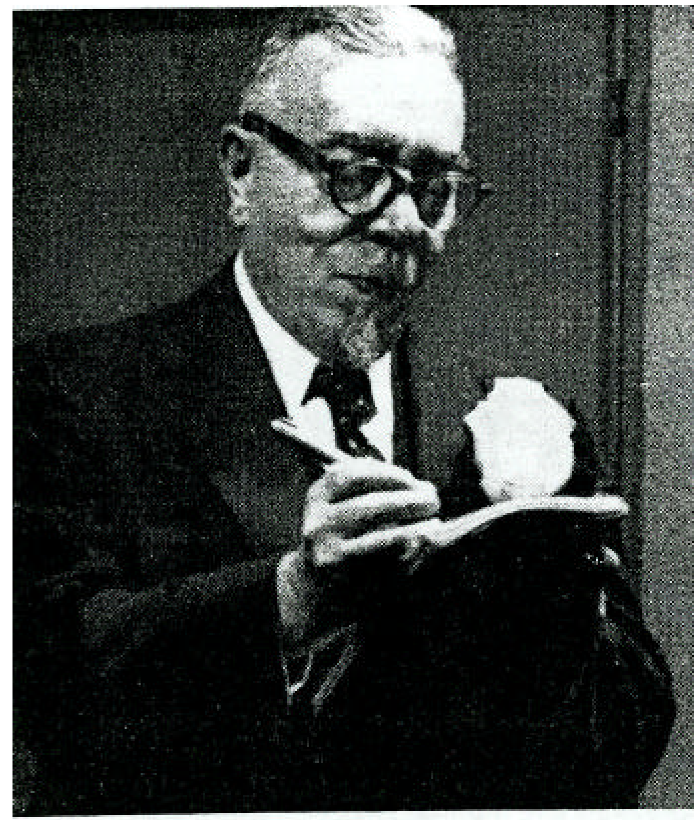
IRE Transactions on Information Theory, March/June1956

The Bandwagon



CLAUDE E. SHANNON

What is Information Theory?

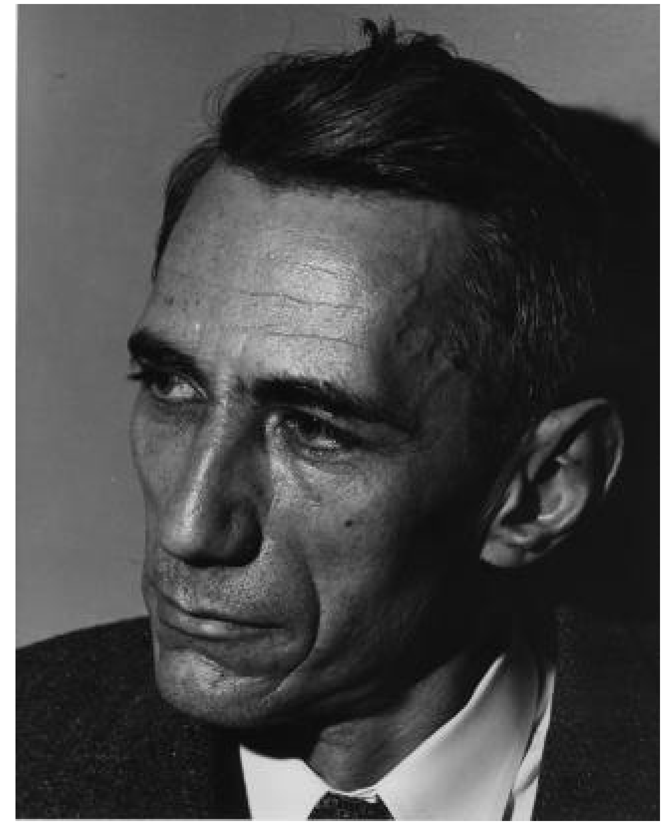


NORBERT WIENER

C. E. Shannon: The Bandwagon, *IRE Transactions on Information Theory*, March 1956

Indeed, the hard core of information theory is, essentially, a branch of mathematics, a strictly deductive system.

Research rather than exposition is the keynote, and our critical thresholds should be raised.



N. Wiener: What is Information Theory? *IRE Trans. on Information Theory*, June 1956

I am pleading in this editorial that Information Theory go back of its slogans and return to the point of view from which it originated: that of the general statistical concept of communication.

I hope that these Transactions may encourage this integrated view of communication theory by extending its hospitality to papers which, why they bear on communication theory, cross its boundaries, and have a scope covering the related statistical theories. In my opinion we are in a dangerous age of overspecialization.

Richard Bellman, Robert Kalaba, 1957:
On the Role of Dynamic Programming in Statistical Communication Theory

R. BELLMAN

R. KALABA

In mathematical terms, let

x = the pure signal emanating from S.
r = the noise associated with the signal.
$x' = F(x, r)$, the input to the communication system.
y = the signal transmitted to the observer by the communication channel. (1)

Let us further write

$$y = T(x') = T(F(x, r)), \tag{2}$$

What Is Optimal?

Lotfi A. Zadeh

Criterion *A*:

- Design D_1 might be better than D_2, and
- Design D_2 might be better than D_3.

Criterion *B*:

- Design D_2 might be better than D_3, and
- Design D_3 might be better than D_1.

Criterion *C*:

- Design D_3 might be better than D_1, and
- Design D_1 might be better than D_2.

L. A. Zadeh, 1962: From Cercuit Theory to System Theory

In: *Proceedings of the IRE*, May 1962, pp. 856-865.

In fact, there is a fairly wide gap between what might be regarded as „animate“ system theorists and „inanimate“ system theorists at the present time, and it is not at all certain that this gap will be narrowed, much less closed, in the near future.

There are some who feel that this gap reflects the fundamental inadequacy of the conventional mathematics – the mathematics of precisely-defined points, functions, sets, probability measures, etc. - for coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. Indeed, the need for such mathematics is becoming increasingly apparent even in the realm of inanimate systems, for in most practical cases the *a priori* data as well as the criteria by which the performance of a man-made system is judged are far from being precisely specified or having accurately-known probability distributions.

L. A. Zadeh, 1963: Optimality and Non-Scalar-Valued Performance Criteria

Consider the constraint set $C \subseteq \mathcal{S}$ is defined by the constraints imposed on system S,
and a partial ordering $\geq$ on $\mathcal{S}$ by associating with each system S in $\mathcal{S}$
the following three disjoint subsets of the set of systems $\mathcal{S}$:

$\mathcal{S}_{>}(S)$:	the subset of all systems which are superior to S.
$\mathcal{S}_{\leq}(S)$:	the subset of all systems which are inferior or equal to S.
$\mathcal{S}_{\sim}(S)$:	the subset of all systems which are not comparable with S.

$$\mathcal{S}_{>}(S) \cup \mathcal{S}_{\leq}(S) \cup \mathcal{S}_{\sim}(S) = \mathcal{S}.$$

L. A. Zadeh, 1963: Optimality and Non-Scalar-Valued Performance Criteria

Definition 1: A system S_0 in C is *noninferior* in C
if the intersection of C and $S_>(S)$ is emty:

$$C \cap S_>(S_0) = \emptyset.$$

Definition 2: A system S_0 in C is *optimal* in C
if C is contained in $S_\leq(S)$:

$$C \subseteq S_\leq(S_0).$$

L. A. Zadeh, 1963: Optimality and Non-Scalar-Valued Performance Criteria

If S is completely ordered by a scalar-valued criterion, then:

$$S_{\sim}(S_0) = \varnothing$$

and

$$S_{>}(S_0) \quad \text{and} \quad S_{\leq}(S_0) \text{ are complementary classes.}$$

Then:

If $C \cap S_{>}(S_0) = \varnothing$, then: $S_{\leq}(S_0) \subseteq C$, and

hence noninferiority and *optimality become equivalent* concepts.

L. A. Zadeh, 1963: Optimality and Non-Scalar-Valued Performance Criteria

Let system *S* be characterized by the vector $\boldsymbol{x} = (x_1, ..., x_n)$,
whose real-valued components represent, say, the values of *n* adjustable parameters of S,
and let *C* be a subset of *n*-dimensional Euclidean space $\mathbf{R}^n$.

Let the performance of system *S* be measured by an *m*-vector

$$\boldsymbol{p}(x) = [p_1(x), ..., p_m(x)]$$

where $p_i(x)$, $i = 1, ..., m$, is a given real-valued function of x.

Then: $S = S' \Leftrightarrow p(x) = p(x')$. That is: $p_i(x) = p_i(x')$, $i = 1, ..., m$.

L. A. Zadeh, 1963: Optimality and Non-Scalar-Valued Performance Criteria

Case:

$S_>(S)$ or, equivalently,
$S_>(x)$ is a fixed cone
with vertex at x.

and the constraint set C is
a closed bounded subset of $\mathbf{R}^n$.

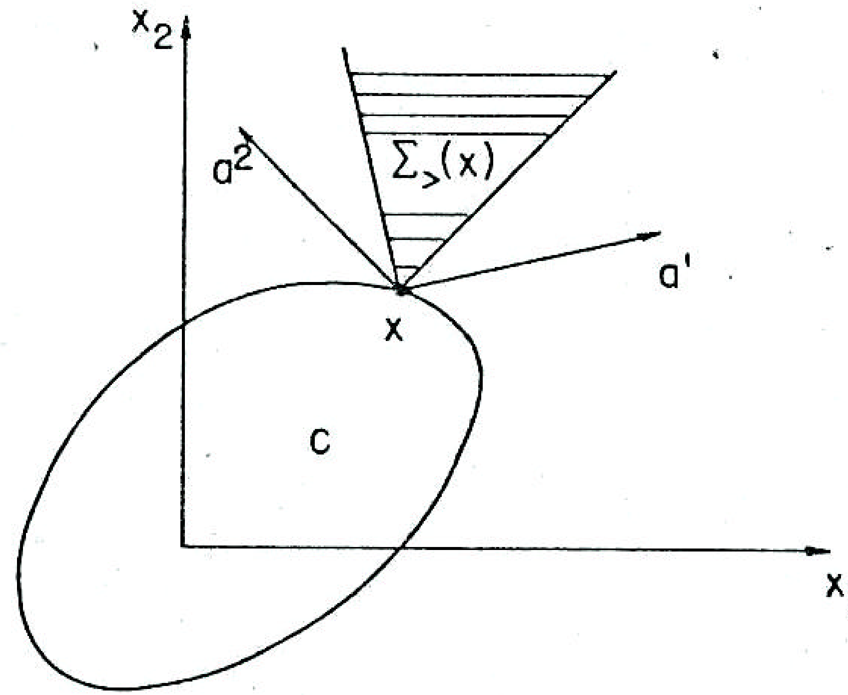


Fig. 1—Illustration of the significance of C and $\Sigma_>(x)$.

Example:

$$p_i(x) = a_i^i x_1 + \ldots + a_n x_n^i,$$

where $a^i = (a_i^i, \ldots, a_n^i)$ is the gradient of $p_i(x)$,

(a constant vector): $a^i = \text{grad } p_i(x)$.

Then: $\Sigma_>(x)$ is the polar cone of the cone spanned by a^i.

L. A. Zadeh, 1963: Optimality and Non-Scalar-Valued Performance Criteria

Definition 1 $\Rightarrow$ *Noninferior* points cannot occur in the interior of the set *C*.

If *C* is a convex set then the set of all noninferior points on the boundary of *C* is the set G of all points x_0, through which hyperplanes separating the set *C* and the set $S_>(x_0)$ can be passed.

The set G is heavy lined on the boundary of *C*.

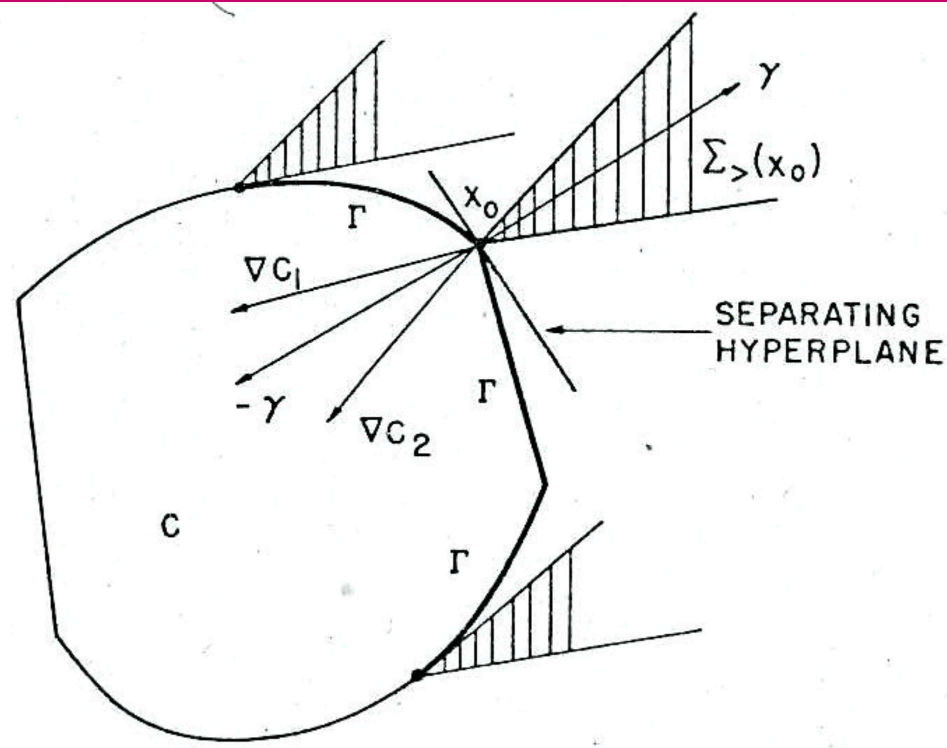


Fig. 2—The set of noninferior points on the boundary of *C*.

Let x_0 be such a point and let g be the normal to the separating hyperplane at x_0, with g directed away from the interior of *C*. Then g belongs to the polar cone of $S_>(x_0)$ since g makes nonobtuse angles with all vectors in $S_>(x_0)$.

1964: Lotfi Zadeh, Talk on Pattern Recognition in Dayton, Ohio (Wright-Patterson Air Base)

R. Bellman, R. Kalaba, L. A. Zadeh, 1964: *Abstraction And Pattern Classification*

Richard Bellman

Robert Kalaba

Lotfi A. Zadeh

MEMORANDUM
RM-4307-PR
OCTOBER 1964

ABSTRACTION AND PATTERN CLASSIFICATION

R. Bellman, R. Kalaba and L. A. Zadeh

This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-700 monitored by the Directorate of Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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The RAND Corporation

Letter: Bellman to Zadeh, September 9, 1964

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS

EDITOR:

RICHARD BELLMAN
The RAND Corporation
1700 Main Street
Santa Monica, California

PUBLISHERS:

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New York 3, New York

9 September 1964

Professor Lotfi Zadeh
Department of Electrical Engineering
University of California
Berkeley 4, California

Dear Lotfi:

I think that the paper is extremely interesting and I would like to publish it in JMAA, if agreeable to you. When I return, or while in Paris, I will write a companion paper on optimal decomposition of a set into subsets along the lines of our discussion.

Cordially,

Richard Bellman

RB:jb

Lotfi A. Zadeh, 1965: A New View of System Theory

Symposium on System Theory,
April 20., 21. and 22. 1965, Polytechnic Institute, Brooklyn.

FUZZY SETS AND SYSTEMS*

L. A. Zadeh
Department of Electrical Engineering, University of California, Berkeley, California

The notion of fuzziness as defined in this paper relates to situations in which the source of imprecision is not a random variable or a stochastic process, but rather a class or classes which do not possess sharply defined boundaries, e.g., the "class of bald men," or the "class of numbers which are much greater than 10," or the "class of adaptive systems," etc.

A basic concept which makes it possible to treat fuzziness in a quantitative manner is that of a fuzzy set, that is, a class in which there may be grades of membership intermediate between full membership and non-membership. Thus, a fuzzy set is characterized by a membership function which assigns to each object its grade of membership (a number lying between 0 and 1) in the fuzzy set.

After a review of some of the relevant properties of fuzzy sets, the notions of a fuzzy system and a fuzzy class of systems are introduced and briefly analyzed. The paper closes with a section dealing with optimization under fuzzy constraints in which an approach to problems of this type is briefly sketched.

$$s_{t+1} = f(s_t, u_t),$$

$$y_t = g(s_t, u_t)$$

$t = 0, 1, 2, \ldots$

S is a fuzzy system if *u(t)* or *y(t)* *or* *s(t)* or any combination are fuzzy sets.

INFORMATION AND CONTROL 8, 338–353 (1965)

Fuzzy Sets*

L. A. ZADEH

Department of Electrical Engineering and Electronics Research Laboratory, University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

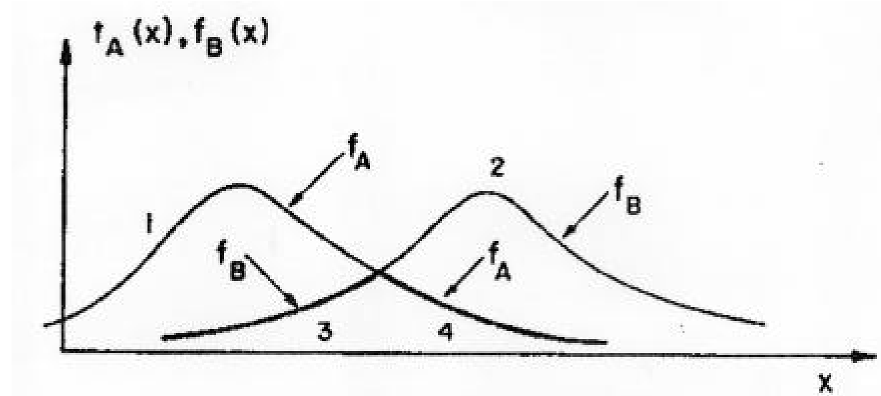
I. INTRODUCTION

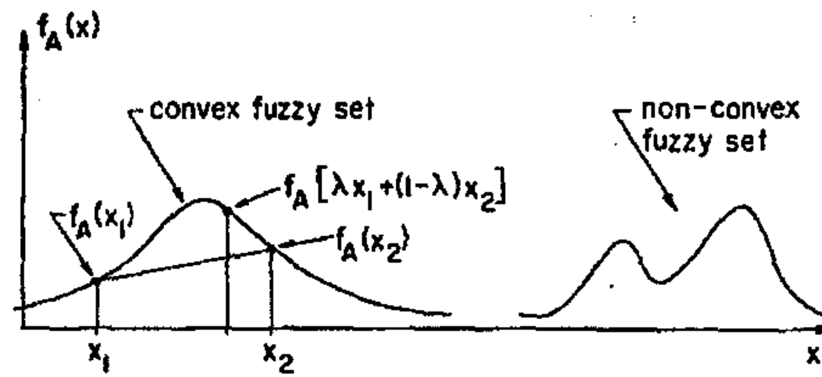
More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1.

Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.

The purpose of this note is to explore in a preliminary way some of the basic properties and implications of a concept which may be of use in

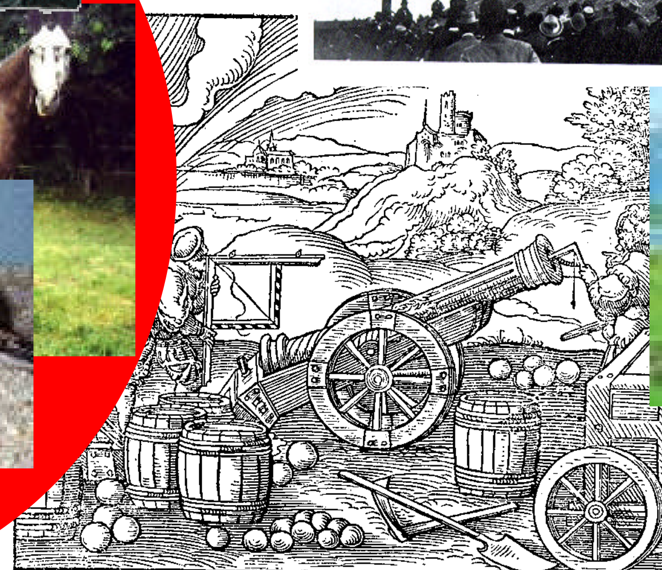
* This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Grant No. AF-AFOSR-139-64 and by the National Science Foundation under Grant GP-2413.





Georg Cantor, 1895/97: Set Theory

Georg Cantor (1845-1918):



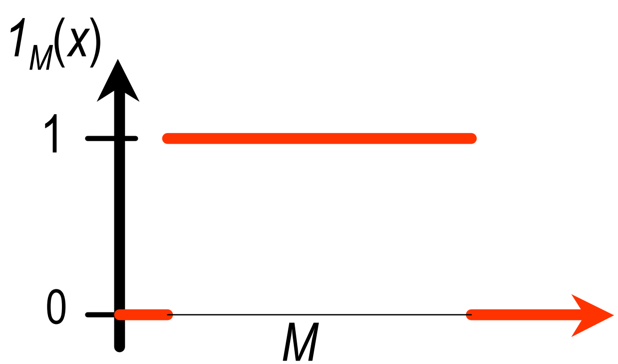
Georg Cantor, 1895/97: Set Theory

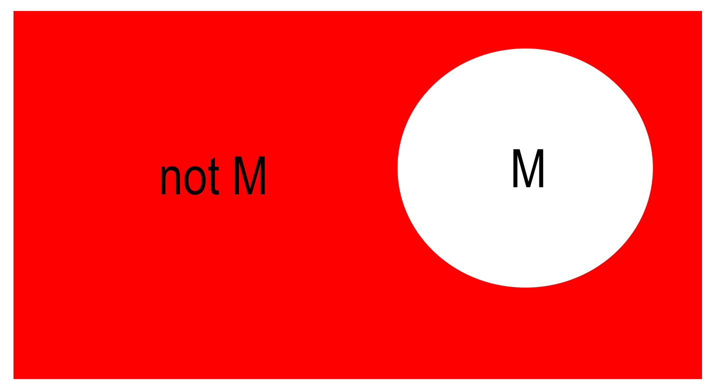
Georg Cantor
(1845-1918):

Definition:

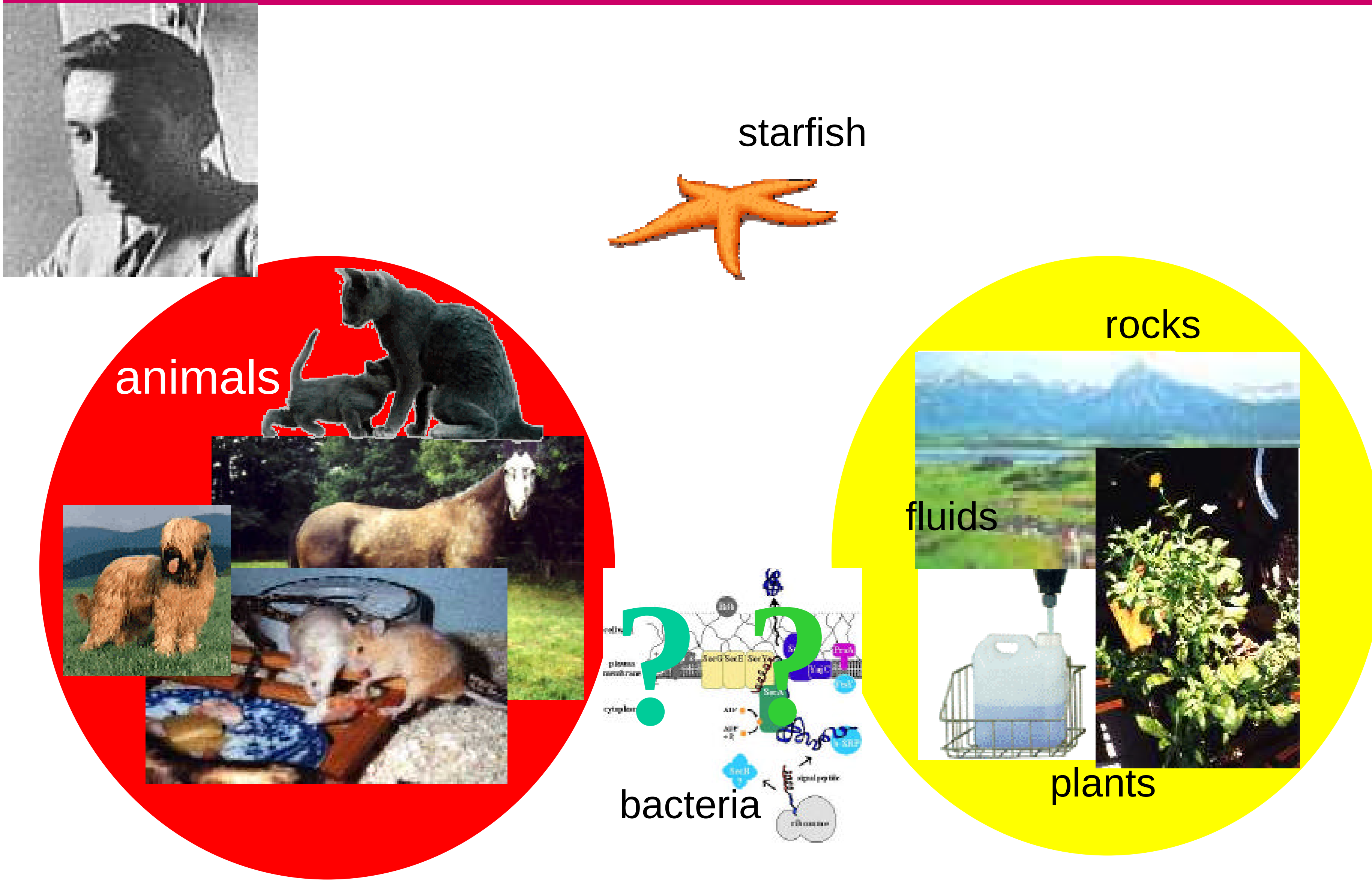
?A set is a collection into a whole *M* of definite and separate objects *m* of our intuition or thought.'

„Unter einer Menge verstehen wir jede Zusammenfassung *M* von bestimmten, wohlunterschiedenen Objekten *m* unserer Anschauung oder unseres Denkens (welche die Elemente von *M* genannt werden) zu einem Ganzen."





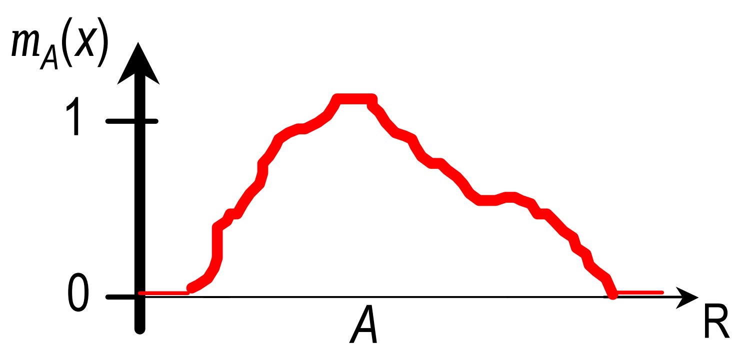
Lotfi A. Zadeh, 1965: Fuzzy Sets

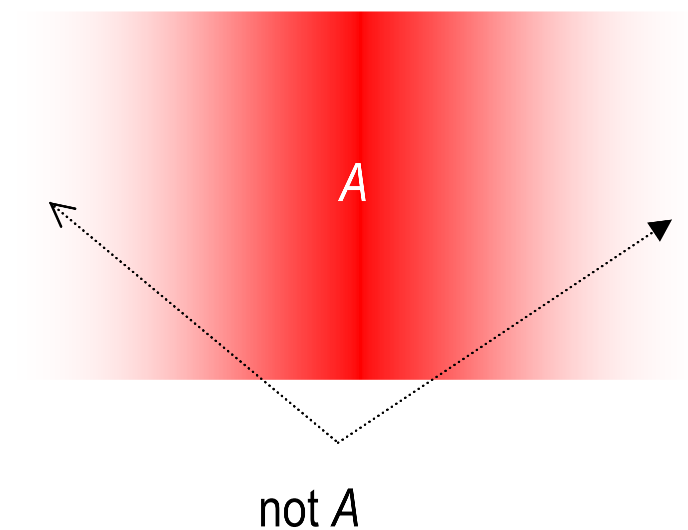


Lotfi A. Zadeh, 1965: Fuzzy Sets

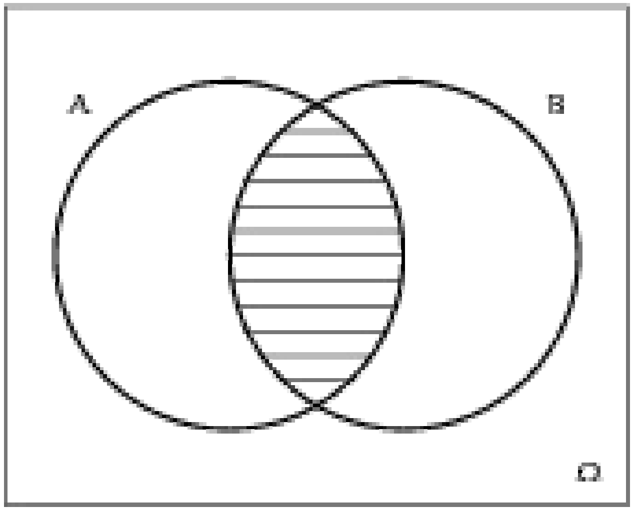
Definition:

„A fuzzy set (class) *A* in *X* is characterized by a membership function (characteristic function) $m_A(x)$ which associates with each point in *X* a real number in the intervall [0,1], with the value of $m_A(x)$ at *x* representing the ‚grade of membership' of x in A."

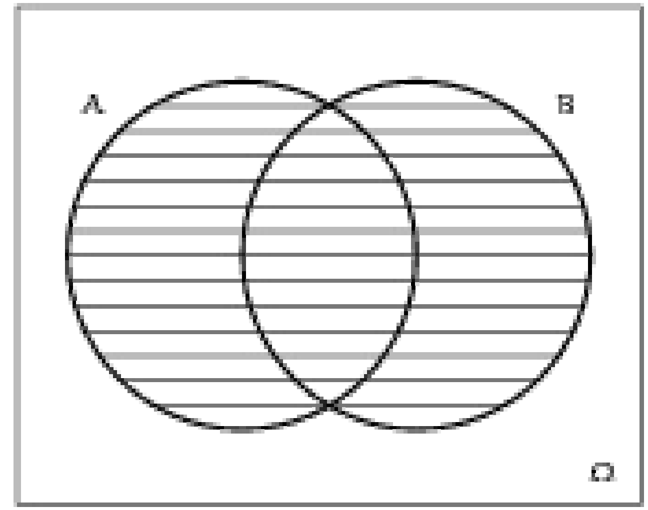




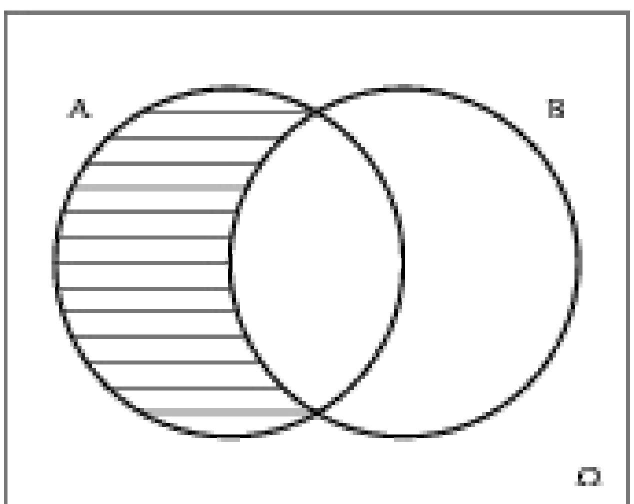
Set Theory



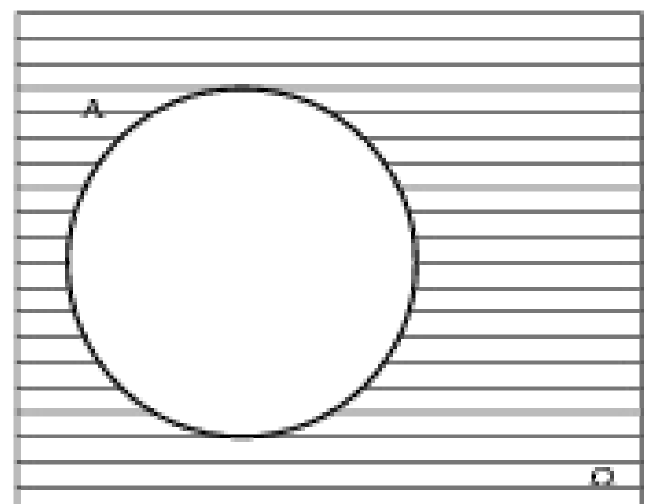
(a) $A \cap B$



(b) $A \cup B$



(c) $A \setminus B$



(d) $\bar{A}$

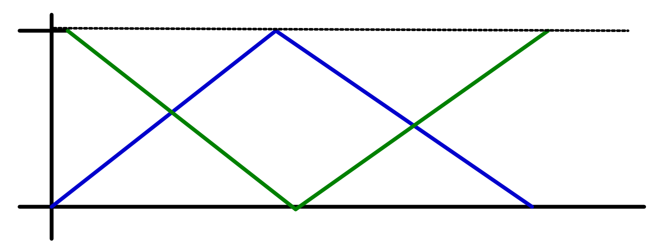
Lotfi A. Zadeh, 1965: Fuzzy Sets

A fuzzy set is *empty* iff: $m_A(x) = 0, \quad x \in X.$

Equal fuzzy sets, *A* = *B*, iff: $m_A(x) = m_B(x), \quad x \in X.$

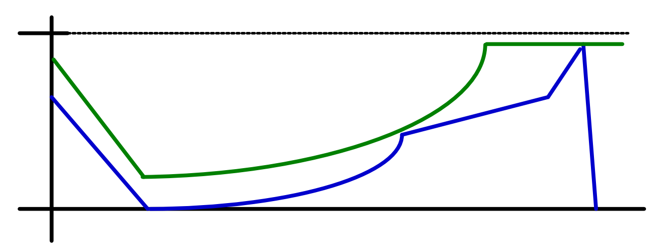
The *complement A'* of a *fuzzy set A* is defined by:

$$m_{A'}(x) = 1 - m_{A'}(x) \; x \in X.$$



Containment: *A Í B* iff:

$$m_A(x) \leq m_B(x), \qquad x \in X.$$

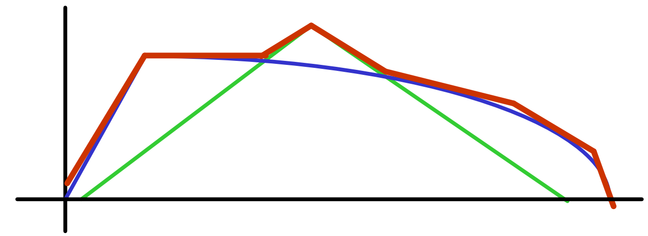


Lotfi A. Zadeh, 1965: Fuzzy Sets

Union $A \cup B$ of two *fuzzy sets*

A and B with resp. membership functions

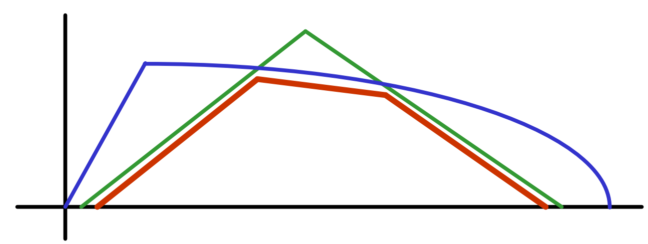
$m_{A \cup B}(x) = \max \{m_A(x), m_B(x)\}, x \in X$



Intersection $A \cap B$ of *fuzzy sets*

A and B with resp. membership functions

$m_{A \cap B}(x) = \min \{m_A(x), m_B(x)\}, x \in X$



Lotfi A. Zadeh, 1965: Fuzzy Sets

Let A and B be two bounded fuzzy sets.

Let H be a hypersurface in $\mathbf{E}^n$ defined by an equation $h(x) = 0$,

with all points x, for which $h(x) = 0$ being on one side of H

and all points x, for which $h(x) = 0$ being on the other side of H.

Let K_H be a number dependent on H such that:

$f_A(x) = K_H$ on one side of H

and $f_A(x) = K_H$ on th other side.

Lotfi A. Zadeh, 1965: Fuzzy Sets

Let M_H be Inf K_H.

The number $D_H = 1- M_H$ will be called the *degree of separation* of *A* and *B* by *H.*

In general: given a family of hypersurfaces $\{H_\lambda\}$ with λ ranging over $\mathbf{E}^m$:

Problem: Find a member of $\{H_\lambda\}$ which realizes the highest possible degree of separation!

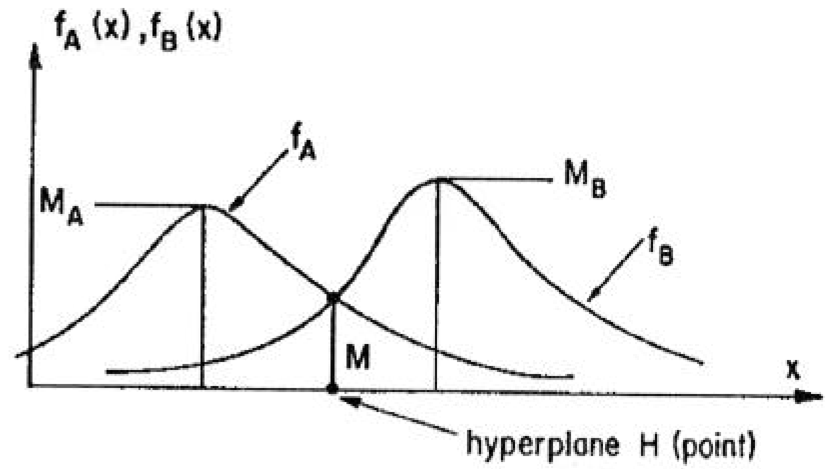
Special case: H_λ are hyperplanes in euclidean space $\mathbf{E}^n$, with λ ranging over $\mathbf{E}^n$:

In this case, we define the degree of separability of *A* and *B by*: $D = 1 – M$, where $M = \text{Inf}_H M_H$ is the infimum of M_H of all hypersurfaces *H*.

Lotfi A. Zadeh, 1965: Fuzzy Sets

The highest degree of separation of two convex fuzzy sets *A* and *B* that can be achieved with a hyperplane in $\mathbf{E}^n$ is one minus the maximal grade in the intersection $A \cap B$.

(Figur: case n = 1.)



Theorem:
Let A and B be bounded convex fuzzy sets in $\mathbf{E}^n$, with maximal grades M_A and M_B, respectively [$M_A = \text{Sup}_x f_A(x)$ and $M_B = \text{Sup}_x f_B(x)$].

Let M be the maximal grade for the intersection $A \cap B$
($M = \text{Sup}_x \text{ Min } [f_A(x), f_B(x)]$).
Then $D = 1 - M$.

Lotfi A. Zadeh, 1965: Fuzzy Sets

"Specifically, let $f_i(x)$ $i = 1, ..., n$, denote the value of the membership function of A_i at x.

Associate with $f_i(x)$ a sieve $S_i(x)$ whose meshes are of size $f_i(x)$.

Then, $f_i(x) \cup f_j(x)$ and $f_i(x) \cap f_j(x)$ correspond, respectively, to parallel and series combinations of $S_i(x)$ and $S_j(x)$. ... ."

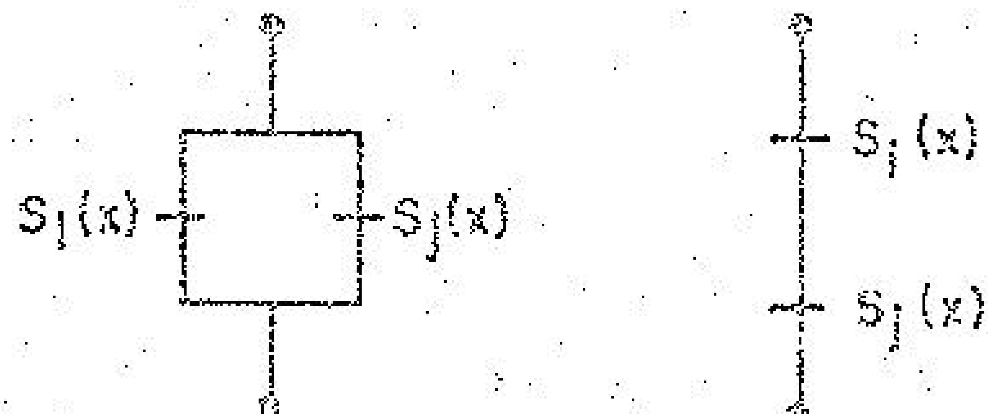


FIG. 2. Parallel and series connection of sieves simulating $\cup$ and $\cap$

Lotfi A. Zadeh, 1965: Fuzzy Sets

“More generally, a well formed expression involving $A_1, \ldots, A_n$, $\cup$ and $\cap$ corresponds to a network of sieves $S_1(x), \ldots, S_n(x)$ which can be found by the conventional synthesis techniques for switching circuits.”

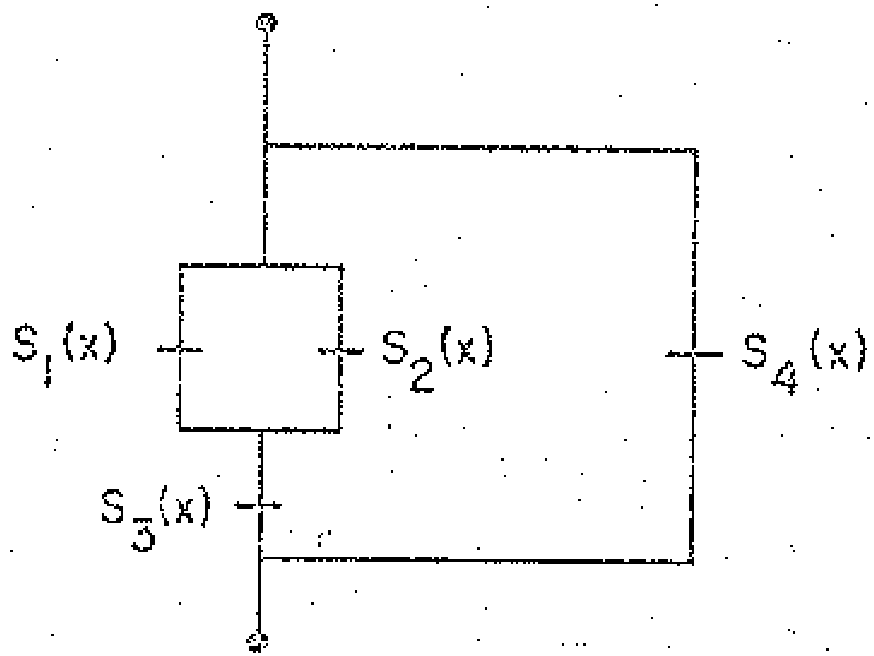


FIG. 3. A network of sieves simultating $\{[f_1(x) \vee f_2(x)] \wedge f_3(x)\} \vee f_4(x)$

First Ph. D Thesis on Fuzzy Sets

Fuzzy Sets and Pattern Recognition

By

Chin-Liang Chang

Grad. (Taiwan Provincial Taipei Institute of Technology) 1958
M.S. (Lehigh University) 1964

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved:

Committee in Charge

Degree conferred .... DEC 1 6 1967
Date

Categories of Fuzzy Sets:
Applications of Non-Cantorian Set Theory

By

Joseph Amadee Goguen, Jr.

A.B. (Harvard University) 1963
M.A. (University of California) 1966

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved:

Committee in Charge

Degree conferred .... JUN 1968
Date

First Papers on Fuzzy Sets (Part 1)

1965: L. A. Zadeh, Fuzzy Sets, *Information and Control*, **8**, pp. 338-353

L. A. Zadeh, Fuzzy sets and systems. In: J. Fox Ed., *System Theory*. Microwave Research Institute Symposia Ser. XV. Brooklyn, New York: Polytechnic Press, pp. 29-37.

1966: L. A. Zadeh, Shadows of fuzzy sets, *Problems in Transmission of Information*, **2**, 37-44 (in Russian).

1968: L. A. Zadeh, Fuzzy algorithms, *Information and Control*, **12**, pp. 94-100.

L. A. Zadeh, Probability measures of fuzzy events, *J. Math. Anal. Appl.*, **23**, 421-427.

1969: L. A. Zadeh, Biological applications of the theory of fuzzy sets and systems. In Proctor, L. D., Ed., *Biocybernetics of the Central Nervous System*. Boston, Mass.: Little, Brown & Co., 199-212.

1971: L. A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sci.*, **3**, pp. 177-200.

L. A. Zadeh, Towards a theory of fuzzy systems. In: R.E. Kalman, N. DeClaris, Eds., *Aspects of Network and System Theory*, New York: Holt, Rinehart & Winston. pp. 469-490.

L. A. Zadeh, Quantitative fuzzy semantics, *Inform. Sci.*, **3**, pp. 159-176.

First Papers on Fuzzy Sets (Part 2)

1972: L. A. Zadeh, Fuzzy languages and their relation to human intelligence. *Proceedings of the International Conference Man, And Computer*, Bordeaux, France. Basel: S. Karger pp. 130-165.

L. A. Zadeh, A new approach to system analysis. In: Marois, M. Ed., *Man and Computer*. Amsterdam: North Holland, pp. 55-94.

L. A. Zadeh, A fuzzy-set-theoretic interpretation of linguistic hedges. *Journal of Cybernetics*, **2**, pp. 4-34.

1973: L. A. Zadeh, Outline of a New Approach to the Analysis of Complex Systems and Decision Processes, *IEEE Transactions on Systems, Man, And Cybernetics*, Vol. SMC-3, No. 1, January 1973, pp. 28-44.

1974: S. Assilian, E. H. Mamdani, Learning Control Algorithms in Real Dynamic Systems, *Proc. 4th Int. IFAC/IFIP Conf. On Digital Computer Appl. To Process Control*, Zürich, March 1974.

1982: Lauritz P. Holmblad and Jens-Jørgen Østergaard: Control of a Cement Kiln by Fuzzy Logic. In: M. M. Gupta and E. Sanchez (eds.): *Fuzzy Information and Decision Processes*, North-Holland, 1982.

TABLE 3

Distribution of year of publication of papers classified as fuzzy

Year	Number
1965	2
1966	4
1967	4
1968	12
1969	22
1970	25
1971	42
1972	58
1973	88
1974	136
1975	227
1976	143 (incomplete)
Total	763

First Papers on Fuzzy Sets (Part 4)

Mozilla Firefox Deutsch | User Support Forum | Mozilla Firefox Hilfe | Plug-in FAQ

BISC

The Berkeley Initiative in Soft Computing

Electrical Engineering and Computer Sciences Department

Berkeley
University of California

Fuzzy Set: 1965 … Fuzzy Logic: 1973 … BISC: 1990 … Human-Machine Perception: 2000 - …

-

Statistics on the impact of fuzzy logic

A measure of the wide-ranging impact of Lotfi Zadeh's work on fuzzy logic is the number of papers in the literature which contain the word "fuzzy" in title. The data drawn from the INSPEC and Mathematical Reviews databases are summarized below. The data for 2000 are not complete.

STATISTICS

INSPEC/fuzzy
1970-1980 : 566
1980-1990 : 2,361
1990-2000 : 23,753
total : 26,680

Math.Sci.Net/fuzzy
1970-1980 : 453
1980-1990 : 2,476
1990-2000 : 8,428
total : 11,357

INSPEC/soft computing
1990-2000: 1,994

Number of citations in the Citation Index: over 11,000.

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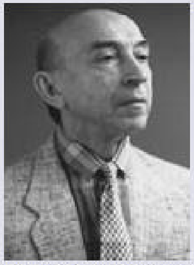
JavaScript and style sheets should be enabled in your browser.



Professor Lotfi A. Zadeh

- Short Curriculum Vitae
- Principal employment and affiliations
- Editorial affiliations
- Advisory committees
- Awards, fellowships, honors
- Achievement and principal contributions
- Summary of principal contributions
- Primary publications
- Statistics on the impact of Fuzzy Logic

Continue

Fuzzy Set: 1965 … Fuzzy Logic: 1973 … BISC: 1990 … Human-Machine Perception: 2000 - …

Composition of Relations

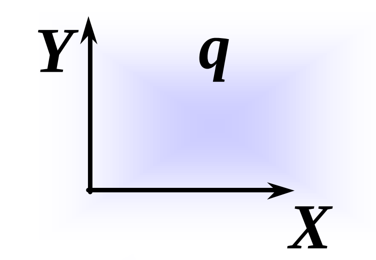
X, *Y* usual sets.

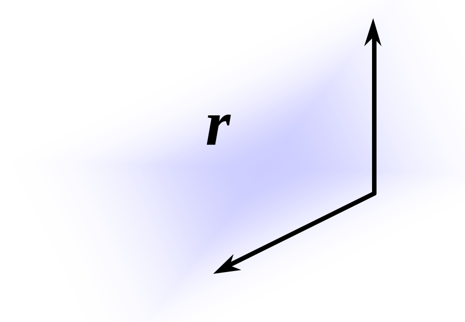
A *relation* *q* of *X* and *Y* is a subset of *X* ´ *Y.*

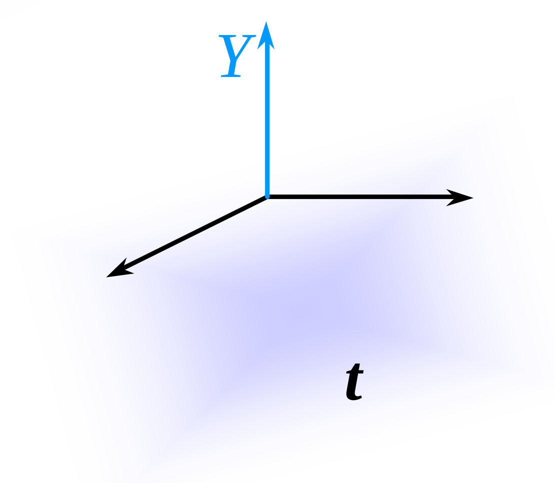
A *relation* *r* of *Y* and *Z* is a subset of *Y* ´ *Z.*

A relation *t* : = *q* ∘ *r* is a subset of *X* ´ *Z.*

t = {(*x*,*z*)| $ *y* : (*x*,*y*) Î *q* Ù (*y*,*z*) Î *r*}







Fuzzy Relations

Let be X, Y usual sets and $X \times Y$ the *Cartesian product*

- $L(X)$: set of all fuzzy sets in X,
- $L(Y)$: set of all fuzzy sets in Y,
- $L(X \times Y)$: set of all fuzzy sets in $X \times Y$.

A *fuzzy relation* R of X and Y is a *fuzzy*-subset of $L(X \times Y)$.

Let be X, Y, Z usual sets and Q, R fuzzy-relations :

- Q in $L(X \times Y)$,
- R in $L(Y \times Z)$.

How to combine Q and R *to a new fuzzy-relation* $T \in L(X \times Z)$?

Composition of Fuzzy Relations

L. A. Zadeh, 1973:
Outline of a New Approach to the Analysis of Complex Systems and Decision Processes

Ù (“and”) ® *min*
V (“or”) ® *max*

- ***Q*** **is fuzzy relation of *X* and *Y*,** ***Q*** **is fuzzy subset of *L*(*X* ´ *Y*),**
- ***R*** **is fuzzy relation of *Y* and *Z*,** ***R*** **is fuzzy subset of *L*(*Y* ´ *Z*).**
- ***T*** **= *Q* ° *R* is fuzzy relation of *X* and *Z*,**

T **is fuzzy subset of *L*(*X* ´ *Z*) with membership function:**

$$m_T(x,z) = \max_{y\,Î\,Y} \min \{m_Q(x,y);\ m_R(y,z)\},\ y\ Î\ Y$$

An Example of the Composition of Fuzzy Relations

X = *Y* = *Z* = *the set of conferences in the world*

Q and R are fuzzy relations of *X* and X *,* *Q and R are* fuzzy subsets of *L*(*X* ´ *X*),

x Q y means " *x* is little compared to *y*"

y R z means " *y* is bigger than *z*"

T = *Q* ∘ *R* is the composition of these fuzzy relations*,*

x T z means "*x* is bigger little than *z*"

An Example of the Composition of Fuzzy Relations

T **is a fuzzy subset of** $L(X \times X)$ **with membership function:**

$$\mu_T(x,z) = \max_{y \in Y} \min \{\mu_Q(x,y); \mu_R(y,z)\},\ y \in Y$$

Then: the conference with the maximal μ_T**-value is:**