UNDERSTANDING OF FUZZY OPTIMIZATION: THEORIES AND METHODS*

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Abstract. A brief summary on and comprehensive understanding of fuzzy optimization is presented. This summary is made on aspects of fuzzy modelling and fuzzy optimization, classification and formulation for the fuzzy optimization problems, models and methods. The importance of interpretation of the problem and formulation of the optimal solution in fuzzy sense are emphasized in the summary of the fuzzy optimization.

Key words. Fuzzy optimization, theory and methods, survey.

1 Introduction

Traditional optimization techniques and methods have been successfully applied for years to solve problems with a well-defined structure/configuration, sometimes known as hard systems. Such optimization problems are usually well formulated by crisply specific objective functions and specific system of constraints, and solved by precise mathematics. Unfortunately, real world situations are often not deterministic. There exists various types of uncertainties in social, industrial and economic systems, such as randomness of occurrence of events, imprecision and ambiguity of system data and linguistic vagueness, etc. which come from many ways[1], including errors of measurement, deficiency in history and statistical data, insufficient theory, incomplete knowledge expression, and the subjectivity and preference of human judgement, etc. As pointed out by Zimmermann[2], various kinds of uncertainties can be categorized as stochastic uncertainty and fuzziness.

Stochastic uncertainty relates to the uncertainty of occurrences of phenomena or events. Its characteristics lie in that descriptions of information are crisp and well defined, however, they vary in their frequency of occurrence. Systems with this type of uncertainty are the so-called stochastic systems, which can be solved by stochastic optimization techniques using probability theory. In some other situations, the decision-maker (DM) does not think the

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commonly-used probability distribution is always appropriate, especially when the information is vague, relating to human language and behavior, imprecise/ambiguous system data, or when the information could not be described and defined well due to limited knowledge and deficiency in its understanding. Such types of uncertainty are categorized as fuzziness which can be further classified into ambiguity or vagueness. Vagueness here is associated with the difficulty of making sharp or precise distinctions, i.e. it deals with the situation where the information cannot be valued sharply or cannot be described clearly in linguistic term, such as preference-related information. This type of fuzziness is usually represented by membership function which reflects the decision-maker's subjectivity and preference on the objects. Ambiguity is associated with the situation in which the choice between two or more alternatives is left unspecified, and the occurrence of each alternative is unknown owing to deficiency in knowledge and tools. It can be further classified into preference-based ambiguity and possibility-based ambiguity from the viewpoint of the ways the ambiguity comes from. The latter is sometimes called imprecision. If the ambiguity arises from the subjective knowledge or objective tools, e.g. 'the processing time is around 2 minutes', it is a preference-based ambiguity, and is usually characterized by a membership function. If the ambiguity is due to incompleteness, e.g. 'the profit of an investment is about 2 dollars, or 1.9–2.1 dollars', it is a possibility-based ambiguity and is usually represented by ordinary intervals, and hence it is characterized by possibility distribution, which reflects the possibility of occurrence of an event or an object. A system with vague and ambiguous information is so-called a soft one in which the structure is ill-defined and it reflects human subjectivity and ambiguity/imprecision. It cannot be formulated and solved effectively by traditional mathematics-based optimization techniques nor probability-based stochastic optimization approaches. However, fuzzy set theory[3,4] which was developed by Zadeh in 1960's and fuzzy optimization techniques[2,5] provide a useful and efficient tool for modelling and optimizing such systems. Modelling and optimization under a fuzzy environment is called fuzzy modelling and fuzzy optimization.

The study on the theory and methodology of the fuzzy optimization has been active since the concept of fuzzy decision and the decision model under fuzzy environments were proposed by Bellman and Zadeh in 1970's[6]. Various models and approaches to fuzzy linear programming[7−16], fuzzy multi-objective programming[17,18], fuzzy integer programming[19,20], fuzzy dynamic programming[21], possibilistic linear programming[22−26] and fuzzy nonlinear programming[27−30] have been developed over the years by many researchers. In the meantime, fuzzy ranking[31], fuzzy set operation, sensitivity analysis[32] and fuzzy dual theory[33], as well as the application of fuzzy optimization to practical problems also represent important topics. Recent surveys on the advancement of the fuzzy optimization have been found in Delgado & Verdegay[34], Fedrizzi[35], Inuiguchi[36,37], Kacprzyk[38], Luhandjula[39]. Especially the systematic survey on the fuzzy linear programming has been made by Rommelfanger[40]. The surveys on other topics of fuzzy optimization like discrete fuzzy optimization and fuzzy ranking have been conducted by Chanas[19] and Bortolan[31], respectively. The classification of uncertainties and of uncertain programming has been made by Liu[41,42]. The latest survey on fuzzy linear programming is provided by Inuiguchi & Ramik[36] from a practical point of view. The possibilistic linear programming is focused and its advantages and disadvantages are discussed in comparison with stochastic programming approach using examples. There are fruitful literatures and broad topics in this area, it is not easy to embrace them all in one paper, hence the above surveys can just introduce and summarize some advancement and achievements of the fuzzy optimization under special cases.

This paper aims to present brief summary of the theory and methods on fuzzy optimization and tries to give readers a clear and comprehensive understanding of knowledge, from the viewpoint of fuzzy modelling and fuzzy optimization, classification and formulation for the fuzzy optimization problems, models and some well-known methods. The importance of interpretation of the problem and formulation of optimal solution in a fuzzy sense are emphasized.

2 Fuzzy Modelling And Fuzzy Optimization

To understand and solve a complex problem under a fuzzy environment effectively, two tasks should be accomplished, i.e. fuzzy modelling and fuzzy optimization. Fuzzy modelling aims to build an appropriate model based upon the understanding of the problem and analysis of the fuzzy information, whereas the fuzzy optimization aims at solving the fuzzy model 'optimally' by means of optimization techniques and tools on the basis of formulation of the fuzzy information in terms of their membership functions and/or possibility distribution functions, etc. Generally speaking, these tasks represent two different processes, however, there are no precise boundaries between them. The whole process for applying fuzzy optimization to solve a complex problem can be decomposed into seven stages as follows:

S1 Understanding the problem. In this stage, the state, constraints and goals of the system, as well as the relationships among them are understood clearly and expressed by sets.

S2 Fuzziness analysis. On the basis of understanding of the background of the problem, such questions, like which kind of fuzzy information (elements) is involved and what the position(e.g. fuzzy goal, fuzzy system of constraints, fuzzy coefficients) it takes, as well as the way (e.g. ambiguity/imprecision in quantity, vagueness in linguistic) in which it is expressed, should be analyzed and summarized. In this stage the fuzzy information is usually expressed in a semantic way.

S3 Development of fuzzy model. Based upon the sub-stages S1 and S2, an appropriate fuzzy optimization model will be built by adopting some mathematical tools, catering for the characteristics of the problem. There are two methods for developing fuzzy models, i.e. using the principles of cause-and-effect and those of transition, and using ordinary equations to express the cause-and-effect relationships. During model development, sets and logic relationships are first established and fuzzified. The optimization model may take the form of fuzzy linear programming, fuzzy nonlinear programming, fuzzy dynamic programming, fuzzy multi-objective programming, or possibilistic linear programming.

S4 Description and formulation of the fuzzy information. On the basis of the stage S2, the fuzzy information including ambiguity and vagueness has been distinguished. What remains to do is to quantify the information in terms of appropriate tools and theory using fuzzy mathematics. In light of the nature and the ways the fuzzy information is expressed, a membership function or a possibility distribution function can be selected to formulate it. The membership function is subjectively determined, and preference-based, which reflects the decision-maker's preference on the objects. It usually applies to the situations involving the human factor with all its vagueness of perception, subjectivity, goals and conception, e.g. fuzzy goals with aspiration, fuzzy constraints with tolerance. Such goals and constraints are expressed vaguely without sharp and thresholds to give the necessary flexibility and elasticity. Nevertheless, the possibility distribution function expresses the possibility measure of occurrence of an event or an object, and it can be constructed in an objective or subjective way. It usually applies to the cases where ambiguity in natural language and/or values is involved, e.g. ambiguous coefficients/parameters in the objective function and/or the system of constraints. These coefficients are considered as possibilistic variables restricted by a possibility distribution. The membership function or the possibility distribution function may take a linear or non-linear form, reflecting the decision-maker's preference and understanding of the problem. This substage takes care of the transition from fuzzy modelling to fuzzy optimization.

S5 Transformation of the fuzzy optimization model into an equivalent or an approximate crisp optimization model. It consists of three procedures, i.e. determination of types of the optimal solution, interpretation and transformation. First of all, the type of the optimal solution is determined, depending on the understanding of the problem and the preference of the decision-makers. That is to say, selection of the type of the optimal solution to a fuzzy model depends absolutely on understanding and definition of the optimal solution in a fuzzy sense. The subsequent task is to propose an appropriate interpretation method and some new concepts to support the understanding and definition of the optimal solution, based on theories and principles on fuzzy mathematics,such as fuzzy ranking, extension principle, fuzzy arithmetics, etc. The interpretation procedure is important for the following procedures. Some well-known interpretations are reviewed in [36]. Finally, the fuzzy model is transformed into an equivalent or approximate crisp optimization model on the basis of the interpretation. For a fuzzy model, different forms of crisp optimization models may be built depending on different types of the optimal solution and interpretations applied.

S6 Solving the crisp optimization model. In light of the characteristics of the crisp optimization model, such as linear or nonlinear; single objective or multiple objectives; decision variable with continuous, discrete or mixed mode, appropriate optimization techniques and algorithms, e.g. traditional heuristic algorithm or intelligent optimization techniques like Genetic Algorithm (GA) [16,29,43], rule-based system approaches [44] or hybrid algorithms, can be adopted or developed for solving the model.

S7 Validity examination. As indicated in [36], the obtained optimal/efficient solution in S6 is not always acceptable, so there is a need to check its validity. If the solution is unreasonable, the fuzzy modelling process and/or the subsequent optimization process should be improved iteratively.

Among the above sub-stages, S4–S6 indicate that the basic procedure of fuzzy optimization is to transform a fuzzy model into a deterministic/crisp optimization one, and the most important task is how to make this transformation. During the transformation, the first thing to do is to understand the problem and then to determine the type of optimal solution, e.g. a deterministic solution or a fuzzy solution, according to the understanding. Then, an appropriate interpretation and some concepts for supporting the understanding and definition of the optimal solution are proposed, and finally a transformation approach can be developed based on the interpretation. The selection of a particular approach to a fuzzy optimization problem depends on several factors including the nature of the problems, decision-maker's preference and the ranking of the objective as well as its evaluation.

3 Classification of a Fuzzy Optimization Problem

An optimization problem consists of two fundamental elements, i.e. a goal or a utility function and a set of feasible domains. As indicated by Dubois & Prade [44], fuzzy optimization refers to the search for extremum of a real-valued function when the function is fuzzily valued, and/or when the domain is fuzzily bounded. With this understanding, a fuzzy optimization problem (FOP) can be described as follows.

Let universe $X = \{x\}$ be a set of alternatives, X_1 a subset or a fuzzy subset of X. The objective/utility function is a mapping $f : X_1 \to L(R)$, where $L(R)$ is a subset or a class of fuzzy subsets of real value set R, the feasible domain is described by a subset or a fuzzy set $C \subset X$, with a membership function $\mu_C(x) \in [0,1]$, which denotes the degree of feasibility of x. In this case, a fuzzy optimization problem may be generally expressed as (FOP)[35,38]:

$$f(x,r) \longrightarrow \max_{x \in C} \tag{1}$$

where r is either a crisp constant or a fuzzy coefficient. Formula (1) means that to find an x 'belonging' to domain C such that $f(x,r)$ can reach a possibly 'maximum', i.e. in a fuzzy sense which can be interpreted in various ways, e.g. the way as explained by Zimmermann[45].

How to interpret the term "belonging" and "maximum" in a fuzzy sense in the formula (1) constitutes the diversity of the FOP, which will be clarified and focused in the section. Hence, the formula (1) is just a sketch of the FOP.

Similarly to deterministic optimization problems, in general, the FOP may be classified into two different types, namely, fuzzy extremum problems and fuzzy mathematical programming problems. This paper mainly discusses the fuzzy mathematical programming problems.

3.1 Classification of the fuzzy extremum problems

The fuzzy extremum problems, i.e. extremum of fuzzy function, are also known as unconstrained fuzzy optimization problems, in which the domain C equals X. The fuzzy extremum problems generally can be described in the following two forms, depending on the definition of the fuzzy function[46].

1) Fuzzy extremum based on the fuzzy function defined from a fuzzy domain to a fuzzy domain. It has the following form:

$$\widetilde{Y} = f(\widetilde{X}, r) \longrightarrow \max / \min \tag{2}$$

where $\widetilde{X} \subset X$ is a fuzzy set in X; $f: X \to R$ is a classical real-valued function from the fuzzy domain $\widetilde{X}$ to the fuzzy domain $\widetilde{Y} \subset R$. The $f(\widetilde{X}, r)$ is a fuzzy function, hence a subset of R. The membership function of the fuzzy function $f(\widetilde{X}, r)$ satisfies:

$$\mu_{\widetilde{Y}}(y) = \sup_{f(x,r)=y} \mu_{\widetilde{X}}(x). \tag{3}$$

Formula (2) means that there exists an x in the fuzzy domain $\widetilde{X}$ of X, at which the crisp function attains "extremum".

2) Fuzzy extremum based on the fuzzy function defined from a crisp domain to a fuzzy domain. It has the form as follows:

$$\widetilde{f}(x,r) \longrightarrow \max / \min \tag{4}$$

where X, Y are the universes, $\widetilde{P}(Y)$ is the set of all fuzzy sets in Y, $\widetilde{f}: X \to \widetilde{P}(Y)$ is a fuzzy function, defined by the membership function $\mu_{\widetilde{f}(x,r)}(y) = \mu_{\widetilde{R}}(x,y)$, and $\mu_{\widetilde{R}}(x,y), \forall (x,y) \in X * Y$ is the membership function of a fuzzy relation. (4) aims to find an x in X such that the function $\widetilde{f}(x,r)$ defined by a fuzzy relation reaches "maximum" or "minimum". The coefficients r in the fuzzy function are usually fuzzy numbers, and the fuzziness of the function comes from the coefficient. Hence this type of fuzzy function is denoted by $f(x, \widetilde{r})$ in what follows for the sake of convenience.

In any forms of the fuzzy extremum problems, the extremum of the function is not a unique, and there are no unique relationships between the extremum of the objective function and the notion of the optimal decision. The solution to the fuzzy extremum problem depends on the ways in which the extremum of the function is interpreted. Possible interpretations of the fuzzy extremum can be found in Dubois and Prade[46]. The concepts of maximizing set[2], maximum and minimum of fuzzy numbers and some integral methods for fuzzy ranking can be applied to solve the fuzzy extremum problems.

3.2 Classification of the fuzzy mathematical programming problems

Fuzzy mathematical programming (FMP) problems are also known as constrained fuzzy optimization problems. It can be generally expressed in the following forms:

$$\begin{array}{l} f(x,r) \longrightarrow \max \\ \text{s.t. } x \in C = \{x \in X \mid g_i(x,s) \widetilde{\leq} 0,\ i = 1, 2, \cdots, m\}. \end{array} \tag{5}$$

In this case, the domain C may be formulated as crisp system of constraints or fuzzy system of constraints in terms of fuzzy equations, fuzzy inequalities, inequalities/equations with fuzzy coefficients, whereas the $f(x,r)$ may be either a crisp objective function or an objective function with fuzzy coefficients. The goal of the problem, C_0, is expressed by $f(x,r) \longrightarrow \max$, which may be a fuzzy goal denoted by $\widetilde{\max}$ or a crisp one.

Recently many methods have been proposed for classifying fuzzy mathematical programming. Zimmermann[47] classified the fuzzy mathematical programming into symmetric and asymmetric models. Luhandjula[39] categorized the fuzzy mathematical programming into flexible programming, fuzzy stochastic programming and mathematical programming with the fuzzy coefficients. Inuiguchi and Ramik[36] further classified the fuzzy mathematical programming into the following three categories in view of the kinds of uncertainties involved in the problems:

- fuzzy mathematical programming with vagueness, i.e. flexible programming;
- fuzzy mathematical programming with ambiguity, i.e. possibilistic programming; and
- fuzzy mathematical programming with vagueness and ambiguity, i.e. robust programming.

In authors' opinion, the formulation and classification of the fuzzy mathematical programming problems depend on what and where the fuzziness are involved. The fuzziness may emerge in the following possible ways:

a) fuzzy goal, i.e. the goal which is expressed vaguely, and usually with an aspiration level, and the target value of the objective function has some leeway, e.g. the target value of the objective function $f(x,r)$ is achieved as 'maximum' as possibly.

b) fuzzy constraints, which represent the system of constraints with tolerances or elasticities in terms of $\widetilde{\leq}, \widetilde{\geq}$ or $\widetilde{=}$.

c) fuzzy coefficients in the objective function and/or the system of constraints.

From the viewpoint of the way the fuzziness emerges and the coefficients involved in the objective function and/or the system of constraints in the problems, fuzzy mathematical programming problems are classified into FMP with crisp coefficients and the FMP with fuzzy coefficients, including:

FMP1-FMP with fuzzy goals C_0 and fuzzy constraints C, i.e.

$$\begin{cases} \widetilde{\max} \quad f(x,r) \\ \text{s.t.} \quad x \in C. \end{cases} \tag{6}$$

FMP2-FMP with fuzzy constraints C, i.e.

$$\begin{cases} \max & f(x,r) \\ \text{s.t.} & x \in C. \end{cases} \tag{7}$$

FMP3-FMP with fuzzy constraints C and fuzzy coefficients in the objective function $f(x,\widetilde{r})$, i.e.

$$\begin{cases} \max & f(x,\widetilde{r}) \\ \text{s.t.} & x \in C. \end{cases} \tag{8}$$

FMP4-FMP with fuzzy goal C_0 and fuzzy coefficients in the system of constraints $C(x,\widetilde{s})$, i.e.

$$\begin{cases} \widetilde{\max} & f(x,r) \\ \text{s.t.} & x \in C(x,\widetilde{s}). \end{cases} \tag{9}$$

FMP5-FMP with fuzzy coefficients in the objective function $f(x,\widetilde{r})$, i.e.

$$\begin{cases} \max & f(x,\widetilde{r}) \\ \text{s.t.} & x \in C(x,s). \end{cases} \tag{10}$$

FMP6-FMP with fuzzy coefficients in the system of constraints $C(x,\widetilde{s})$, i.e.

$$\begin{cases} \max & f(x,r) \\ \text{s.t.} & x \in C(x,\widetilde{s}). \end{cases} \tag{11}$$

FMP7-FMP with fuzzy coefficients in the objective function $f(x,\widetilde{r})$ and the system of constraints $C(x,\widetilde{s})$.

$$\begin{cases} \max & f(x,\widetilde{r}) \\ \text{s.t.} & x \in C(x,\widetilde{s}). \end{cases} \tag{12}$$

Here the problems FMP1, FMP3, FMP4 and FMP7 are referred to as symmetric ones, while the FMP2, FMP5 and FMP6 as asymmetric problems, with regard to fuzziness. That means a problem is classified as symmetric or asymmetric from the viewpoint of fuzziness involved in the goal (or objective function) and/or the system of constraints. The symbol $f(x,\widetilde{r})$ representing the objective function with the fuzzy coefficients is used to distinguish from the fuzzy goal C_0. The notation $C(x,\widetilde{s})$ is used to represent the system of constraints with the fuzzy coefficients in order to distinguish from the fuzzy constraints C. This classification is adopted in the rest of the paper.

The fuzzy goal and fuzzy constraints are characterized by a preference-based membership function. In comparison with the category by Inuiguchi and Ramik[36], FMP1 and FMP2 are in the category of flexible programming problems. The fuzzy coefficients in the objective function and in the system of constraints may be characterized by a preference-based membership function and a possibility distribution function. When a fuzzy coefficient is formulated by a possibility distribution function, it is viewed as a possibilistic variable restricted by the possibility distribution. In this case, FMP5, FMP6 and FMP7 are the so-called possibility programming problems, denoted by PMP5, PMP6 and PMP7 respectively hereafter, and FMP3 and FMP4 are robust programming problems, denoted by PMP3 and PMP4 respectively.

3.3 Classification of the fuzzy linear programming problems

Owing to the simplicity of linear programming formulation and the existence of some developed software for optimization, linear programming has been an important and most frequently applied Operations Research technique for real life problems. Since the introduction of fuzzy sets theory into traditional linear programming problems by Zimmermann[45] and the fuzzy decision concept proposed by Bellman and Zadeh[6], the fuzzy linear programming (FLP) has been developed in a number of directions with successful applications. It has been an important area of the fuzzy optimization. Hence, classification of the fuzzy linear programming problems is emphasized as follows. Traditional linear programming problems can be presented in the following general form:

$$\max\ c^{\mathrm{T}}x \ \ \text{s.t. } Ax \le b,\ x \ge 0, \tag{13}$$

where $c^{\mathrm{T}} = (c_1, c_2, \cdots, c_n), A = (A_{ij})_{mn}, b = (b_1, b_2, \cdots, b_m)^{\mathrm{T}}, x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ are benefit coefficient vector, technical coefficient matrix, resources vector and decision variable vector, respectively.

The formulation of a linear programming problem under fuzzy environment depends on what and where the fuzziness is introduced. In general, fuzziness may be initiated in fuzzy linear programming problems in the following ways:

(a) the fuzzy goal, i.e. the maximum of the linear objective function is expressed vaguely and usually with an aspiration level, and it has flexibility, e.g. the target value of the objective function $c^{\mathrm{T}}x$ is deemed 'maximum' as possible and pursues an aspiration level.

(b) the fuzzy constraints, i.e. linear system of constraints expressed by fuzzy relations in terms of fuzzy equations or/and fuzzy inequalities.

(c) the objective function with the fuzzy benefit coefficients $\widetilde{c}_i$, and

(d) the linear system of constraints with the fuzzy technical coefficients $\widetilde{A_{ij}}$ and/or fuzzy resources/thresholds $\widetilde{b}_i$.

Based on the above cases, the fuzzy linear programming problems can be categorized as follows.

Category I– FLP with crisp coefficients. In this type of the FLP, the goal and/or the system of constraints is/are formulated by decision-makers in a vague and subjective way. The goal and the system of constraints are called the fuzzy goal and the fuzzy constraints respectively. This type of FLP includes.

FLP1-FLP with the fuzzy goals and the fuzzy constraints, i.e. the goal of the objective function is formulated vaguely, e.g. in terms of $\widetilde{\max}$, and the linear system of constraints are defined by fuzzy relations ($\widetilde{\le}$) with tolerances, e.g. FLP as defined by Zimmermann[44].

FLP2-FLP with the fuzzy constraints, i.e. the linear system of constraints are defined by fuzzy relation ($\widetilde{\le}$) with tolerances.

Category II–FLP with fuzzy coefficients. In this type of the FLP, some or all of the coefficients are ambiguous, and can usually be expressed by fuzzy numbers. This type of the FLP comprises the backbone of a FLP, and it includes

FLP3-FLP with fuzzy constraints and fuzzy objective coefficients $\widetilde{c}_i$,

FLP4-FLP with fuzzy goal and fuzzy technical coefficients $\widetilde{A}_i$ and/or the fuzzy resources/ thresholds $\widetilde{b}_i$,

FLP5-FLP with fuzzy objective coefficients, i.e. the benefit coefficients $\widetilde{c}_i$ in the objective function are fuzzy numbers,

FLP6-FLP with fuzzy technical coefficients and fuzzy thresholds, i.e. the technical coefficients $\widetilde{A}_i$ and threshold $\widetilde{b}_i$ are fuzzy numbers, and

FLP7-FLP with fuzzy coefficients, i.e. the benefit coefficients, technical coefficients and resources/thresholds, are all fuzzy numbers.

The detailed formulation of the above classes of the FLP is given in [5]. In the sense that fuzzy constraints are defined as fuzzy inequalities with tolerances, they are equivalent to fuzzy resources/thresholds in FLP. If fuzzy coefficients are modelled by possibility distribution, the corresponding FLP is a possibility linear programming (PLP) problem. Under this circumstance, corresponding to the classification by Inuiguchi[36], FLP1-FLP2 are the so-called the flexible programming, FLP3-FLP4 are the robust programming, and FLP5-FLP7 are the possibilistic programming.

Other methods of classification of FLP can be found in [36, 39, 40]. In this paper fuzzy linear programming is distinguished from the possibilistic linear programming.

4 Brief Summary of Solution Methods for FOP

Since the concept of the fuzzy decision was proposed by Bellman and Zadeh[6], fuzzy optimization has received much attention, and various models and methods have been proposed by many researchers. Recent surveys on fuzzy optimization techniques can be found in [34, 36, 38, 39, 40, 44], focusing on special category of the fuzzy mathematical programming. Owing to the increasing work on the fuzzy mathematical programming, it is impossible to embrace all of the techniques in a paper, hence we will just have a brief summary on the techniques for FMP with vagueness and FMP with the fuzzy coefficients characterized by the membership functions. The approaches to the possibilistic programming problems POP5–POP7 have been summarized in [36]. This brief summary tries to emphasize the understanding and interpretation of the problem and the optimal solution in a fuzzy sense.

4.1 Symmetric approaches[6,39,45,48]

Symmetric approach is an important approach to the fuzzy optimization problems, especially for FMP1. The word "symmetric" used here comes originally from the symmetric model by Zimmermann[45]. The symmetric approaches here cited by many researchers[39] usually refer to the approaches proposed by Bellman and Zadeh[6], Tanaka[48] and Zimmermann[45] to FMP1 firstly, and they are then extended to represent a type of approaches to symmetric mathematical programming models in the sense that the goals and the system of constraints involved in the problem are dealt with in a symmetric way with regard to fuzziness. It means that the scope of the symmetric and the asymmetric approach is made from the perspective of the ways in which the goal and the system of constraints are treated, and not from the viewpoint of the problem itself. The symmetric/asymmetric way in which the goals and the system of constraints are treated is understood to be the same concept as symmetric/asymmetric model. In this sense, the symmetric or asymmetric approach is named according to the symmetric or asymmetric model, and not to the symmetric or asymmetric problem. Symmetric approaches based on fuzzy decision and on the concept of non-dominated relation are summarized as follows.

4.1.1 Symmetric approaches based on the fuzzy decision[6]

This type of approaches are developed originally to deal with decision making problems with fuzzy goals and fuzzy constraints, i.e. FMP1, based on the concept of the fuzzy decision, as proposed by Bellman and Zadeh[6]. In the viewpoint of Bellman and Zadeh, a symmetry between the goals and the constraints is an important feature in decision making under fuzzy environment, and the fuzzy goals and the fuzzy constraints can be considered to play the same roles in the problem, and hence can be dealt with symmetrically. The fuzzy decision is defined as a fuzzy set of alternatives resulting from the intersection of the goals and the constraints. By introducing the fuzzy decision D, the solution to FMP1 can be interpreted as the intersection of the fuzzy goal C_0 and the fuzzy constraints C, i.e. $D = C_0 \cap C$, where $\cap$ is a conjunctive operator, which have different alternatives and different meanings in practical situations. In terms of the membership function, the fuzzy decision can be formulated as:

$$\mu_D(x) = \mu_{C_0}(x) \bigcap \mu_C(x), \quad \forall x \in X, \tag{14}$$

where μ_{C_0} and μ_C are the membership functions of the fuzzy goals and the fuzzy constraints respectively, and preferences are involved.

A maximizing decision x^* is then defined to be an alternative with the highest membership in the fuzzy decision D, i.e. $\mu_D(x^*) = \max \mu_D(x), \forall x \in X$.

More generally, maximizing decision x^* can be determined by

$$\mu_D(x^*) = \bigcup_{x \in X} \mu_D(x). \tag{15}$$

The maximizing decision x^* is the optimal solution in a sense that it can be interpreted in different ways, depending on the definitions of the operators $\cap$ and $\cup$. The operator $\cap$ may be extended to various forms of conjunctive operators, such as minimum operator, weighted sum of the goals and the constraints, multiplication operator, mean value operator, bounded product, Hamacher's min operator, etc., and $\cup$ can be substituted by algebraic sum, bounded sum, Yager's max operator[49], etc. which are summarized in [5] in detail. Among these operators, the Max-Min operator is commonly used in practice. The selection of the operators depends on the preference of the decision-maker and the problem-context and semantic interpretation.

This approach provides a framework for solving fuzzy optimization problems with fuzzy goals and fuzzy constraints, and it is well known as the fundamental of decision making under a fuzzy environment. Since then, various forms of symmetric approaches[2,15,29,45,48,50,51] have been developed by applying different combinations of the operators. Among them, Tanaka[48] extended Bellman and Zadeh's approach to tackle multiobjective fuzzy mathematical programming problems. The tolerance approach proposed by Zimmermann[45] is one of the most important and practical approaches. By using piecewise linear membership functions to represent fuzzy goal and fuzzy constraints, the original problem can then be translated into a linear programming model. A maximizing decision among the fuzzy decision set can be achieved by solving the linear programming. In addition, the decision-maker may capture some essential features on other solutions in the neighborhood of the maximizing decision. Along this line, Verdegay[33] and Chanas[52] propose parametric programming techniques to obtain the whole fuzzy decision set and complete fuzzy decision set respectively.

Apart from FMP1, this type of approaches can also apply to the symmetric problems FMP3, FMP4 and FMP7, in which fuzzy coefficients are characterized by membership functions. These fuzzy coefficients are embedded into the objective function and/or the system of constraints, and their membership functions $\mu_{\widetilde{r}}$ and $\mu_{\widetilde{s}}$ reflect the preference of the decision-maker. When applying the symmetric approaches to these problems, the fuzzy objective function and the fuzzy system of constraints are treated as the fuzzy goal and the fuzzy constraints respectively in a symmetric way. Firstly, $\mu_{f(x,\widetilde{r})}$ and $\mu_{C(x,\widetilde{s})}$, the membership functions of the fuzzy objective function and the fuzzy system of constraints, can be obtained via $\mu_{\widetilde{r}}$ and $\mu_{\widetilde{s}}$ using the extension principle, and then similar procedures can be applied by substituting μ_{C_0} and μ_C with $\mu_{f(x,\widetilde{r})}$ and $\mu_{C(x,\widetilde{s})}$ respectively. In addition, this approach can apply to the solution of asymmetric problem FMP2. Werners [51] developed a symmetric approach to linear programming problems with fuzzy resources by treating the goal of the problem in the same way as the fuzzy constraints are treated.

This approach can be applied to the cases with single objective or multiple objectives, in the forms of linearity or nonlinearity. The types of optimal solutions to these approaches can be expressed in different forms, such as the fuzzy decision[6], maximizing decision[45,51], fuzzy optimal solution[15,16,29,50], depending on the operators and the interpretation applied.

4.1.2 Symmetric approach based on non-dominated alternation

This approach is developed for solving FMP1, in which the fuzzy goal is expressed in a fuzzy utility function $\varphi(x, y) : X * Y \longrightarrow [0, 1]$, and the fuzzy constraints are expressed in fuzzy preference relations, denoted by $\mu : Y * Y \longrightarrow [0, 1]$, where X and Y are a set of alternatives and a universal set of estimates respectively based on the concept of fuzzy strict preference relations and non-dominated alternatives[53,54]. In this case, given an alternative $x \in X$, the function φ gives the corresponding utility value $\varphi(x, y)$ in the form of a fuzzy set in Y. The basic rationale of this approach is as follows: Firstly, $\forall \bar{x} \in X, \bar{x} \neq x$, a fuzzy strict preference relation R^s in X is defined using the original fuzzy relation μ in Y. The membership function $\mu_R^s(\bar{x}, x)$ of R^s, representing the degree that $\bar{x}$ is strictly preferred to x, is defined as follows:

$$\mu_R(\bar{x}, x) = \sup_{y_1, y_2 \in Y} \min\{\varphi(\bar{x}, y_1), \varphi(x, y_2), \mu(y_1, y_2)\}, \tag{16}$$

$$\mu_R^s(\bar{x}, x) = \max\{0, \mu_R(\bar{x}, x) - \mu_R(x, \bar{x})\}, \tag{17}$$

where μ_R is a fuzzy preference relation induced in X. The elements of the general scheme are given as

$$e(x, C_0, C) = \mu_R^s(\bar{x}, x), \tag{18}$$

$$K(e(x, C_0, C)) = 1 - \sup_{\bar{x} \in X} \mu_R^s(\bar{x}, x) = \mu^{ND}(x), \tag{19}$$

$$TK(x) = \mu^{ND}(x), \tag{20}$$

where $\mu^{ND}(x)$ is the degree to which x is non-dominated by any other elements $\bar{x}$, e.g. for some x such that $\mu^{ND}(x) = \alpha$, it means that this element is dominated by other elements to a degree not higher than α.

In this sense, the original FMP is stated as the following problem

$$\max_{x \in X} \ \mu^{ND}(x), \tag{21}$$

which can be solved by transforming it to an equivalent semi-infinite programming model. The optimal solution is understood in the sense of non-dominated alternatives.

It can be seen from the (16)–(21) that the fuzzy goal and fuzzy constraints are treated in the same way as indicated in Bellman and Zadeh's approach, and hence it is a symmetric approach.

4.2 Asymmetric approaches[14,39,53,55]

In contrast to the symmetric approaches, the asymmetric approaches here refer to the type of approaches to the asymmetric mathematical model in the sense that the goals (or the objective functions) and the system of constraints are treated in an asymmetric way with regard to the fuzziness, i.e. only one of the two constituents is treated as fuzziness is and the counterpart as crispness no matter what fuzziness involved in the problems. In this sense, the asymmetric approaches can not only solve the asymmetric problem FMP2, FMP5 and FMP6, but also solve the symmetric problems FMP1, FMP3, FMP4 and FMP7, in which the goals and the system of constraints are treated in an asymmetric way. We first focus our attention to the approaches to FMP1. When solving the FMP1, the asymmetric approaches treat the fuzzy goal C_0 and the fuzzy constraints C in an asymmetric way, and usually via the following asymmetric form[39]:

$$\max_{x \in C} \ \mu_{C_0}(x), \tag{22}$$

where $\mu_{C_0}(x)$ is a crisply defined compatibility function. Similarly, the other symmetric problems FMP3, FMP4 and FMP7 can be treated in the same way (22) by substituting the μ_{C_0} and/or C with $\mu_{f(x, \widetilde{r})}$ and $C(x, \widetilde{s})$ respectively. On the other hand, the asymmetric problems

FMP2 and FMP6 have the same form as (22), and the fuzzy dual problem of FMP5 can also be expressed in the form of (22). Hence, the approaches here for FMP1 can also be applied to the problems FMP2–FMP7.

The problem (22) is meaningless in mathematics, and hence the optimal solution of this problem should be understood in a fuzzy sense, which constitutes the fundamental part of the approaches. One possible interpretation is by the concept of maximizing set, which is a fuzzy set and reflects the compatibility of elements in the support of the feasible domain C with the fuzzy goal C_0. Other possible ways of interpretation and definition of the optimal solution include fuzzy maximizing decision, maximum decision, fuzzy solution, α-optimal solution and fuzzy optimal solution set. Various approaches are available depending on the possible interpretation and the definition of the optimal solution, some of which are summarized as follows:

1) Fuzzy maximizing decision approach. According to the definition of the maximizing set, a maximizing set M is a fuzzy set, the membership function of which reflects the compatibility degree of the fuzzy goal C_0 and the support set S_C of the fuzzy feasible set C. The fuzzy maximizing decision M is a maximizing set and can be characterized by $\mu_M(x)$ as follows:

$$\mu_M(x) = \frac{\mu_{C_0}(x) - \inf\limits_{x \in S_C} \mu_{C_0}(x)}{\sup\limits_{x \in S_C} \mu_{C_0}(x) - \inf\limits_{x \in S_C} \mu_{C_0}(x)} \tag{23}$$

where S_C is a support set of the fuzzy set C. In comparison with the problem (22) and (2), one can see that (22) is a special case of the fuzzy extremum problem. Hence, this approach can also be applied to the fuzzy extremum problem (2). The fuzzy maximizing decision M is regarded as the optimal solution in the sense of a fuzzy set, the membership function of which reflects the compatibility degree of the fuzzy goal and the support set of the fuzzy constraints. The fuzzy maximizing decision approaches are commonly applied to solve asymmetric problems like FMP2, FMP6 and fuzzy extremum problems. It can also be applied to FMP5 through transformation into its dual problem.

2) Crisp maximum decision approach. This approach originally is developed to solve asymmetric problems, and it comes from the idea that the objective should be also fuzziness owing to the fuzziness involved in the feasible domain. Hence, symmetric approach based on the fuzzy decision can be also applied to (22) by regarding the fuzzy maximizing decision as the fuzzy decision. It aims to achieve the maximum degree of intersection between the fuzzy maximizing decision M and the fuzzy feasible set C. The alternatives with the highest degree in the fuzzy maximizing decision are interpreted as the optimal solutions, i.e. $\mu_M(x^*) = \max\{\mu_M(x), | \, x \in S_C\}$. They are crisp solutions.

When applying the fuzzy maximizing decision approach and the crisp maximum decision approach to solve the FMP2 and FMP6, the solution can be obtained by substituting μ_{C_0} and C with $f(x, r)$ and $C(x, \widetilde{s})$ respectively.

3) Fuzzy solution approach[14,53,54,55]. This approach is applied when one wants to know the extent to which the uncertain solution reflects the uncertainty of the problem's setting, especially to the asymmetric problems with respect to the fuzziness, i.e. FMP2, FMP5 and FMP6. An important concept of the approach is the fuzzy solution which is a fuzzy set. Fuzzy solution can be expressed in various forms depending on the formulation of the membership function which results in various forms of the fuzzy solution approaches. Among them, Orlovski [53,54] firstly proposed the concept of fuzzy solution to the problems (22), and two methods are developed to formulate the fuzzy solutions denoted by $Sol1$ and $Sol2$ using an α-level cut set of the fuzzy feasible domain and the Pareto optimal solution, respectively. The concrete forms of the fuzzy solutions are defined by the membership functions in the form of (24) and (27) respectively. Verdegay[55,56] investigated a fuzzy solution for fuzzy mathematical programming

problems FLP2 and FLP5 based on the concept of α-optimal solution. The fuzzy solution is understood as the optimal solution in the sense that it optimizes the objective function under a preferred level set of the fuzzy constraints, i.e. α-optimal solution to the sub-problem defined on the α-level cut set of the fuzzy domain C. Werners proposed a formulation for the fuzzy solution to FLP2 and FLP6, and named it fuzzy set 'decision'[2]. It is interpreted as a fuzzy optimal solution set, which is a union of the sets of α-optimal solutions to the sub-problem, and it has a different formulation from that of Verdegay's. Tanaka and Asai[14] developed fuzzy solution for fuzzy LP with fuzzy coefficients in the system of constraints using α-level cut set. The fuzzy solution with the widest spread is understood as the optimal solution in the sense that it satisfies the system of constraints to a given degree. In general, the fuzzy solutions can be obtained using parametric programming techniques or multiobjective programming. The possibility and necessity optimal solution sets[36] can take the form of fuzzy solutions. The fuzzy solution expressed in various forms is regarded as the optimal solution in this approach.

$$\mu_{Sol1} = \begin{cases} \mu_{C_0}(x) & \text{if } \ x \in \bigcup_{k\in[0,1]} V(k), \\ 0 & \text{else.} \end{cases} \tag{24}$$

where

$$V(k) = \{x \in X \mid \mu_{C_0}(x) = \max_{t\in D^k} \ \mu_{C_0}(t)\}, \tag{25}$$

$$D^k = \{x \in X \mid \mu_C(x) \geq k\}. \tag{26}$$

$$\mu_{Sol2} = \begin{cases} \mu_{C_0}(x) & \text{if } \ x \in E, \\ 0 & \text{else,} \end{cases} \tag{27}$$

where E is a set of efficient solutions of the multiobjective programming

$$\max_{x\in X} \ \{\mu_{C_0}(x), \mu_C(x)\}.$$

4.3 Possibility and necessity measure-based approaches[36,57]

The symmetric and asymmetric approaches are summarized mainly on the fuzzy optimization problems with vagueness and fuzzy coefficients characterized by membership functions. The approaches to solving the possibilistic mathematical programming (PMP) problems can be found in the survey[36] and the Chapter 4 in [5]. Among them, the approaches based on possibility and necessity measures are important approaches to PMP. They are briefly summarized as follows.

As indicated in the previous section, the fuzzy coefficients in the possibilistic programming problems are viewed as the possibilistic variables restricted by the possibility distributions. Under this circumstance, no matter how the objective or the system of constraints with the fuzzy coefficients is possibilistic function, its values are also ambiguous, and could not be determined uniquely. Hence, how to formulate and how to measure these values in an appropriate way are important constituents of the approaches to solving this category of problems. To do this, a specific interpretation should be introduced and developed based on the possibility theory[4,57]. The possible interpretations are summarized by Inuiguchi and Ramik in a recent survey[36]. Among the interpretations, two basic concepts are the possibility measure and the necessity measure.

Given a possibilistic variable a restricted by a fuzzy set A with a possibility distribution μ_A, the possibility measure and the necessity measure, denoted by $\pi_A(B)$ and $N_A(B)$ respectively, represent the possibility degree and the necessity degree of the event that a is in the fuzzy set B, i.e. the extent of possibility and the extent of certainty that a is in the fuzzy set B. They are defined as follows [36]:

$$\pi_A(B) = \sup_a \min(\mu_A(a), \mu_B(a)), \tag{28}$$

$$N_A(B) = \inf_a \max(1 - \mu_A(a), \mu_B(a)). \tag{29}$$

Based on these two concepts, two possibilistic function values, or a possibilistic function value and a real value can be ranked with an index, e.g. $Pos(a \leq g)$ or $Nes(a \leq g)$ in the sense of the possibility degree or the certainty degree. Here the indices $Pos(a \leq g)$ and $Nes(a \leq g)$ defined as follows indicate the degree of possibility and the degree of certainty to which the value a restricted by the possibility distribution μ_A is not greater than g. $Pos(a \geq g)$ and $Nes(a \geq g)$ can be defined and understood in a similar way. The selection of the indices depends on the form of the goal (e.g. max or min), the inequality relations involved in the system of constraints and the decision-maker's attitude.

$$Pos(a \leq g) = \pi_A((-\infty, g)) = \sup\{\mu_A(r), r \leq g\}, \tag{30}$$

$$Nes(a \leq g) = N_A((-\infty, g)) = 1 - \sup\{\mu_A(r) \mid r > g\}. \tag{31}$$

Using these indices, the possibilistic objective function and the system of constraints can be formulated by an appropriate interpretation. This interpretation reflects the decision-maker's preference on the degree of possibility and certainty, and the attitude to the treatment of the objective function and the system of constraints. From an attitude point of view, i.e. the symmetric attitude or the asymmetric attitude to the treatment of the objective function and the system of constraints, the approaches to solve the PMP can be classified into asymmetric and symmetric approaches, which are introduced briefly as follows.

4.3.1 Asymmetric approaches to PMP5 & PMP6

These approaches are developed to solve the PMP5 and the PMP6, in which the fuzzy objective function and the system of constraints are treated separately. When applying this type of approaches, firstly define an appropriate index based on the possibility measure and the necessity measure. The succeeding procedure is to understand the problem and try to find an appropriate interpretation so as to transform the possibilistic programming model into a crisp one using the concepts. Different interpretations result in various approaches to the problem.

1. Fractile approach to PMP5

The fractile approach originates from the Kataoka's model[58] for solving stochastic programming problems, and it can be applied to the treatment of the possibilistic objective function and the system of constraints. The two important concepts of the fractile approach are p-possibility fractile and p-necessity fractile, which are defined as the smallest values of u satisfying $Pos(a \leq u) \geq p$ and $Nes(a \leq u) \geq p$ respectively.

If the decision-maker has more interest in the objective function, i.e. one pursues a maximum objective function with a high certainty, then the maximum objective function with a high certainty can be interpreted as a p-necessity fractile in the following equivalent form:

$$Nes(f(x, \widetilde{r}) \geq u) \geq p, \tag{32}$$

where the p is a preferred value reflecting the decision-maker's desire or preference on the certainty degree of the objective function.

In this case, PMP5 can be solved by transforming (32) into the following equivalent model:

$$\begin{aligned} \max \quad & u \\ \text{s.t.} \quad & Nes(f(x,\widetilde{r}) \geq u) \geq p, \\ & x \in C(x,s). \end{aligned} \tag{33}$$

The solution to (33) is an optimal solution to PMP5 in the sense of p-necessity fractile that the objective function is not less than u^* at a certainty degree with p.

2. Asymmetric approach to PMP6

Similarly, in the case where the decision-maker goes towards more to a higher degree of satisfaction of the constraints, it can be interpreted that the problem aims to pursue a maximum objective at a higher certainty degree of satisfying the constraints. This certainty degree is not the one in the sense of p-necessity fractile. With this interpretation, PMP6 can be solved by transforming it into an equivalent crisp model as follows:

$$\begin{aligned} \max \quad & f(x,r) \\ \text{s.t.} \quad & Nes(C(x,\widetilde{s})) \geq p, \end{aligned} \tag{34}$$

where p is a preferred value reflecting the decision-maker's preference on the certainty degree of the system of constraints.

The solution to (34) is an optimal solution to PMP6 in the sense that the system of constraints are satisfied with a certainty degree not less than p.

Similarly, the objective function can be treated with p-possibility fractile, and the system of constraints can also be treated in terms of the possibility degree, when dealing with the PMP5 and PMP6, respectively.

Apart from the fractile approach to the FMP5, the modality approach[36] can also treat the objective function such that the decision-maker puts more emphasis on a maximum certainty degree of which the objective function is not less than a preferred level.

4.3.2 Symmetric approaches to the PMP7

The fractile approach can not only be applied to solve asymmetric problems, i.e. the PMP5 and the PMP6, but it can also work with the symmetric problem PMP7 using an appropriate interpretation. In some cases, the decision-maker not only pursues the objective, but also is concerned with the satisfaction with the system of constraints. It can be interpreted that the problem aims to pursue a maximum objective with a high possibility degree at a higher certainty degree of satisfying the constraints. The possibility degree and the certainty degree can be understood in the way of p-possibility(necessity) fractile. Owing to various combination of the possibility measures and the necessity measures involved in the interpretation, various symmetric models and approaches can be developed to solve PMP7. For simplicity, the p-possibility fractile and the necessity degree are used to treat the objective function and the system of constraints respectively, while PMP7 can be solved by transforming it into the following model:

$$\begin{aligned} \max \quad & u \\ \text{s.t.} \quad & Pos(f(x,\widetilde{r}) \geq u) \geq p_1, \\ & Nes(C(x,\widetilde{s})) \geq p_2, \end{aligned} \tag{35}$$

where p_1 and p_2 are the preferred levels by the decision-maker.

If neither of the possibility degree nor the necessity degree is the one in the sense of p-fractile, and the objective function is treated as in the modality approach, PMP7 can be understood in terms of

$$
\begin{aligned}
&\max \quad p \\
&\text{s.t.} \quad Pos(f(x,\widetilde{r}) \geq u) \geq p, \\
&\qquad Nes(C(x,\widetilde{s})) \geq p,
\end{aligned} \tag{36}
$$

where u is the preferred level of the objective function.

4.4 Interactive satisfying solution approach[17,59,60,61]

The interactive satisfying solution approach is an important type of approaches to the fuzzy optimization problems, especially to fuzzy multi-objective programming problems through an interactive fuzzy optimization procedure. The satisfying solution, compromise solution and Pareto optimal solution are understood as the optimal solutions to these problems. With this approach, the solution is determined step by step in an interactive process. Many procedures of this type of approach can be found in [17, 60, 61].

4.5 Generalized approach by Angelov

The generalized approach, originally proposed by Angelov[62], is viewed as a new approach to fuzzy optimization problems on the basis of the generalization of Bellman-Zadeh's concept[6]. It directly solves the fuzzy optimization problems through a parametric generalization of intersection of fuzzy sets and a generalized defuzzification procedure called BADD[63] without the step of transforming the model into a crisp one. It can be outlined as follows.

Step 1 Specifying α and β, where $\alpha \in [0,\infty)$ reflects the credibility of every fuzzy solution, and $\beta \in [0,\infty)$ is the degree of strength of the flexible conjunction.

Step 2 Construction of fuzzy decision D as

$$
\mu_D(x) = \frac{\mu_{C_0}(x)\mu_C(x)}{\beta + (1-\beta)(\mu_{C_0}(x) + \mu_C(x) - \mu_{C_0}(x)\mu_C(x))}. \tag{37}
$$

Step 3 Determination of a crisp solution x_0 as

$$
x_0 = \sum_{j=1}^{N} \frac{\mu_{D_j}^{\alpha}(x_j)}{\sum_{i=1}^{N}\mu_{D_i}^{\alpha}(x_i)} x_j, \qquad N = Card(x). \tag{38}
$$

With these procedures, a family of parametric crisp solutions of the FOP can be obtained, via the variations of α and β; whereas in Bellman-Zadeh's method, the decision with maximal degree of membership is taken. In this sense, Bellman-Zade's approach can be considered as a special case of this approach. The fuzzy solutions and the crisp solutions can be understood as the optimal solution to the FOP.

4.6 Fuzzy genetic algorithm

Buckley[64] proposes a fuzzy genetic algorithm to solve the following type of fuzzy maximum problems approximately

$$
\max \ F(\widetilde{X}), \tag{39}
$$

where $\widetilde{X}$ is any type of fuzzy subset in $[0,M], M>0$ and F is a crisply defined map.

The fundamental of the fuzzy genetic algorithm is that, first of all, define a measurement function $m(F(\widetilde{y})) = \theta$; and then discretize $\widetilde{X}$, i.e.

$$
\widetilde{X} = (x_0, x_1, x_2, \cdots, x_N), \quad x_i = \mu_{\widetilde{X}}(z_i)
$$

$$z_i = i * M/N, \quad i = 0, 1, 2, \cdots, N.$$

Under this circumstance, the original fuzzy optimization problem (39) may be stated as how to determine $x_i, i = 0, 1, 2, \cdots, N$ such that $m(f(\widetilde{x})) = \theta \ \rightarrow$ max, to which a Genetic Algorithm can be applied, and an approximate optimal solution can be achieved.

4.7 Genetic-based fuzzy optimal solution method

Based on the method proposed by Zimmermann[45], a genetic-based fuzzy optimal solution ([15, 16, 29, 50]) is interpreted as the neighboring domain of an optimal solution, in which every solution is acceptable, i.e. it is an optimal solution in a fuzzy sense. Using this method a family of solutions with acceptable degree of membership can be found through genetic search, and the solutions preferred by the decision maker under different criteria can be achieved by means of the human-computer interactions. This method has been applied to fuzzy linear, quadratic and nonlinear programming problems of the types FMP1 and FMP4. Recently some other intelligent-based fuzzy optimization approaches[43] have been found popular.

4.8 Penalty function based approach[28,65]

This approach is first proposed by Lodwick and Jamison[65] to solve fuzzy constrained optimization problems. The penalty functions are imposed on the objective as a penalty when the fuzzy constraints are 'violated'. It is useful in computation and reflects the practical scene. The authors[28] consider the penalty in the fuzzy nonlinear programming problems with fuzzy resources, and suggest some properties on the fuzzy optimal solution set of this model. A genetic-based approach for finding the maximum decision is developed.

Apart from the above approaches to the fuzzy optimization problems, parametric techniques[66], dual approach[33], fuzzy dual decompose approach[18] and differential equation approach[67] are also proposed by many researchers. In addition, convergence analysis, stability analysis and sensitivity analysis of the algorithm for the FOP are also applied for the fuzzy optimization.

5 Concluding Remarks

On the basis of our previous paper [68], an extensive study on fuzzy optimization is conducted in this paper, which leads to the following concluding remarks that the basic procedure of fuzzy optimization problems is to transform a fuzzy model into a crisp one, and the most important thing is how to make this transformation have an appropriate and reasonable interpretation. During the transformation, the first thing to do is to understand the problem and interpret the optimal solution, and then try to find an appropriate interpretation, and propose some concepts and theory to support the interpretation, and finally transform the fuzzy model into a crisp one. The interpretation and formulation are the key constituent parts of the approaches, and they also bridge the gap between the fuzzy optimization and the application in solving practical problems.

It is noted that owing to limitation of the author's knowledge and the space it allows, this paper could not list a huge number of literatures, among which some invaluable ones are neglected.

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References

[1] F. Simon, Uncertainty and imprecision: modeling and analysis, *Jour. of Oper. Res. Soc.*, 1995, **46**: 70–79.

[2] H. -J. Zimmermann, *Fuzzy Set Theory and Its Applications*(2rd Ed.), Kluwer-Nijhoff, Hinghum, 1991.

[3] L. A. Zadeh, Fuzzy sets, *Information and Control*, 1965, **8**: 338–353.

[4] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1978, **1**: 3–28.

[5] Y. J. Lai and C. L. Hwang, *Fuzzy Mathematical Programming Lecture Notes in Economics and Mathematical Systems* 394, Springer-Verlag, Berlin, 1992.

[6] R. E. Bellman and L. A. Zadeh, Decision making in a fuzzy environment, *Management Science*, 1970, **17**: B-141–164.

[7] S. Fang, C. F. Hu, et al., Linear programming with fuzzy coefficients in constraints, *Computers and Mathematics with Applications*, 1999, **37**: 63–76.

[8] S. C. Fang, and G. Li, Solving fuzzy relation equations with a linear objective function, *Fuzzy Sets and Systems*, 1999, **103**: 107–113.

[9] H. Hamacher, H. Leberling and H. J. Zimmermann, Sensitivity analysis in fuzzy linear programming, *Fuzzy Sets and Systems*, 1978, **1**(1): 269–281.

[10] S. Han, H. Ishii and S. Fujii, One machine scheduling problem with fuzzy due dates, *European Journal of Operational Research*, 1994, **79**: 1–12.

[11] H. Ishibuchi, N. Yamamoto, T. Murata and H. Tanaka, Genetic algorithms and neighborhood search algorithm for fuzzy flow shop scheduling problems, *Fuzzy Sets and Systems*, 1994, **67**(1): 81–100.

[12] H. Rommelfanger, Fuzzy linear programming and applications, *European Journal of Operations Research*, 1996, **92**(3): 512–527.

[13] J. Ramik and H. Rommelfanger, Fuzzy mathematical programming based on some new inequality relations, *Fuzzy Sets and Systems*, 1996, **81**(1): 77–87.

[14] H. Tanaka, and K. Asia, Fuzzy solution in fuzzy linear Programming, *IEEE Trans. on SMC.*, 1984, **14**(2): 325–328.

[15] D. Wang, An inexact approach for linear programming with fuzzy objective and resource, *Fuzzy Sets and Systems*, 1997, **8**(1): 61–68.

[16] D. Wang and S. C. Fang, A genetic-based approach for aggregate production planning in fuzzy environment, *IEEE Trans. on SMC* (Part A), 1997, **12**(5): 636–645.

[17] M. Sakawa and H. Yano, An interactive fuzzy satisfacing method for multi-objective nonlinear programming problems with fuzzy parameters, *Fuzzy Sets and Systems*, 1989, **30**(10): 221–238.

[18] M. Sakawa and H. Yano, Fuzzy dual decomposition method for large-scale multiobjective non-linear programming problem, *Fuzzy Sets and Systems*, 1994, **67**: 19–27.

[19] S. Chanas, D. Kuchta, Discrete fuzzy optimization, in *Fuzzy Sets in Decision Analysis Operations Research and Statistics–The Handbook of Fuzzy Sets Series*(Slowinski, Ed.,) Kluwer Academic, Dordrecht, 1998, pp. 249–276.

[20] M. Stoica, et al., Fuzzy integer programming, in H.-J. Zimmermann, L. A. Zadeh and B. R. Gaines Ed. *Fuzzy Sets and Decision Analysis*, North-Holland, Amsterdam, 1984, pp. 123–132.

[21] J. Kacprzyk, and A. O. Esogbue, Fuzzy dynamic programming: Main developments and applications, *Fuzzy Sets and Systems*, 1996, **81**(1): 31–46.

[22] D. Dubois, Linear programming with fuzzy data, in J. C. Bezdek Ed., *Analysis of Fuzzy Information*, CRC Press, Boca Raton, FLa, 1987, pp. 241–263.

[23] Y. J. Lai and C. L. Hang, A new approach to some possibilistic linear programming problems, *Fuzzy Sets and Systems*, 1992, **49**: 121–133.

[24] J. Ramik, and J. Rimanek, Inequality relation between fuzzy numbers and it's use in fuzzy optimization, *Fuzzy Sets and Systems*, 1985, **16**: 123–138.

[25] H. Rommelfanger, Hanuschek R. and J. Wolf, Linear programming with fuzzy objective, *Fuzzy Sets and Systems*, 1989, **29**: 31–48.

[26] H. Tanaka and K. Asai, Fuzzy linear programming problems with fuzzy numbers, *Fuzzy Sets and Systems*, 1984, **13**: 1–10.

[27] J. Liu, and SC Fang, Solving nonlinear optimization problems with fuzzy relation equation constraints, *Fuzzy Sets and Systems*, 2001, **119**: 1–20.

[28] J. Tang and D. Wang, A non-symmetric model for fuzzy nonlinear programming problems with penalty coefficients, *Computers & Operations Research*, 1997, **24**(8): 717–725.

[29] J. Tang, D. Wang and Richard Y K Fung, Model and method based on GA for non-linear programming problems with fuzzy objective and resources, *International Journal of Systems Science*, 1998, **29**(8): 907–913.

[30] J. F. C. Trappey, C. R. Liu and T. C. Chang, Fuzzy non-linear programming: Theory and application in manufacturing, *International Journal of Production Research*, 1988, **26**(5): 957–985.

[31] G. Bortolan, R. Degani, A review of some methods for ranking fuzzy subsets, *Fuzzy Sets and Systems*, 1985, **15**: 1–19.

[32] R. Ostermark, Sensitivity analysis of fuzzy linear programs: An approach to parametric interdependence, *Kybernetes*, 1987, **16**: 113–120.

[33] J. L. Verdegay, A dual approach to solve the fuzzy linear programming problems, *Fuzzy Sets and Systems*, 1984, **14**(1): 131–141.

[34] M. Delgado, S. Kacprzyk, J. L. Verdegay, M. A. Vila, *Fuzzy Optimization: Recent Advances*, Physica Verlag, Berlin, 1994.

[35] M. Fedrizzi, J. Kacprzyk and J. L. Verdegar, A survey of fuzzy optimization and mathematical programming, in: M. Fedrizzi, J. Kacprzyk, et al. Eds. *Interactive Fuzzy Optimization-Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, 1991, pp. 15–28.

[36] M. Inuiguchi, J. Ramik, Possibility linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, *Fuzzy Sets and Systems*, 2000, **111**: 3–28.

[37] M. Inuiguchi, Fuzzy linear programming: what, why and how ? Tatra, *Mountain Math. Publ.*, 1997, **13**, 123–167.

[38] J. Kacprzyk, and S. A. Orlovski, Fuzzy optimization and mathematical programming: A brief introduction and survey, in *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, D. Publishing Company, 1987, pp. 50–72.

[39] M. K. Luhandjula, Fuzzy optimization: an appraisal, *Fuzzy Sets and Systems*, 1989, **30**(3): 257–282.

[40] H. Rommelfanger, R. Slowinski, Fuzzy linear programming with single and multiple objective functions, in *Fuzzy Sets in Decision Analysis Operations Research and Statistics–The Handbook of Fuzzy Sets Series*, Slowinski R. Ed., Kluwer Academic, Dordrecht, 1998, pp. 179–207.

[41] B. Liu, *Uncertain Programming*, xii+248 pages, John Wiley & Sons, New York, 1999.

[42] B. Liu, *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.

[43] F. Jivenez, J. L. Verdegay, Solving fuzzy solid transportation problems by an evolutionary algorithm based parametric approach, *European Journal of Operations Research*, 1999, **117**: 485–510.

[44] D. Dubois and H. Prade, Decision making under fuzzy constraints and fuzzy criteria mathematical programming vs rule-based system approach, in M. Delgado, et al. Ed. *Fuzzy Optimization-Recent Advances*, Physica-Verlag, Heidelberg, 1994, pp. 21–32.

[45] H.-J. Zimmermann, Description and optimization of fuzzy system, *Int. J. of General System*, 1976, **2**: 209–216.

[46] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, New York, Toronto, 1980.

[47] H.-J. Zimmermann, Application of fuzzy set theory to mathematical programming, *Information Sciences*, 1985, **36**: 29–58.

[48] H. Tanaka, T. Okudu and K. Asai, On fuzzy mathematical programming, *Cybernet*, 1974, **3**: 37–46.

[49] R. R. Yager, Mathematical programming with fuzzy constraints and a preference on the objectives, *Kybernetics*, 1979, **9**: 109–114.

[50] J. Tang and D. Wang, An interactive approach based on GA for a type of quadratic programming problems with fuzzy objective and resources, *Computers & Operations Research*, 1997, **24**(5): 413–422.

[51] B. Werners, An interactive fuzzy programming systems, *Fuzzy Sets and Systems*, 1987, **23**: 131–147.

[52] S. Chanas, Using parametric programming in fuzzy linear programming, *Fuzzy Sets and Systems*, 1983, **11**(3): 243–251.

[53] S. A. Orlovski, On programming with fuzzy constraints sets, *Kybernets*, 1977, **6**: 197–201.

[54] S. A. Orlovski, On formulation of general fuzzy mathematical problems, *Fuzzy Sets and Systems*, 1980, **3**(1): 311–321.

[55] J. L. Verdegay, Fuzzy mathematical programming, in M. Gupta and E. Sanchez, Eds., *Fuzzy Information and Decision Processes*, North-Holland, Amsterdam, 1982, pp. 231–237.

[56] J. L. Verdegay, Application of fuzzy optimization in operational research, *Control and Cybernetics*, 1984, **13**: 229–239.

[57] D. Dubois & H. Prade, *Possibility Theory–An Approach to Computerized Processing of Uncertainty*, New York: Plenum Press, 1988.

[58] I. M. Stancu-Minasian, *Stochastic Programming with Multiple Objective Functions*, Reidel, Dordrecht, 1984.

[59] M. Fedrizzi, J. Kacprzyk, etc. Ed., *Interactive fuzzy optimization*, Springer-Verlag, Berlin, 1991.

[60] R. Slowinski, J. Teghem, *Stochastic vs Fuzzy Approaches to Multiobjective Mathematical Programming Under Uncertainty*, Kluwer Academic, Dordrecht, 1990, 249–262.

[61] H. Rommelfanger, FULPAL-An interactive method for solving multi-objective fuzzy linear programming, in: Slowinski R. and Teghem J. (Eds.), *Stochastic vs Fuzzy Approaches to Multiobjective Mathematical Programming Under Uncertainty*, Kluwer Academic, Dordrecht, 1990, pp. 279–299.

[62] P. Angelov, A generalized approach to fuzzy optimization, *International Journal of Intelligent Systems*, 1994, **9**(3): 261–268.

[63] D. P. Filev and R. R. Yager, A generalized defuzzification method via BAD distribution, *International Journal of Intelligent Systems*, 1991, **6**: 687–697.

[64] J. J. Buckley, and Y. Hayashi, Fuzzy genetic algorithm and applications, *Fuzzy Sets and Systems*, **61**(2): 129–136.

[65] W. A. Lodwick & K. D. Jamison, A computational method for fuzzy optimization, in *Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics and Neural Network Approach* (ed by B M Ayyub and M M Gupta), Kluwer Academic Publisher, Boston, 1998, 291–300.

[66] C. Carlsson, and P. Korhonen, A parametric approach to fuzzy linear programming, *Fuzzy Sets and Systems*, 1986, **20**(1): 17–30.

[67] F. M. Ali, A differential equation approach to fuzzy non-linear programming problems, *Fuzzy Sets and Systems*, 1998, **93**(1): 57–61.

[68] J. Tang, D. Wang and R. Y. K. Fung, A survey on fuzzy modeling and fuzzy optimization, in *Proceedings of 8th Fuzzy Sets Association World Congress*(*IFSA* 99), Taiwan, 1999, pp. 532–536.