Mathl Modelling, Vol. 9, No. 6, pp. 447–459, 1987
Printed in Great Britain.
0270-0255/87 $3.00 + 0.00

LINEAR PROGRAMMING WITH FUZZY SETS: A GENERAL APPROACH

S. T. WIERZCHON

Institute of Computer Science, Polish Academy of Sciences, P.O. Box 22, 00-901 Warsaw PKiN, Poland

Abstract—The aim of this paper is to give a tutorial presentation of the main questions concerning the fuzzy linear programming problems. The key idea is that fuzzy set theory allows concise characterization of an environment within which an agent operates. Depending on the specific requirements the fuzzy description can be converted into a deterministic model. The methods of how to translate such a fuzzy description into a concrete mathematical model are the main theme of the paper. Interpretative aspects of the resulting models are discussed.

1. INTRODUCTION

There exists a pretty large gap between mathematical thinking and common thinking. The former employs well-defined notions only, requiring a high standard of precision in all quantities in processing, while the latter operates with provisional judgements, vague notions and partly recognized relationships. In general, when solving practical problems, what we have at our disposal can be referred to as the evidence from which information must be extracted. Here, by evidence we mean the raw material from which judgements of facts are made, and information is the meaningful interpretation and correlation of data allowing us to make decisions.

Thus, what a practitioner really needs is a set of tools enabling him to cope with evidence, i.e. to extract and organize all the information from his provisional knowledge about a problem.

This task can be approached in a number of ways but the aim of this paper is to show that fuzzy sets theory allows quite precise conceptualization by which one can think of and manipulate the reality.

We restrict our attention to the linear programming (LP) problems. It appears that the fuzzy approach allows a unified treatment of various LP tasks. To provide this unicity, in Section 2 we propose a look at the problem of how to define such a task. In this context, a decision maker (DM) is forced to reflect primarily on questions like "What does it mean that an alternative satisfies a given criterion?" "How can I test whether it satisfies this criterion?" and so on. The resulting mathematical model is now a product of a particular representation of the DM's aspirations, possibilities and desires. Using fuzzy sets and choosing an appropriate representation we are able to recover almost all propositions described in the literature on fuzzy linear programming (FLP). This is demonstrated in Sections 4 and 5. To make the paper self-contained, in Section 3 we display a number of basic concepts needed for further considerations; this material is mainly devoted to the unacquainted reader.

2. DEFINING AN LP PROBLEM

The aim of this section is to outline a conceptual background for the "fuzzification" of the standard LP problem

$$\begin{aligned} \max \quad & G_j\colon \mathbf{c}_j\mathbf{x}, & j = 1,\ldots,J \\ \text{s.t.} \quad & \mathbf{a}_i\mathbf{x} \leqslant b_i, \mathbf{x} \geqslant \mathbf{0}, & i = 1,\ldots,I, \end{aligned} \tag{1}$$

where $\mathbf{x}$ is an $(M \times 1)$ column vector, $\mathbf{c}_j$ and $\mathbf{a}_i$ are the $(1 \times M)$ row vectors and b_is are real numbers.

Problem (1) is a particular case of the baseline model considered in Ref. [1] and it can be viewed as a mathematical representation of the task defined as

"Do the best under the circumstances".

To reach such a representation we proceed via the four steps listed below. (Of course these steps cannot be misguided by the system theoretic procedure for model building. We focus here only on the aspects making problem (1) a meaningful representation of the verbally formulated task.)

A. Determine a set $\mathscr{K}$ of criteria, or aspects, by means of which it is decided whether a given alternative is the solution. $\mathscr{K}$ is decomposed into two disjoint and nonempty sets $\mathscr{G}$ and $\mathscr{F}$. The criteria forming the set $\mathscr{G}$ are used to evaluate the quality or attractivity of a given alternative, while the criteria forming the set $\mathscr{F}$ are used to evaluate the feasibility or realizability of an alternative.
B. Identify a set of operational characteristics or measurement procedures visualizing how the system behaves with respect to each criterion K in $\mathscr{K}$. To denote that an operational characteristic concerns criterion G_j in $\mathscr{G}$ (resp. F_i in $\mathscr{F}$) we will write g_j (resp. f_i). In terms of problem (1) this step corresponds to: (a) the corroboration that all the characteristics are linear functions of the variable $\mathbf{x}$; and (b) the determination of numbers constituting the vectors $\mathbf{c}_j$ and $\mathbf{a}_i$.
C. Define what it means that an alternative satisfies a given criterion. In the case of the deterministic problem (1) we use two such definitions. The first of them exploits what we will call the satisfiability region $B_i = (-\infty, b_i]$. Referring to this notion we can write

$$\mathbf{x} \operatorname{sat} F_i \quad \text{iff} \quad f_i(\mathbf{x}) \in B_i. \tag{2}$$

Here "sat" and "iff" are shortenings of "satisfies" and "if and only if", respectively. Observe that the satisfiability region models what was called "circumstances" in our verbal formulation of problem (1). Introducing $B_i = [\underline{b}_i, \bar{b}_i]$ we model the requirement $\underline{b}_i \leqslant f_i(\mathbf{x}) \leqslant \bar{b}_i$; similarly, using $B_i = \{b_i\}$ we get the equality-type constraint $f_i(\mathbf{x}) = \{b_i\}$ and so on. The second definition assigns a meaning to the term "best alternative", which usually leads to the optimization problem. Note that this definition can be considered in a sense as a counterpart of definition (2): the satisfiability region is defined here in a more sophisticated (nonevident) but still meaningful way.
D. Determine a decision rule, i.e. a rule employed in choosing the solution to problem (1). The most natural seems to be

$$\mathbf{x} \in \mathbf{S} \quad \text{iff} \quad (\forall K \in \mathscr{K}) \mathbf{x} \operatorname{sat} K, \tag{3}$$

where by $\mathbf{S}$ we mean a set of alternatives being the solution to our decision problem. Unfortunately, although reasonable, this rule is not always applicable immediately. A constructive remedy is to "soften" rule (3) to a form which enables us to consider alternatives "supposedly" satisfying given criteria:

$$\mathbf{x} \in \mathbf{S} \quad \text{iff} \quad (\forall K \in \mathscr{K}) \mathbf{x} \widetilde{\operatorname{sat}} K, \tag{4}$$

where $\widetilde{\text{sat}}$ stands for "supposedly satisfies".

The general remark is that the successful realization of all the steps requires a relatively large portion of information; in particular, the DM must be able to isolate relevant criteria, identify the operational characteristics and he must be able to define a way in which the satisfiability of a given criterion will be tested. Consider a simple example taken from Ref. [2].

Suppose the DM is interested in allocating M crops over area A in a way which provides high total benefit, and that he assumes that the success depends on the possibility of irrigation of this area only. In this case the DM isolates only three criteria (Step A): K_1 = benefit; K_2 = water demand; and the commonsense criterion, K_3 = total area of land allocated to all the crops. Denoting $\mathbf{x} = (x_1, \ldots, x_M)$ with x_M = the area allocated to the Mth crop, he defines next (Step B) three operational characteristics h_1, h_2 and h_3 regarding the consequences of choosing a concrete alternative. Finally (Step C), he defines what it means that **x** satisfies a given criterion. Depending on the DM's aspirations, desires and actual circumstances the definitions may take the following forms:

$$
\begin{array}{llll}
\text{(a)} & \mathbf{x} \text{ sat } K_1 & \text{iff} & h_1(\mathbf{x}) = \max\limits_{\mathbf{y}} h_1(\mathbf{y}), \\
\text{(a')} & \mathbf{x} \text{ sat } K_1 & \text{iff} & h_1(\mathbf{x}) \geqslant B, \\
\text{(b)} & \mathbf{x} \text{ sat } K_2 & \text{iff} & h_2(\mathbf{x}) \leqslant W, \\
\text{(b')} & \mathbf{x} \text{ sat } K_2 & \text{iff} & h_2(\mathbf{x}) = \min\limits_{\mathbf{y}} h_2(\mathbf{y}), \\
\text{(c)} & \mathbf{x} \text{ sat } K_3 & \text{iff} & h_3(\mathbf{x}) \leqslant \mathbf{A},
\end{array}
$$

where A, B, and W are prespecified numbers. Choosing appropriate definitions the initial problem can be represented as the set of inequalities, as a single- or multiple-objective mathematical programming (MP) problem. For instance, the satisfiability of K_2 can be defined by condition (b) when the water resources are fully recognized or by condition (b') in the case of a water deficit. Observe also that in ordinary OR language the first definition will be classified as a "constraint" while the second as an "objective". These names are in fact irrelevant to the DM as he is interested in whether **x** satisfies K_2 or not.

When testing the satisfiability of a given criterion, the following situations may occur:

(i) A criterion K must be satisfied rigidly but the DM is not able to determine the satisfiability region definitely.

(ii) An operational characteristic is only partly recognized, i.e. the DM is able to show a weak relationship between a given alternative and a possible consequence.

(iii) The form of a given characteristic is known (e.g. it is linear) but its parameters cannot be determined precisely.

In practice a combination of these situations may occur.

The theory of fuzzy sets allows us to cope with all these situations in a unified and efficient way. A general recipe—proposed by Bellman and Zadeh [3]—is to introduce numerical degrees of truth assessing the extent to which the proposition "x sat K" seems to be credible or plausible. These degrees are determined "at hand" or are inferred from the evidence that the DM has at his disposal.

3. FUZZY SET TOOLS FOR REPRESENTING EVIDENCE

According to L. A. Zadeh, the meaning of a partly recognized concept can be represented by means of a so-called membership function μ mapping a universe of discourse into the unit interval.

To explain this idea suppose an operational characteristic f_i is such a partly recognized concept, i.e. the DM cannot assign to an alternative **x** an exact value $z = f_i(\mathbf{x})$ but he has evidence enabling him to state that the value of f_i in **x** is ABOUT z. This last term induces a fuzzy set $\tilde{Z}$ (the tilde is used to indicate that Z is a fuzzy set) of $\mathscr{R}$, the set of real numbers, characterized by the membership function $\mu_{\tilde{Z}}$. $\tilde{Z}$ can be imagined as an elastic constraint acting on the values that may be assigned to $f_i(\mathbf{x})$. $\mu_{\tilde{Z}}(r)$ is interpreted as the degree to which the constraint ABOUT z represented by $\tilde{Z}$ is satisfied when r is assigned to $f_i(\mathbf{x})$:

$$\mu_{\tilde{Z}}(r) = \text{Poss}(f_i(\mathbf{x}) = r), \tag{5}$$

where Poss is the possibility measure [3].

Caution. Since any crisp (nonfuzzy) subset is a special kind of fuzzy set, hereafter we will not distinguish—if not necessary—between fuzzy and crisp sets.

3.1. Fuzzy numbers

For practical purposes the fuzzy subsets of $\mathscr{R}$ are classified as fuzzy numbers provided that they are convex, unimodal, normalized and have upper semi-continuous membership functions (for details see Refs [4, 5]). The set of real fuzzy numbers will be denoted $F(\mathscr{R})$.

An important type of fuzzy numbers are the so-called LR fuzzy numbers (consult Ref [4]), characterized by membership functions of the form:

$$\mu_Z(r) = \begin{cases} L(r) & \text{for } -\infty \leqslant \underline{z} \leqslant r \leqslant z \\ R(r) & \text{for } z \leqslant r \leqslant \bar{z} + \infty \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where L (resp. R) is a nondecreasing (resp. nonincreasing) function such that $L(z) = R(z) = 1$, and $L(\underline{z}) = R(\bar{z}) = 0$. Here $\underline{z}$ (resp. $\bar{z}$) is said to be the lower (resp. upper) bound of Z and z is referred to as the main value of Z. An LR fuzzy number Z will be denoted

$$Z = (\underline{z}, z, \bar{z})_{LR}. \tag{7}$$

A most useful—from a practical standpoint—example of an LR fuzzy number is the triangular fuzzy number with the linear functions L and R; such a number will be denoted $(\underline{z}, z, \bar{z})_{\triangle}$.

Remark 1. When $\mu_Z(r) = 1$ for $r \in [z_*, z^*]$, then this Z is referred to as the flat fuzzy number or fuzzy interval. In this case we write $(\underline{z}, z_*, z^*, \bar{z})_{LR}$.

A binary operation $*$: $\mathscr{R} \times \mathscr{R} \to \mathscr{R}$ can be extended to the operation $\circledast$: $F(\mathscr{R}) \times F(\mathscr{R}) \to F(\mathscr{R})$ as follows [4]:

$$\mu_{A \circledast B}(r) = \sup_{\substack{u,v \in \mathscr{R} \\ u*v = r}} \min(\mu_A(u), \mu_3(v)). \tag{8}$$

In particular,

$$(\underline{a}, a, \bar{a})_{LR} \oplus (\underline{b}, b, \bar{b})_{LR} = (\underline{a} + \underline{b}, a + b, \bar{a} + \bar{b})_{LR} \tag{8'}$$

and

$$r \odot (\underline{a}, a, \bar{a})_{LR} = \begin{cases} (r\underline{a}, ra, r\bar{a})_{RL} & \text{for } r \geqslant 0 \\ (r\bar{a}, ra, r\underline{a})_{LR} & \text{for } r \leqslant 0; \end{cases} \tag{8''}$$

equation (8′) defines the addition of two LR fuzzy numbers and equation (8″) defines the multiplication of an LR fuzzy number by a real number.

3.2. Fuzzy functions

By a fuzzy function we mean a mapping $f: Y \to F(\mathscr{R})$ assigning fuzzy numbers to the points in Y. We can interpret such a mapping in two, equivalent, ways. The first way provides expression (5). Writing $f(y_0) = Z \in F(\mathscr{R})$, we impose an elastic constraint on possible values of $f(y_0)$, and $\mu_Z(r)$ expresses the possibility (or the degree of ease) that $f(y_0)$ takes the value r. In the second approach, f is considered as a fuzzy relation $\varnothing_f$: $Y \times R \to [0, 1]$ and $\varnothing_f(y, r)$ determines the degree of compatibility between the cause y and a possible result $r = f(y)$. For fixed y_0, $\varnothing_f(y_0, r)$ reduces to the fuzzy set Z. Although both interpretations are equivalent, in some circumstances [see case (iii) in Section 2] the second interpretation seems more convenient.

From a mathematical standpoint the fuzzy mapping can be treated as a generalized multivalued mapping. The main repercussion of such a standpoint is that to determine a co-image of a fuzzy set A we must specify two sets called the upper (A^*) and lower (A_*) inverses of A [6], defined as follows:

$$\mu_{A^*}(y) = \text{Inter}(Z, A) = \sup_{\mu \in \mathscr{R}} \min(\mu_Z(u), \mu_A(u)) \tag{9}$$

and

$$\mu_{A_*}(y) = \text{Incl}(Z, A) = \inf_{u \in \mathscr{R}} \max(1 - \mu_z(u), \mu_A(u)), \tag{10}$$

where $Z = f(y)$, Inter(Z, A) is the degree of intersection of Z with A and Incl(Z, A) is the degree of inclusion of Z in A. In particular, when Z and A are crisp sets then

$$\text{Inter}(Z, A) = \begin{cases} 1 & \text{when } Z \cap A \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{Incl}(Z, A) = \begin{cases} 1 & \text{when } Z \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

One can verify the following relationships, true for each Z and A:

$$\text{Incl}(Z, A) = 1 - \text{Inter}(Z, A^c) \tag{11}$$

and

$$\text{Incl}(Z, A) \leqslant \text{Inter}(Z, A). \tag{12}$$

Here A^c stands for the complement of A and $\mu_{A^c}(y) = 1 - \mu_A(y)$ for each y. Similarly, the condition $\mu_A(y) \leqslant \mu_B(y)$ for each y means that the fuzzy set A is included in B in the "hard" sense; to denote this we will write $A \subseteq_h B$. Thus, taking into account relations (11) and (12) we state that

$$A_* = Y \setminus (A^c)^* \quad \text{and} \quad A_* \subseteq_h A^*,$$

i.e. we recover the "traditional" properties of the upper and lower inverses. The interested reader will find more details concerning these generalized notions in Ref. [6].

Employing the identity $\max(a, b) = (a + b + |a - b|)/2$ we verify that when $A \subseteq_h B$ then Incl$(A, B) \geqslant 0.5$. Unfortunately, the converse statement is not true which causes problems in inferring the hard containment of A in B. Defining (see Ref. [2])

$$\text{Incl}_1(Z, A) = \begin{cases} \inf\limits_{y \in I(Z,A)} \mu_A(y) & \text{when } I(Z, A) \neq \varnothing \\ 1 & \text{otherwise,} \end{cases} \tag{13}$$

where $I(Z, A) = \{y \in \mathscr{R} : \mu_Z(y) > \mu_A(y)\}$ we get an index possessing the desired property, i.e.

$$\text{Incl}_1(Z, A) = \begin{cases} 1 & \text{when } Z \subseteq_h A \\ \alpha \in (0, 1) & \text{when supp}\, A \subseteq \text{supp}\, B \\ 0 & \text{otherwise.} \end{cases}$$

Here supp A denotes the support of the fuzzy set A, i.e. the topological closure of the (crisp) set $\{y \in \mathscr{R}\colon \mu_A(y) > 0\}$. One can verify that

$$\begin{aligned} \text{Incl}_1(Z, A) = \alpha \text{ iff } \alpha &= \sup\{\beta \in [0, 1]\colon Z_\beta \subseteq A_\beta\} \\ &= \sup\{\beta \in [0, 1]\colon Z_\beta \cap (A^c)_\beta = \varnothing\}, \end{aligned}$$

where $A_\beta = \{y \in \mathscr{R} : \mu_A(y) \geqslant \beta\}$ denotes the β-cut of the fuzzy set A.

This last definition is in a sense "dual" to the definition of Inter(Z, A), namely

$$\text{Inter}(Z, A) = \alpha \text{ iff } \alpha = \sup\{\beta \in [0, 1]: Z_\beta \cap A_\beta \neq \varnothing\}.$$

Similarly, one can verify that

$$\text{Incl}(Z, A) = \alpha \text{ iff } \alpha = 1 - \sup\{\beta \in [0, 1]: Z_\beta \cap (A^c)_\beta \neq 0\}.$$

4. A SOLUTION TO A FUZZY MATHEMATICAL PROGRAMMING (FMP) PROBLEM OF TYPE 1

An FMP problem is of type 1 when the DM agrees to test the satisfiability of all criteria forming the set $\mathscr{K}$ by means of the satisfiability regions. Symbolically, a problem of type 1 can be written in the form

$$\begin{aligned} &h_k(\mathbf{x}) \in B_k, \quad k = 1, 2, \dots, N, \\ &\mathbf{x} \geqslant \mathbf{0}, \end{aligned} \tag{14}$$

where some or all of the entities are fuzzy and $N = \text{card}(\mathscr{K})$.

Consider a single line of problem (14), say $h_j(\mathbf{x}) \in B_j$, $j \in \{1, \dots, N\}$, and let us write $h(\mathbf{x}) \in B$ for simplicity.

Introducing a fuzzy function we model the situation when the performance of the system under consideration with respect to the corresponding criterion is only partly recognized. Similarly, the fuzzy set B means some ambiguity concerning the satisfiability of this criterion.

As we have argued, the shape of the set B models the DM's aspirations and it can be given "at hand" or it must be inferred from other, possibly fuzzy, data.

The first situation occurs for instance when the DM considers as satisfactory all the alternatives placing the values of the operational characteristic in the "vicinity" of some threshold b, being a real number, without exceeding it. Here "vicinity" is a vague concept and its meaning must be defined by the DM [cf. case (i) from Section 2]. This approach was suggested in Refs [7, 8].

The second case corresponds, for instance, to a situation when an alternative $\mathbf{x}$ is taken as satisfactory if the value of an operational characteristic h is "above" some threshold $\tilde{b}$ and $\tilde{b}$ is a fuzzy number.

Suppose, for example, that our jth criterion K = "profit" and denote by u the desired benefit. In this case the fuzzy number $\tilde{b}$ can be interpreted in the following way. The DM will be: (1) unsatisfied when $u \leqslant \underline{b}$; (2) quite satisfied when $u = b$; and (3) fully satisfied when $u = \bar{b}$. (Here $\underline{b} < b < \bar{b}$ are prespecified real numbers.) Within the limits of the DM's best knowledge it is possible to attain the value b and it is very hard to attain $\bar{b}$. Hence, a strategy directed towards attaining the value $u \in [b, \bar{b}]$ requires some effort and this effort increases when u approaches the upper bound $\bar{b}$. (In the case of the "agricultural" model discussed in Section 2 this effort concerns some agrotechnical and/or organizational undertakings.) In effect, the resulting satisfaction in attaining the profit value u increases when u changes from $\underline{b}$ to b and decreases (because of the expenditure involved in the undertakings mentioned above) when u changes from b to $\bar{b}$. Denote by $\mu_{\tilde{b}}(u)$ the degree of this resulting satisfaction.

Now the term ABOVE $\tilde{b}$ will be modelled in accordance with the DM's aspirations. Referring to the above example, we can expect that an ambitious (or "greedy") agent would define his satisfiability region as

$$\mu_{\underline{B}}(u) = \inf_{r \geqslant u}(1 - \mu_{\tilde{b}}(r)) = \begin{cases} 0 & \text{when } u \leqslant b \\ 1 & \text{when } u \geqslant \bar{b} \\ 1 - R_{\tilde{b}}(u) & \text{otherwise,} \end{cases} \tag{15}$$

i.e. he will be unsatisfied with attaining the profit value $u \leqslant b$ and he will be fully satisfied when $u \geqslant \bar{b}$. Similarly, a less ambitious (or "lazy") agent would define his satisfiability region as

$$\mu_{\underline{B}}(u) = \sup_{r \leqslant u} \mu_{\tilde{b}}(r) = \begin{cases} 0 & \text{when } u \leqslant \underline{b} \\ 1 & \text{when } u \geqslant b \\ L_{\tilde{b}}(u) & \text{otherwise.} \end{cases} \quad (16)$$

(Recall that $R_{\tilde{b}}$ and $L_{\tilde{b}}$ denote the right and left reference functions, respectively, characterizing the fuzzy threshold $\tilde{b}$.)

The membership functions defined above characterize what was termed by Dubois and Prade [5] as the fuzzy intervals of numbers certainly greater than $\tilde{b}$, denoted $(\tilde{b}, +\infty)$, and of numbers possibly not less than $\tilde{b}$, denoted $[\tilde{b}, +\infty)$, respectively. We can also define the complements of these intervals, i.e.

$$\mu_{(-\infty,\tilde{b})}(u) = 1 - \mu_{[\tilde{b},+\infty)}(u), \qquad \mu_{(-\infty,\tilde{b}]}(u) = 1 - \mu_{(\tilde{b},+\infty)}(u). \quad (17)$$

The last two intervals are appropriate for modelling the term BELOW $\tilde{b}$.

Observe that both $\mu_{\underline{B}}$ and $\mu_{\bar{B}}$ have the general form

$$\mu(u) = \begin{cases} 0 & \text{when } u \leqslant b^- \\ \alpha \in (0,1) & \text{when } b^- < u < b^+ \\ 1 & \text{otherwise,} \end{cases} \quad (18)$$

where, for instance, $b^- = \underline{b}$ when we use expression (16). A membership function of this form was applied by Zimmermann [7] to assign a meaning to the statement $f(\mathbf{x}) \gtrsim b$, with $\gtrsim$ being a fuzzy counterpart of the crisp $\geqslant$ relation.

Having defined the satisfiability region B, the DM can construct a set of admissible alternatives. Define

$$\mathbf{X}_k^v(\alpha) = \{\mathbf{x} \in \mathbf{X} : v(h_k(\mathbf{x}), B_k) \geqslant \alpha\} \quad (19)$$

to be the set of alternatives satisfying the kth criterion with the degree not less than α; $v \in \{\text{Inter}, \text{Incl}, \text{Incl}_1\}$.

Our first observation is that when h_k is a real-valued function, then

$$v(h_k(\mathbf{x}), B_k) = \mu_{B_k}(h_k(\mathbf{x})), \qquad v \in \{\text{Inter}, \text{Incl}, \text{Incl}_1\},$$

i.e. we obtain Zimmermann's approach. When h_k is an interval-valued mapping (i.e. with each $\mathbf{x}$ in $\mathbf{X}$ there is associated a closed and crisp interval on the real line) and B_k is a crisp interval then, for $v \in \{\text{Incl}, \text{Incl}_1\}$, we have

$$v(h_k(\mathbf{x}), B_k) = \begin{cases} 1 & \text{when } h_k(\mathbf{x}) \subseteq B_k \\ 0 & \text{otherwise,} \end{cases}$$

i.e. we obtain Soyster's approach to inexact programming [9]. Finally, when h_k is a fuzzy mapping and B_k is an LR fuzzy number then, using as v the Incl_1 index, we get a counterpart of the fuzzy inexact programming discussed by Negoita [10].

Suppose now that the DM has decided to consider as admissible the alternatives from the set $\mathbf{X}_k^{\text{Inter}}(\alpha)$. According to the basic relation (11), he is certain that any alternative $\mathbf{x}$ from this set is inadmissible to an extent not greater than $1 - \alpha$; more precisely,

$$\text{if } \mathbf{x} \in \mathbf{X}_k^{\text{Inter}}(\alpha) \quad \text{then } \text{Incl}(h_k(\mathbf{x}), B_k^c) \leqslant 1 - \alpha.$$

Similarly,

$$\text{if } \mathbf{x} \in \mathbf{X}_k^{\text{Incl}}(\alpha) \text{ then } \text{Inter}(h_k(\mathbf{x}), B_k^c) \leqslant 1 - \alpha.$$

Since $\text{Incl}(h_k(\mathbf{x}), B_k^c)$ is not greater than $\text{Inter}(h_k(\mathbf{x}), B_k^c)$, the alternatives from the set $\mathbf{X}_k^{\text{Incl}}(\alpha)$ are more "acceptable" than those from the set X Inter(α). In other words, when using the Inter index the DM prefers a risky strategy, while when using the Incl index he prefers a careful strategy. The choice between these indices may also depend on the importance of a given criterion: when this criterion is important it is better to use the Incl index, and when the consequences of its "violation" are not too extreme we may use the Inter index. This last observation allows a simple and attractive combination of different criteria with different degrees of importance (note that the extent of the set of admissible alternatives may be also controlled by the definition of the satisfiability region B_k, as we discussed earlier). In the case of a very important criterion the DM may be interested in considering alternatives for which the support of the fuzzy number $h_k(\mathbf{x})$ is surely contained in the support of the satisfiability region. Such a requirement is satisfied when he uses the Incl_1 index instead of the index Incl.

Now we are ready to solve problem (14), a fuzzified version of problem (1). To do this we refer to Bellman–Zadeh's rule as presented in Ref. [10], which now takes the form

$$\begin{aligned} &\max \alpha \\ &v(h_k(\mathbf{x}), B_k) \geqslant \alpha, \quad k = 1, 2, \ldots, N, \\ &\mathbf{x} \geqslant \mathbf{0}, \quad \alpha > 0. \end{aligned} \tag{20}$$

Assume for simplicity that all coefficients specified in problem (1) are triangular fuzzy numbers. According to equations (8′) and (8″), the value of each operational characteristic is also a triangular fuzzy number, e.g. $g_j(\mathbf{x}) = (\underline{\mathbf{c}}_j\mathbf{x}, \mathbf{c}_j\mathbf{x}, \bar{\mathbf{c}}_j\mathbf{x})_\triangle$, where, for instance, $\underline{\mathbf{c}}_j$ denotes the vector containing the lower bounds $\underline{c}_{jm}$, $m = 1, 2, \ldots, M$. It is a simple exercise to derive the following deterministic equivalents of our fuzzified problem.

(i) When v = Inter, problem (20) takes the form

$$\begin{aligned} &\max \alpha \\ \text{s.t.}\ &(\bar{\mathbf{c}}_j - \alpha(\bar{\mathbf{c}}_j - \mathbf{c}_j))\mathbf{x} \geqslant b_j^- + \alpha(b_j^+ - b_j^-), \quad j = 1, \ldots, J, \\ &(\underline{\mathbf{a}}_i + \alpha(\mathbf{a}_i - \underline{\mathbf{a}}_i))\mathbf{x} \leqslant b_i^+ - \alpha(b_i^+ - b_i^-), \quad i = 1, \ldots, L, \\ &\mathbf{x} \geqslant \mathbf{0}, \quad \alpha > 0. \end{aligned} \tag{21}$$

Here $b_j^- = \underline{b}_j$ (resp. b_j) and $b_j^+ = b_j$ (resp. $\bar{b}_j$) when for B_j we take the fuzzy interval $[\tilde{b}_j, +\infty)$ (resp. $(\tilde{b}_j, +\infty)$). Similarly, $b_i^- = \underline{b}_i$(resp. b_i) and $b_i^+ = b_i$ (resp. $\bar{b}_i$) when for B_i we take the fuzzy interval $(-\infty, \tilde{b}_i)$ (resp. $(-\infty, \tilde{b}_i]$). The same remark applies to the remaining cases (ii) and (iii).

(ii) When v = Incl, problem (20) takes the form

$$\begin{aligned} &\max \alpha \\ \text{s.t.}\ &(\mathbf{c}_j - \alpha(\mathbf{c}_j - \underline{\mathbf{c}}_j))\mathbf{x} \geqslant b_j^- + \alpha(b_j^+ - b_j^-), \qquad j = 1, \ldots, J, \\ &(\mathbf{a}_i + \alpha(\bar{\mathbf{a}}_i - \mathbf{a}_i))\mathbf{x} \leqslant b_i^+ - \alpha(b_i^+ - b_i^-), \qquad i = 1, \ldots, I, \\ &\mathbf{x} \geqslant \mathbf{0}, \qquad \alpha \geqslant 0. \end{aligned} \tag{22}$$

(iii) When $v = \text{Incl}_1$, problem (20) takes the form

$$\begin{aligned}
&\max \alpha \\
&\text{s.t.}\quad (\underline{\mathbf{c}}_j + \alpha(\mathbf{c}_j - \underline{\mathbf{c}}_j))\mathbf{x} \geqslant b_j^- + \alpha(b_j^+ - b_j^-) \\
&\qquad (\bar{\mathbf{a}}_i - \alpha(\bar{\mathbf{a}}_i - \mathbf{a}_i))\mathbf{x} \leqslant b_i^+ + \alpha(b_i^+ - b_i^-) \\
&\qquad \underline{\mathbf{c}}_j\mathbf{x} \geqslant \mathbf{b}_j^- \\
&\qquad \bar{\mathbf{a}}_i\mathbf{x} \leqslant b_i^+ \\
&\qquad \mathbf{x} \geqslant \mathbf{0}, \quad \alpha \in [0,1], \quad j = 1,\ldots,J, \quad i = 1,\ldots,I. \qquad (23)
\end{aligned}$$

It should be stressed that the approach presented here was initiated by Dubois [11] and, independently, by Wierzchoń *et al.* in Ref. [12].

5. A SOLUTION TO A FUZZY MATHEMATICAL PROGRAMMING (FMP) PROBLEM OF TYPE 2

An FMP is of type 2 if it reduces to the form

$$\begin{aligned}
&\max \tilde{g}_j(\mathbf{x}), \quad j = 1,\ldots,J, \\
&\text{s.t.}\quad \mathbf{x} \in \mathbf{X} \qquad (24)
\end{aligned}$$

where $\tilde{g}_j$s are fuzzy functions and **X** is a crisp set of admissible alternatives.

According to our earlier nomenclature this problem arises when the DM is (a) able to point out unambiguously some set of "acceptable" alternatives and (b) interested in finding an alternative maximizing some operational characteristics over this **X** but (c) these characteristics are only partly recognized.

Assume for a moment that $J = 1$ and write $\tilde{g}(x)$ instead of $\tilde{g}_1(x)$.

To give a meaning to the term "maximal value" of such an imprecise characteristic $\tilde{g}$, observe that in the crisp case (when the characteristic is a real function we will write g instead of $\tilde{g}$) an alternative $\mathbf{x}_0$ is said to be optimal iff there is no $\mathbf{x}$ in **X** s.t. $g(x) > g(x_0)$ or, equivalently, for any $\mathbf{x}$ in **X** we have $g(\mathbf{x}) \leqslant g(\mathbf{x}_0)$. This corresponds to the two approaches exploited in the classical theory of choice: we can treat $\mathbf{x}_0$ as the non-dominated or dominating alternative, respectively (see Ref. [13] for details).

Denote $w = g(\mathbf{x}_0)$ and define $B_0 = [w, +\infty)$, $B_0^c = (-\infty, w)$. When g is unimodal over **X** we can reformulate the above statements as follows. An alternative $\mathbf{x}_0$ is optimal when

(i) there is no $\mathbf{x}$ in **X**, $\mathbf{x} \neq \mathbf{x}_0$, s.t. $g(\mathbf{x}) \in B_0$

or

(ii) for any $\mathbf{x}$ in **X**, $\mathbf{x} \neq \mathbf{x}_0$, we have $g(\mathbf{x}) \in B_0^c$.

This last observation makes it possible to assign a precise meaning to the notion of a maximal value of an imprecise characteristic. Denote again by $v(\tilde{g}(\mathbf{x}), B)$ an index used to evaluate the degree with which the value of $\tilde{g}(\mathbf{x})$ can be treated as belonging to the set B. Now we are able to give the "softened" versions of the earlier definitions.

Definition 1

An alternative $\mathbf{x}^0$ is referred to as α–v-nondominated when there exists a set $B^0 = [w^0, +\infty)$ such that

$$v(\tilde{g}(\mathbf{x}^0), B^0) = \alpha$$

and

$$v(\tilde{g}(\mathbf{x}), B^0) \leqslant \alpha, \quad \mathbf{x} \in \mathbf{X}.$$

Definition 2

An alternative $\mathbf{x}_0$ is said to be α–v-dominating when there exists a set $B_0 = (-\infty, w_0]$ such that

$$v(\tilde{g}(\mathbf{x}_0), B_0) = \alpha$$

and

$$v(\tilde{g}(\mathbf{x}), B_0) \geqslant \alpha, \quad \mathbf{x} \in \mathbf{X}.$$

The first definition means that $\mathbf{x}^0$ belongs to B^0 with the maximal degree while the latter chooses an alternative that belongs to B_0 with the minimal degree of membership. When $\tilde{g}$ is a real function then $\mathbf{x}_0 = \mathbf{x}^0$.

Proposition 1

Let $v(.,.) = \text{Inter}(.,.)$. An alternative $\mathbf{x}^0$ is α–Inter-nondominated when it is a solution to the following MP problem:

$$\begin{aligned} &\min w^0 \\ &\text{s.t.} \quad R_{\tilde{g}(\mathbf{x})}(w^0) \leqslant \alpha \\ &\qquad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1. \end{aligned} \tag{25}$$

An alternative $\mathbf{x}_0$ is α–Inter-dominating when it solves the following MP problem:

$$\begin{aligned} &\max w_0 \\ &\text{s.t.} \quad L_{\tilde{g}(\mathbf{x})}(w_0) \geqslant \alpha \\ &\qquad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1. \end{aligned} \tag{26}$$

Here $L_{\tilde{g}(\mathbf{x})}$ and $R_{\tilde{g}(\mathbf{x})}$ stand for the left and right reference functions of the fuzzy number $\tilde{g}(\mathbf{x}) = (\underline{g}(\mathbf{x}), g(\mathbf{x}), \bar{g}(\mathbf{x}))_{LR}$, respectively.

Proof of this proposition follows immediately from the nature of the Inter index and from the fact that $L_{\tilde{g}\mathbf{x}}(w)$(resp. $R_{\tilde{g}}(w)$) is a nondecreasing (resp. nonincreasing) function of its argument w.

Corollary 1

When L and R are invertible then the MP problems reduce to the following:

$$\begin{aligned} &\max R^{-1}_{\tilde{g}(\mathbf{x})}(\alpha) \\ &\text{s.t.} \quad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1, \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\max L^{-1}_{\tilde{g}(\mathbf{x})}(\alpha) \\ &\text{s.t.} \quad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1. \end{aligned}$$

In particular, when $\tilde{g}(\mathbf{x}) = \mathbf{cx}$ and all the components of $\mathbf{c}$ are triangular fuzzy numbers then to find the α–Inter-nondominated alternative we should solve the following parametric LP problem:

$$\begin{aligned} &\max (\bar{\mathbf{c}} - \alpha(\bar{\mathbf{c}} - \mathbf{c}))\mathbf{x} \\ &\text{s.t.} \quad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1; \end{aligned} \tag{27}$$

and to find the α–Inter-dominating alternative we should solve the problem

$$\begin{aligned} &\max (\underline{\mathbf{c}} + \alpha(\mathbf{c} - \underline{\mathbf{c}}))\mathbf{x} \\ &\text{s.t.} \quad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1. \end{aligned} \tag{28}$$

Corollary 2

When $\mathbf{x}^0$ is the α–Inter-nondominated alternative then, for any $\mathbf{x}$ in $\mathbf{X}$,

$$\text{Incl}(\tilde{g}(\mathbf{x}), (-\infty, w^0(\alpha)]) \geqslant 1 - \alpha. \tag{29}$$

Similarly, when $\mathbf{x}_0$ is the α–Inter-dominating alternative then, for any $\mathbf{x}$ in $\mathbf{X}$,

$$\text{Incl}(\tilde{g}(\mathbf{x}), [w_0(\alpha), +\infty)) \leqslant 1 - \alpha. \tag{30}$$

Here $w^0(\alpha)$ is the value found by solving problem (25); solving problem (26) we find the value $w_0(\alpha)$.

This corollary shows that $\mathbf{x}^0$ is α–Inter-nondominated when it intersects with the set $[w^0(\alpha), +\infty)$ with the maximal (equal to α) degree and simultaneously, when it is included in the set $(-\infty, w^0(\alpha)]$ with the minimal (equal to $1 - \alpha$) degree; we can apply symmetrical reasoning to the interpretation of $\mathbf{x}_0$.

With Proposition 1, by application of the basic identity (11), we have the following.

Proposition 2

Let $v(.,.) = \text{Incl}(.,.)$. An alternative $\mathbf{y}^0$ is α–Incl-nondominated when it solves the following MP problem:

$$\begin{aligned} &\max u^0 \\ &\text{s.t.}\quad L_{\tilde{g}(\mathbf{x})}(u_0) \geqslant 1 - \alpha \\ &\qquad\ \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1; \end{aligned} \tag{31}$$

and an alternative $\mathbf{y}_0$ is α–Incl-dominating when it is a solution to the problem

$$\begin{aligned} &\min u_0 \\ &\text{s.t.}\quad R_{\tilde{g}(\mathbf{x})}(u^0) \leqslant 1 - \alpha \\ &\qquad\ \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha \leqslant 1. \end{aligned} \tag{32}$$

Analysing both propositions we obtain the following.

Corollary 3

An alternative $\mathbf{x}^0$ is α–Inter-nondominated iff it is $(1 - \alpha)$–Incl-dominating and an alternative $\mathbf{x}_0$ is α–Inter-dominating iff it is $(1 - \alpha)$–Incl-nondominated.

It can be easily verified that the first part of Proposition 1 covers the approach proposed by Orlovski [2] (see also Ref. [14]). Similarly, the second part of Proposition 2 includes the approach suggested by Verdegay [15].

Proposition 3

Let $v(.,.) = \text{Incl}_1(.,.)$. An alternative $\mathbf{z}^0$ is Incl_1-nondominated when it is a solution to the problem $\max_{\mathbf{x}\in\mathbf{X}} \underline{g}(\mathbf{x})$ and the alternative solving the problem $\max_{\mathbf{x}\in\mathbf{X}} \bar{g}(\mathbf{x})$ is Incl_1-dominating.

This proposition follows immediately from the definition of the Incl_1 index.

Corollary 4

An alternative $\mathbf{z}^0$ is Incl_1-nondominated iff it is 1–Incl-nondominated (or 0–Inter-dominating) and an alternative $\mathbf{z}_0$ is Incl_1-dominating iff it is 1–Incl-dominating (or 0–Inter-nondominated).

Our discussion hitherto can easily be extended to the case of $J > 1$ objective functions. We can define, for example, the set $S^{\text{ND}}_{\alpha_1,\ldots,\alpha_J}$ of nondominated alternatives as follows:

$$S^{\text{ND}}_{\alpha_1,\ldots,\alpha_J} = \{\mathbf{y} \in \mathbf{X}\text{: there is no } \mathbf{x} \text{ in } \mathbf{X} \text{ s.t. } v(\tilde{g}_k(\mathbf{x}), B_k^0) > v(\tilde{g}_k(\mathbf{y}), B_k^0) \text{ for some } k \in \{1,\ldots,J\} \text{ and } v(\tilde{g}_j(\mathbf{x}), B_k^0) \geqslant v(\tilde{g}_j(\mathbf{y}), B_k^0)\ \forall j \neq k\}.$$

For example, when all $\tilde{g}_j$s have all the right reference functions invertible and $v(\ldots) = \text{Inter}(\ldots)$, then we can refer to the classical multiobjective programming problem by saying that $\mathbf{y}^0 \in S^{\text{ND}}_{\alpha_1,\ldots,\alpha_J}$ when it is a solution to the problem

$$\max\,[R^{-1}_{\tilde{g}_1(\mathbf{x})}(\alpha_1), \ldots, R^{-1}_{\tilde{g}_J(\mathbf{x})}(\alpha_J)]$$
$$\text{s.t.}\quad \mathbf{x} \in \mathbf{X}, \quad 0 < \alpha_j \leqslant 1, \quad j = 1, \ldots, J. \tag{33}$$

To see that such a definition has a sense, suppose $\mathbf{x}^0$ is an efficient (Pareto) solution to problem (33). Then, for any $\mathbf{x} \notin S^{\text{ND}}_{\alpha_1,\ldots,\alpha_J}$, we have

$$w_k(\mathbf{x}) = R^{-1}_{\tilde{g}_k(\mathbf{x})}(\alpha_k) < R^{-1}_{\tilde{g}_k(\mathbf{x}^0)}(\alpha_k) = w_k(\mathbf{x}^0) \quad \text{for some } k$$

and

$$w_j(\mathbf{x}) = R^{-1}_{\tilde{g}_j(\mathbf{x})}(\alpha_j) \leqslant R^{-1}_{\tilde{g}_j(\mathbf{x})}(\alpha_j) = w_j(\mathbf{x}^0) \quad \forall j \neq k.$$

Taking into account that $R_{\tilde{g}_j(\mathbf{x})}(w_j(\mathbf{x})) = \alpha_j$ and that the right reference function is now a strictly decreasing function, and defining $B^0_j = [w_j(\mathbf{x}^0), +\infty)$ we state that, for any $\mathbf{x} \notin S^{\text{ND}}_{\alpha_1,\ldots,\alpha_J}$,

$$\text{Incl}(\tilde{g}_k(\mathbf{x}), B^0_k) < \alpha_k$$
$$\text{Incl}(\tilde{g}_j(\mathbf{x}), B^0_j) \leqslant \alpha_j, \quad j \neq k.$$

In closing let us mention "optimization" of a fuzzy function over a fuzzy domain $\tilde{\mathbf{X}}$. Using the method of approximation of a fuzzy set described by Negoita [10, Section 3-1], we can replace this $\tilde{\mathbf{X}}$ by a crisp set $\mathbf{X}_{\text{apr}}$. In this way our initial task can be reduced to problem (24). Another possible method relies on the extraction of a crisp set $\mathbf{X}_\beta$ being the β-cut of the fuzzy set $\tilde{\mathbf{X}}$. Again such an approach reduces our initial task to problem (24).

6. CONCLUDING REMARKS

The paper can be summarized as follows. To represent an LP problem mathematically the DM must ask himself what does it really mean that a given alternative is the solution? To answer this question, usually he proceeds via the four steps presented in Section 2. In the presence of perfect information the obtained solution is of a dogmatic nature. As any real-life problem is rarely perfectly recognized, and in particular the DM aspires to attain a set of (implicitly stated) goals, the dogmatic representation of his problem seems to be questionable.

Thus the DM should try to build up a representation that reflects his diverse desires, aspirations and his subjective opinions. Admitting the fuzzy representation, the DM gains the spectrum of models discussed in previous sections.

The nondogmatic character of the fuzzy models allows us to fit them accurately to actual circumstances, making it possible to investigate exhaustively the effects of particular decisions.

Acknowledgement—This work was partly supported by the International Institute for Applied Systems Analysis, Laxenburg, Austria.

REFERENCES

1. J. P. Ignizio, *Linear Programming in Single and Multiple Objective Systems.* Prentice-Hall, Englewood Cliffs, N.J. (1982).
2. S. A. Orlovski, *Problems of Decision-making with Fuzzy Information.* Nauka, Moscow (1981). In Russian.
3. L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets Syst.* **1**, 3–29 (1978).
4. D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications.* Academic Press, New York (1980).
5. D. Dubois and H. Prade, Ranking fuzzy numbers in the setting of possibility theory. *Inf. Sci.* **30**, 183–224 (1983).
6. D. Dubois and H. Prade, Twofold fuzzy sets and rough sets. In *Proc. 2nd Napoli Mtg on The Mathematics of Fuzzy Systems*, Napoli (Edited by A. Di Nola and A. G. S. Ventre) (1985).
7. H. J. Zimmermann, Fuzzy programming and linear programming with several objective functions. *Fuzzy Sets Syst.* **1**, 45–56 (1978).

8. H. Hamacher, H. Leberling and H. J. Zimmermann, Sensitivity analysis in fuzzy linear programming. *Fuzzy Sets Syst.* **1**, 269–281 (1978).
9. A. L. Soyster, Convex programming with set-inclusive constraints. Applications to inexact linear programming. *Opns Res.* **21**, 1154–1157 (1973).
10. C. V. Negoita, *Management Applications of Systems Theory*. Birkhauser Verlag, Basel (1979).
11. D. Dubois, Linear programming with fuzzy data. In *The Analysis of Fuzzy Information*, Vol. III (Edited by J. C. Bezdek). CRC Press, Boca Raton, Fla (1986).
12. S. T. Wierzchoń, J. Kindler and S. Tyszewski, A class of solutions to the fuzzy linear programming problems (submitted).
13. M. A. Aizerman and A. V. Malishevski, General theory of best variant choice: some aspects. *IEEE Trans. Automn Contr.* **AC-26**, 1030–1040 (1981).
14. S. A. Orlovski, Multiobjective programming problems with fuzzy parameters. *Contr. Cybern.* **13**, 175–184 (1984).
15. J. L. Verdegay, Applications of fuzzy optimization in operational research. *Contr. Cybern.* **13**, 229–240 (1984).