Fuzzy Sets and Systems 30 (1989) 257–282
North-Holland

FUZZY OPTIMIZATION: AN APPRAISAL

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Dedicated to Professor H.-J. Zimmermann

Received July 1986
Revised September 1987

This paper takes a general look at core ideas that make up the burgeoning body of Fuzzy mathematical programming emphasizing the methodological view.

Although Fuzzy mathematical programming has enjoyed a rapidly increasing acceptance within the scientific community, some technical hurdles exist to hinder a unanimity. Reasons for this as well as possible ways for improvement are also discussed.

Keywords: Fuzzy, Optimization, Decision-making.

1. Introduction

Mathematical disciplines have traditionally arisen in response to the need to solve problems in the real world. Fuzzy mathematical programming, following this traditional pattern, has been developed in connection with the necessity to meet requirements of decision making in practice.

As witnessed by discussions in professional journals, many decision models outline wrongly the problem to be tackled. One of the reasons of this mismatching is the fact that most of these models and methods are unsuitable for decision situations in which intrinsic or informational imprecision plays a pivotal role.

Although probabilistic theories (Kall [39], Vaja [96]) claim to model decision making under uncertainty, there is a qualitatively different kind of imprecision which is not covered by these apparatus, that is: inexactness, ill-definedness, vagueness. Situations where doubt arises about the exactness of concepts, correctness of statements and judgements, degree of credibility, have little to do with the occurrence of events, the back-bone of probability theory.

Fuzzy sets theory offers a proper framework for coming to grips with such situations involving non-stochastic imprecision.

A look at the existing literature clearly indicates its contribution in enhancing the descriptive power of numerous decision making models (Mathematical programming: Zimmermann [106], Negoita [56], Dubois [18]; Utility theory: Kickert [40]; Knowledge engineering: Baldwin [2], Sugeno [86], etc.) by letting irreductible imprecision be taken into account.

The purpose of this paper is to review the state of knowledge in the area of Fuzzy mathematical programming and to provide a perspective on potential research directions. An exhaustive review is not possible within the scope of a single article as the subject is still being vigorously pursued by many workers. On the other hand picking and reviewing an individual segment would not do justice to the richness of the subject. Thus I shall try to glance at significant areas with a decision perspective in view. This does not imply a value judgement on topics and results which are not reported.

Let me mention that existing fuzzy mathematical programming approaches proceed along the lines to be given below.

First the set P of admissible alternatives is specified. Next, the result $e(x, 0, C_1, \ldots, C_m)$ representing the position of each potential alternative $x \in P$ vis-a-vis to the goal (0) and the constraints (C_i, $i = 1, \ldots, m$) is elicited.

Let now a compatibility function on the set I of possible results be given by $K: I \to R$, i.e. $K(e(x, 0, C_1, \ldots, C_m))$ is the degree of compatibility of x with the goal and constraints of the problem.

Finally a transformation T which assigns a real valued function on P to any compatibility function is defined. The resulting problem is to maximize $TK(x) = F(x)$ on P.

As will become apparent in the sequel, existing points of view differ on the choice of P, K and T and on the kind of solution sought.

The paper is organized as follows: Section 2 deals with flexible programming, i.e. mathematical programming problems with crisp parameters, the objective and constraints of which are ill stated. Mathematical programming problems with fuzzy parameters are taken up in Section 3. Extensions to situations where fuzziness and randomness are combined in the scope of a mathematical program as well as to multiple objective programming problems are discussed in Section 4. We end up with a discussion on technical hurdles which hamper the rapid elevation of Fuzzy optimization from the academic to the practical realm and to ways for improvements.

2. Flexible programming

In solving their problems, Deciders generally grapple with technological, environmental and competitive factors which interact in a complicated fashion. In such a turbulent environment, the formulation of the problem in terms of dichotomous 'yes or no' statements yields often inconsistencies which are expressed by the vacuousness of the feasible set. So it is worthwhile to discuss how to tolerate some leeways in the formulation of goals and constraints of a mathematical program. This is the main topic of flexible programming, the subject matter of this section.

2.1. *Problem formulation*

Consider the mathematical program

$$\begin{aligned} &\widetilde{\min}\, f(x), \\ &g_i(x) \lesssim b_i, \quad i = 1, \ldots, m, \\ &x \in X = \{x \in R^n \mid x \geq 0\}, \end{aligned} \tag{P1}$$

where f and g_i ($i=1,\ldots,m$) are real functions of n variables and '$\sim$' indicates that the Decider is no longer able to specify clearly the goal and constraints of his problem because it is just impossible or because he wants to allow himself some leeways. Speaking informally, (P1) reads: Find $x \in X$ such that $f(x)$ is maintained to a lower level while $g_i(x) \leq b_i$ ($i=1,\ldots,m$) are met as well as possible.

Immediately the question arises of what the best or the more satisfying alternative in this context means and how to single out such an alternative. The business of the remainder of this section is devoted to these questions and related problems.

2.2. *Symmetrical approaches*

2.2.1. *Symmetrical approaches based on the concept of decision in a fuzzy environment*

For convenience, the goal of (P1) is replaced by a constraint of the form: $f(x) \lessapprox z_0$, where z_0 is some prescribed threshold. Furthermore, membership functions of fuzzy sets C_i ($i=0,\ldots,m$) representing the Decider's aspirations for the attainment of the goal and of the constraints should be made explicit and defined in some proper way. The proper way is open for the model builder. For practical purposes, (piecewise) linear, polynomial, hyperbolic functions are mostly used. How to elicit membership functions of involved fuzzy sets falls outside the scope of this paper. Interested readers are referred to Kuz'min [41] Yager [101] where axiomatical and empirical considerations in connection with this problem are discussed.

Keeping in mind notations of the general construction scheme of Section 1, let

$$e(x, 0, C_1, \ldots, C_m) = (u_{C_0}(x), \ldots, u_{C_m}(x))$$

where $u_{C_0}(x)$ is the membership function of the fuzzy set C_0 representing the ill-defined goal and u_{C_i} ($i=1,\ldots,m$) fuzzy membership functions of fuzzy sets representing ill-formulated constraints.

Define K and T as follows:

$$K(u_{C_0}(x), \ldots, u_{C_m}(x)) = \min_i u_{C_i}(x),$$

$$\begin{array}{lll} T: \mathscr{L} \to \mathscr{M} & & \\ \quad K & TK = F: X \to R & \\ & \qquad\qquad x & \min_i u_{C_i}(x) \end{array}$$

where $\mathscr{L} = \{K: I \to R\}$ and $\mathscr{M} = \{F: X \to R\}$.

The deterministic program resulting from this construction is

$$\max_{x \in X} \min_i u_{C_i}(x). \qquad \text{(P1')}$$

(P1') is nothing but the deterministic counterpart of a flexible program obtained via the Bellman and Zadeh concept of decision making in a fuzzy environment (Negoita [55], Tanaka et al. [92], Zimmermann [104]).

The fuzzy decision is the fuzzy set D characterized by the membership function

$$u_D(x) = \min_i u_{C_i}(x).$$

$(P1')$ reflects the pure rationality principle (the Homo economicus attitude) consisting in optimizing a utility function (here the compatibility function $u_D(x)$) over the set of feasible actions. Slightly different versions of (P1′) may be obtained by considering other triangular norms or compensatory operators (to avoid the ultra conservative character of triangular norms) in the definition of the compatibility function.

Naturally the question arises of how to find an optimal solution of (P1′). The following results offer a basis for putting (P1′) in a form which is convenient for computational purposes.

Proposition 2.1.

$$\bigvee_{x\in X} u_D(x) = \bigvee_{\substack{\alpha_1,\ldots,\alpha_m\\ \alpha_i\in[0,1]}} [(\alpha_1 \wedge \cdots \wedge \alpha_m) \vee u_{C_0}(x) \vee u_{C_m^{\alpha}}(x)]$$

where $\vee$ *and* $\wedge$ *denote* max *and* min *respectively and*

$$C_i^{\alpha_i} = \{x \in X \mid u_{C_i}(x) \geq \alpha_i\}.$$

Proposition 2.2. x^0 *is optimal for the program*

$$\max_{x\in X} \min_{i=0,1} (u_{C_i}(x))$$

if and only if x^0 *is optimal for*

$$\max u_{C_0}(x), \qquad u_{C_0}(x) \leq u_{C_1}(x), \quad x \in X.$$

For the general case ((P1) has m constraints, $m > 1$) the reader is referred to Tanaka et al. [92] where among other things a discussion on how Proposition 2.1 may be exploited to find a solution of the initial flexible program, is reported.

In a more practical level Zimmermann [104] has treated the linear case. Using piecewise-linear functions to represent involved soft goals and constraints, the original problem is translated into a linear program. See also Dyson [22] for a discussion on these matters and Dubois [18] for linkups between soft constraints and fuzzy thresholds.

Zimmermann's proposal offers the maximizing alternative in the Fuzzy decision set. In addition the Decider may want to capture some essential features on other alternatives in the neighbourhood of the maximizing solution. Along this line, some efforts have been devoted to elicit the whole fuzzy decision set via parametric programming techniques (Verdegay [98], Chanas [13]).

The above mentioned ideas have been adequately adapted to deal with situations where variables are required to be integer (Stoica and Fabian [85], Ignizio [36]) or Booleans (Zimmermann and Pollatchek [108]).

Let me also mention that Zimmermann's approach has been outstandingly successful in solving concrete problems: Air regulation problem (Sommer and Pollatchek [82]), Media selection (Wiedey and Zimmermann [100]), Diet and production problems (Afgoun [1], Lebbah [43]), Logistics (Ernst [23]), Maintenance scheduling (Holtz and Desonki [35], Inventory planning (Rinks [68]).

Duality relationships that allow interesting economic interpretations have also been derived (Hamacher et al. [29], Rödder and Zimmermann [69]). An indisputable advantage of the symmetrical approach for a flexible program briefly outlined in this subsection is the existence of user-friendly mathematical programming packages that may be used to solve resulting deterministic programs.

Nevertheless if one wants to be more realistic, the incorporation of non-linear membership functions into the model in a way to translate more faithfully the desiderata of the Decider should be envisaged. Early steps along this line may be found in Nakamura [54]. Furthermore much more is needed in connection with the problem of yielding the whole fuzzy decision. How to reduce the support of the fuzzy decision set by assigning appropriate bounds to the objective function, how to deal with more general membership functions are, among others, questions which merit more attention.

Let us now turn to the case where the Decider is only able to specify some subjective preferences ordering on the set of potential actions.

2.2.2. *Symmetrical approaches based on the concept of 'nondominated action'*

Consider the standard flexible program (P1) and assume that the Decider is only able to give subjective articulations of his preferences via a set of estimates and a fuzzy preference relation. Define elements of the general construction scheme (see Section 2) as follows:

$$P = X, \qquad e(x, 0, C_1, \ldots, C_m) = N^s(\bar{x}, x) \quad \forall \bar{x} \in X, \bar{x} \neq x,$$

where $N^s(\bar{x}, x)$ is a fuzzy strict preference relation induced from the original fuzzy relation (see Orlovski [60, 61] for further details). $N^s(\bar{x}, x)$ is the degree to which $\bar{x}$ is strictly preferred to x.

$$K(e(x, 0, C_1, \ldots, C_m)) = 1 - \sup_{\bar{x} \in X} N^s(\bar{x}, x) = N^{\mathrm{ND}}(x),$$

$$TK(x) = N^{\mathrm{ND}}(x).$$

The problem resulting from these definitions (see Section 2) is

$$\max N^{\mathrm{ND}}(x), \quad x \in X. \tag{P2}$$

On account of conventional operations of negation and union of fuzzy sets (Zadeh [103]), $N^{\mathrm{ND}}(x)$ is nothing but the degree to which x is non-dominated by any other action $\bar{x} \in X$.

Proposition 2.3. *If (x^0, k^0) is optimal for the program*

$$\begin{aligned} &\min k \\ &N^s(\bar{x}, x) \leq k, \quad \forall \bar{x} \in X, \bar{x} \neq x, \\ &x \in X, \end{aligned} \tag{P2'}$$

then x^0 is optimal for (P2).

(P2′) is a semi-infinite mathematical program that can be solved by the three-phase algorithm (Glashoff and Gustafson [28]) or by cutting plane method (Blankenship and Falk [6], Timsi and Kerri [94]). Under some assumptions (P2′) can be solved by existing mathematical programming packages. This is the content of the following:

Proposition 2.4. *If N^s is convex in the second argument and X is a bounded polyhedric set the extreme points of which define the set X', then*

$$\min_{x\in X}\ \max_{\bar{x}\in X} N^s(\bar{x}, x) = \min_{x\in X}\ \max_{\bar{x}\in X'} N^s(\bar{x}, x). \tag{1}$$

As (P2) is equivalent to the first member of (1), it is clear that a solution of (P2) may be obtained by solving the following mathematical program:

$$\max k, \quad (x, k) \in X'',$$

where

$$X'' = \{(x, k) \in X \times R \mid k \geq N^s(\bar{x}, x), \bar{x} \in X'\}.$$

The usefulness and interest of the outlined method for solving a flexible program would be enhanced if there would be some appropriate techniques to capture the Decider's preferences.

A still open question in connection with the above approach is that of finding an analytical representation of N^s and N^{ND} in a way to solve resulting deterministic programs.

Methods discussed in the two previous subsections have the common feature of considering constraints and goals as similar concepts; hence the name symmetrical approaches.

Let us now turn to situations where this symmetry is not justified.

2.3. *Assymmetrical approaches*

Consider the following definitions:

$$P = \{x \in X \mid u_{C_i}(x) \geq \alpha_i, i = 1, \ldots, m\}$$

where u_{C_i} $(i = 1, \ldots, m)$ are as previously membership functions of fuzzy sets representing constraints of (P1) and α_i $(i = 1, \ldots, m)$ are thresholds fixed to express the Decider's attainments for the constraints.

$$e(x, 0, C_1, \ldots, C_m) = (u_{C_0}(x), u_{C_1}(x), \ldots, u_{C_m}(x)),$$
$$K(e(x, 0, C_0, C_1, \ldots, C_m) = u_{C_0}(x) \quad \text{and} \quad TK(x) = u_{C_0}(x).$$

The resulting program is then

$$\begin{aligned} &\max u_{C_0}(x),\\ &u_{C_i}(x) \geq \alpha_i, \quad i = 1, \ldots, m,\\ &x \geq 0. \end{aligned} \tag{P3}$$

(P3) reflects an asymmetrical attitude, i.e. the fuzzy set representing the goal is considered as a utility function that ranges potential actions and the fuzzy sets representing the constraints determine the admissible set. If u_{C_i} have some desired properties, (P3) may be solved by existing mathematical programming packages.

We have an entirely different type of problem if P is considered as a fuzzy set of X denoted here D and $e(x, 0, C_1, \ldots, C_m)$, K, and T are defined as previously. The resulting problem is:

$$\max u_{C_0}(x), \quad x \in D. \tag{P3'}$$

(P3′) is mathematically meaningless because of the fuzziness surrounding the feasible set and the question emerges of how the optimal solution of this problem should be understood. One possible interpretation of a solution of (P3′) is via the concept of maximizing set (Werners [99], Zimmermann [106]), i.e. a fuzzy set which reflects the compatibility of elements in the support of the feasible set D with the fuzzy set representing the objective (C_0).

For instance a fuzzy maximizing decision M may be characterized by the following membership functions:

$$u_M(x) = \begin{cases} 0 & \text{if } u_{C_0}(x) \leq \inf\limits_{x \in S_D} u_{C_0}(x), \\ \dfrac{u_{C_0}(x) - \inf_{x \in S_D} u_{C_0}(x)}{\sup_{x \in S_D} u_{C_0}(x) - \inf_{x \in S_D} u_{C_0}(x)} & \text{if } \inf\limits_{x \in S_D} u_{C_0}(x) \leq u_{C_0}(x) \leq \sup\limits_{x \in S_D} u_{C_0}(x), \\ 1 & \text{if } \sup\limits_{x \in S_D} u_{C_0}(x) \geq u_{C_0}(x), \end{cases}$$

where S_D is the support of D, i.e. $S_D = \{x \in X \mid u_D(x) > 0\}$.

An alternative with a higher membership degree in the fuzzy set intersection of M and D may be regarded as an optimal solution of (P3′).

In an attempt to reflect the fuzzy nature of the problem (P3′), Orlovski [59] has proposed to find a fuzzy solution for (P3′).

The fuzzy set Sol1 characterized by the membership function

$$u_{\text{Sol1}}(x) = \begin{cases} \max\limits_{k \in V(k)} & k \text{ if } x \in \bigcup\limits_{k \in [0,1]} V(k), \\ 0 & \text{elsewhere}, \end{cases}$$

where

$$V(k) = \left\{ x \in X \mid u_{C_0}(x) = \max_{t \in D^k} u_{C_0}(t) \right\}$$

and

$$D^k = \{x \in X \mid u_D k(x) \geq k\}$$

may be regarded as a solution of (P3′). A justification of this may be found elsewhere (Orlovski [59], Negoita [55]).

Proposition 2.5. *If* $x \in u_{\mathrm{Sol1}}^{-1}]0,1]$ *then* $u_{\mathrm{Sol1}}(x) = u_{C_0}(x)$.

Owing to this proposition, $u_{\mathrm{Sol1}}(x)$ can simply be written as

$$u_{\mathrm{Sol1}}(x) = \begin{cases} u_{C_0}(x) & \text{if } x \in \bigcup_{k \in [0,1]} V(k), \\ 0 & \text{elsewhere.} \end{cases}$$

An alternative to Sol1 is based on the concept of optimality in the sense of Pareto. Let me denote such a solution Sol2.

$$u_{\mathrm{Sol2}}(x) = \begin{cases} u_{C_0}(x) & \text{if } x \in E, \\ 0 & \text{elsewhere,} \end{cases}$$

where E is the set of efficient solutions of the multiobjective program

$$\max(u_D(x), u_{C_0}(x)), \quad x \in X.$$

Further details on these matters may be found in [55, 59].

In practice, $u_{\mathrm{Sol1}}(x)$ and $u_{\mathrm{Sol2}}(x)$ may be obtained via parametric programming techniques (Verdegay [97]) and multiple objective programming methods (Philip [65]) respectively.

The above fuzzy solutions seem more appropriate for flexible programs where the goal is fuzzy while constraints are crisp and vice versa. Whether to choose an approach for a given flexible program can not be determined theoretically. This choice depends on the real situation and on the aims of the Decider.

2.4. *Discussion*

The main lesson of this section is that different point of views proposed in the literature for dealing with a flexible program may be encompassed in a unified framework (The general construction of Section 1).

Solutions obtained are either deterministic, satisfying or fuzzy. Approaches yielding deterministic solutions are interesting from a computational standpoint as resulting problems are generally mathematical programs about which a great deal is known. Nevertheless they may be criticized in that the approximation of a fuzzy mathematical program by a deterministic one may falsify the real problem in some direction.

In addition the pure rationality principle on which these methods are based is valid in situations where the Decider has full and exact knowledge about the decision problem, but worthless in a fuzzy context. So, much more is needed to translate more faithfully a fuzzy problem into a deterministic one.

Fuzzy solutions seem to our opinion more realistic in that they reflect the fuzzy nature of the problem and provide more insights. It would be interesting to put forward methods giving fuzzy solutions which are less prohibitively costly from a computational point of view than those discussed here.

Another interesting axe for further inquiry is that of exploiting appropriately the bounded rationality principle (Simon [79]) to propose interactive methods for yielding solutions which are feasible and optimal to some desired extent (satisfying solutions).

3. Mathematical programming problems with fuzzy parameters

Usually, coefficients of the objective and constraints of a mathematical program are supposed to be fixed characteristics of the modelled reality. This condition is not fulfilled in many practical situations, e.g. when data are demands, technological coefficients, available capacities, cost rates and so on. This section is concerned with problems arising when some or all coefficients of the problem are restricted by some fuzzy restrictions, i.e. are possibilistic variables (Zadeh [103]). Such models seem to be quite typical for practical situations when the parameters are obtained from experts.

As non-linear programming problems with fuzzy parameters have hitherto not received further scrutiny in the literature, we restrict ourselves to the linear case. Readers interested in Fuzzy non-linear programming may find some preliminary material in Dumitru and Luban [21] and Baptistella [4].

The problem to be considered in this section is that of finding a solution for the program

$$\max \hat{c}x, \qquad \hat{A}x \subseteq \hat{b}, \quad x \geqslant 0, \tag{P4}$$

where $\hat{A}$ is an $m \times n$ matrix, $\hat{b}$ an m-vector and $\hat{c}$ an n-vector, the components of which are characterized by fuzzy restrictions.

Similarly to approaches for solving flexible programming problems, the construction scheme described in Section 1 constitutes the main apparatus for deriving deterministic counterparts of (P4).

The large number of papers in this field can be explained by the diversity of assumptions made: What is fuzzy in the problem (the fuzziness enters at the goal and/or the constraint level), the shape of involved possibility distributions, the criterion used for comparing fuzzy quantities, the type of solution which is sought (deterministic, fuzzy or satisfying).

We will first restrict our considerations to the case when the coefficients of the objective function are real numbers.

3.1. *Deterministic objective function*

As inexact programming problems will appear as subproblems in analyzing and solving some particular cases of (P4), we briefly focus on these problems.

An inexact program is a problem of the type

$$\begin{aligned} &\max cx, \\ &x_1A_1 + \cdots + x_nA_n \subseteq B, \\ &x_j \geqslant 0, \quad j = 1, \ldots, n, \end{aligned} \tag{P4'}$$

where A_j $(j = 1, \ldots, n)$ and B are non-empty convex sets of R^m and '+' is to be interpreted as follows: If A and B are two sets then $A + B = \{a + b \mid a \in A$ and $b \in B\}$.

Here are the key facts concerning (P4′) (Soyster [83]):

Proposition 3.1. *The set*

$$Q = \{x \in R^n \mid x_1 A_1 + \cdots + x_n A_n \subseteq B, x_j \geq 0, j = 1, \ldots, n\}$$

is a convex set of R^n. *Furthermore, if* $B = \{y \in R^n \mid y \leq b\}$ *then*

$$Q = Q' = \{x \in R^n \mid \bar{A}x \leq b\}$$

where $\bar{A} = (\bar{a}_{ij})$ *and* $\bar{a}_{ij} = \sup_{a_j \in A_j} a_{ij}$ *with* a_{ij} *being the i-th component of* a_j.

According to the above statement, (P4′) is a convex program about which a great deal is known (Lagrangian, Kuhn and Tucker methods, ...). If B has the above form then (P4′) is equivalent to a linear program.

We are now prepared to handle the following particular case of (P4′)

3.1.1. *Mathematical program with fuzzy sets inclusive constraints*

Consider the following mathematical program:

$$\begin{aligned} &\max cx, \\ &x_1 \hat{A}_1 + \cdots + x_n \hat{A}_n \subseteq \hat{B}, \qquad \text{(P4'')} \\ &x_j \geq 0, \quad j = 1, \ldots, n, \end{aligned}$$

where $\hat{A}_j$ $(j = 1, \ldots, n)$ and $\hat{B}$ are fuzzy sets of R^m, '+' denotes the extension of addition of crisp sets to fuzzy sets and '$\subseteq$' the inclusion between fuzzy sets.

(P4″) is termed robust program (Negoita [55]) and the interpretation which goes with this program is that $\hat{B}$ is a maximum tolerance of the fuzziness of $\sum x_i \hat{A}_i$.

The following result [55] provides a direction on which one can proceed in order to find a solution of (P4″).

Proposition 3.2. *If the fuzzy set* $\hat{B}$ *is such that the image of* $u_{\hat{B}}$ *denoted here* $\operatorname{Im} u_B = \{r_1, \ldots, r_p\}$ *with* $0 = r_1 < \cdots < r_p < 1$ *then* $x = (x_1, \ldots, x_n) \geq 0$ *is feasible for* (P4″) *if and only if*

$$x_1 \hat{A}_1^{r_i} + \cdots + x_n \hat{A}_n^{r_i} \subseteq \hat{B}^{r_i}, \quad i = 1, \ldots, p. \qquad \text{(P4''')}$$

The dimension of the resulting problem reduces substantially if $\hat{A}_i$ $(i = 1, \ldots, n)$ and $\hat{B}$ are vectors, the components of which are fuzzy numbers of the L–R type (Dubois and Prade [19, 20] i.e. there are well shaped functions L, R and real numbers m_{ij}, m_i, α_{ij}, α_i, β_{ij}, β_i such that

$$u_{\hat{a}_{ij}}(x) = \begin{cases} L\left(\dfrac{m_{ij} - x}{\alpha_{ij}}\right) & \text{if } x \leq m_{ij},\ \alpha_{ij} > 0, \\[2ex] R\left(\dfrac{x - m_{ij}}{\beta_{ij}}\right) & \text{if } x \geq m_{ij},\ \beta_{ij} > 0, \end{cases}$$

$$u_{\hat{B}}(x) = \begin{cases} L\left(\dfrac{m_i - x}{\alpha_i}\right) & \text{if } x \leq m_i,\ \alpha_i > 0, \\[2ex] R\left(\dfrac{x - m_i}{\beta_i}\right) & \text{if } x \geq m_i,\ \beta_i > 0. \end{cases}$$

Symbolically $\hat{a}_{ij}$ and $\hat{b}_i$ are denoted $(m_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ and $(m_i, \alpha_i, \beta_i)_{LR}$ respectively.

Proposition 3.3. *If components $\hat{a}_{ij}$ of $\hat{A}_i$ and $\hat{b}_i$ of $\hat{B}$ are fuzzy numbers of the* L–R *type denoted as previously then $x = (x_1, \ldots, x_n)$ is feasible for* (P4″) *if and only if x is feasible for the system*

$$\sum_j m_{ij}x_j = m_i, \quad i = 1, \ldots, m,$$
$$\sum_j \alpha_{ij}x_j \leq \alpha_i, \quad i = 1, \ldots, m,$$
$$\sum_j \beta_{ij}x_j \leq \beta_i, \quad i = 1, \ldots, m.$$

The finiteness of Im $u_{\hat{B}}$ as well as the fact that components of $\hat{A}_i$ and $\hat{B}$ must be fuzzy numbers of the same type are too restrictive to be realistic. The following result based on the decomposition theorem [55] may be helpful in solving (P4″) without imposing the above mentioned restrictions.

Proposition 3.4. *$x = (x_1, \ldots, x_n) \in X$ is feasible for* (P4″) *if and only if*

$$x_1\hat{A}_1^r + \cdots + x_n\hat{A}_n^r \subseteq \hat{B}^r \quad \forall r \in]0, 1] \tag{2}$$

where $\hat{A}_j^r$ and $\hat{B}^r$ are the r-cut of $\hat{A}_j$ and $\hat{B}$ respectively.

By virtue of Proposition 3.4, constraints of the fuzzy program (P4″) may be replaced by (2) and $x_j \geq 0$ $(j = 1, \ldots, n)$.

The resulting problem is a semi-infinite program which may be solved by available methods (discretization, three-phase algorithm or cutting plane methods; see Glashoff and Gustafson [28], Timsi and Kerri [94]). Assuming that $\hat{a}_{ij}$ and $\hat{b}_i$ are characterized by convex possibility distributions simplifies matters considerably. As a matter of fact in this case $\hat{a}_{ij}^r$ and $\hat{b}_i^r$ are real closed intervals; denote them by $[\hat{a}_{ij}^{\mathrm{L}}, \hat{a}_{ij}^{\mathrm{U}}]$ and $[\hat{b}_i^{\mathrm{L}}, \hat{b}_i^{\mathrm{U}}]$ respectively.

Owing to interval arithmetic, the equivalent deterministic program of (P4″) is the linear program:

$$\max cx,$$
$$\sum_j \hat{a}_{ij}^{\mathrm{L}}x_j \geq \hat{b}_i^{\mathrm{L}}, \quad i = 1, \ldots, m,$$
$$\sum_j \hat{a}_{ij}^{\mathrm{U}}x_j \leq \hat{b}_i^{\mathrm{U}}, \quad i = 1, \ldots, m,$$
$$x_j \geq 0, \quad j = 1, \ldots, n.$$

We turn now to in(equalities)-constrained problems which seem to be quite typical for practical situations especially when parameters are obtained from experts.

3.1.2. *Incorporating fuzzy components in a mathematical programming framework*

Consider first the following mathematical program:

$$\max cx,$$
$$A_i x \, T \, \hat{b}_i, \quad i = 1, \ldots, m, \tag{P5}$$
$$x \geq 0.$$

In some particular cases, (P5) may be conveyed into a flexible program and solved by methods described in Section 2. Assume that T represents equality and $\hat{b}_i = (b_i, b_i^-, b_i^+)$, i.e. it is characterized by the possibility distribution $u_{\hat{b}_i}$ the support of which is $[b_i^-, b_i^+]$; furthermore $u_{\hat{b}_i}$ increases on $[b_i^-, b_i]$, $u_{\hat{b}_i}(b_i) = 1$ and it decreases on $[b_i, b_i^+]$. Then (P5) may be interpreted as the flexible program:

$$\max cx,$$

$$A_i(x) \cong b_i, \quad i = 1, \ldots, m,$$

$$x \geqslant 0,$$

where $\cong$ is interpreted as follows: The degree $u_i(x)$ to which x satisfies the fuzzy constraint $A_i x = b_i$ is $u_{\hat{b}_i}(A_i x)$.

The above described approach is also valid when $(T, \hat{b}_i)$ is $(\leqslant, (b_i, -\infty, b_i^+))$ or $(\geqslant, (b_i, b_i^-, +\infty))$. For further details on these matters see ÓhÉigeartaigh [58].

An alternative which lends itself better for solving (P5) in the case when T and $\hat{b}_i$ are arbitrary, consists in restricting on alternatives which are possibly and/or necessarily feasible to some desired extent. Keeping this in mind, the feasible set and the objective function of the resulting deterministic problem may be defined as follows:

$$P = \{x \in X \mid \mathrm{Poss}(A_i x \, T \, \hat{b}_i) \geqslant \alpha_i, \; i = 1, \ldots, m\}$$

where α_i denotes an appropriate fixed threshold,

$$F(x) = cx,$$

or

$$P = \{(x, k) \in X \times I \mid \mathrm{Poss}(cx \geqslant \hat{c}^0) \geqslant k, \; \mathrm{Poss}(A_i x \, T \, \hat{b}_i) \geqslant k; \; i = 1, \ldots, m\},$$
$$F(x, k) = k,$$

$\hat{c}^0$ being a target level fixed by the Decider.

A need to more realism has led some researchers to cope with the problem of finding a fuzzy solution of (P5). A fuller exposition of these ideas for the case when $\hat{b}$ are characterized by trapezoidal possibility distributions may be found in Tanaka et al. [89–91]. It would be interesting to extend these ideas to more general possibility distributions.

Let us now consider the case where fuzziness enters at the technological matrix level. It is clear that resulting problems will heavily depend on approaches chosen for comparing fuzzy quantities. Here is among other things a comparison pattern.

Definition. Consider two fuzzy quantities $\hat{a}$ and $\hat{b}$. Then $\hat{a} \leqslant \hat{b}$ if and only if $\sup \hat{a}^\alpha \leqslant \sup \hat{b}^\alpha$ and $\inf \hat{a}^\alpha \leqslant \inf \hat{b}^\alpha \; \forall \alpha \in [0, 1]$.

The following result due to Ramik and Rimanek [67] is helpful for obtaining equivalent deterministic programs of a possibilistic program when coefficients of the technological matrix as well as those of the second member are characterized by nicely behaving functions under the above comparison criterion.

Proposition 3.5. *Assume that $\hat{a}_{ij}$ $(j = 1, \ldots, n)$ and $\hat{b}_i$ $(i = 1, \ldots, m)$ are fuzzy intervals of the type* LR, *i.e. generalizations of fuzzy numbers of the type* LR *where the mean values are intervals instead of real numbers (see Section* 3.1.1), *symbolically denoted*

$$(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{\mathrm{L}_i\mathrm{R}_i}, \quad (p_i, q_i, \gamma_i, \delta_i)_{\mathrm{L}_i\mathrm{R}_i}.$$

Then $x \in X$ is feasible for

$$\begin{aligned} &\max cx, \\ &\hat{A}_i x \leqslant \hat{b}_i, \quad i = 1, \ldots, m, \\ &x \geqslant 0, \end{aligned} \tag{P6}$$

if and only if x is a solution of the following system:

$$\begin{aligned} &-e_{\mathrm{L}_i}\left(\sum \alpha_{ij}x_j - \gamma_i\right) \leqslant p_i - \sum m_{ij}x_j, \\ &-d_{\mathrm{L}_i}\left(\sum \alpha_{ij}x_j - \gamma_i\right) \leqslant p_i - \sum m_{ij}x_j, \\ &e_{\mathrm{R}_i}\left(\sum \beta_{ij}x_j - \delta_i\right) \leqslant q_i - \sum n_{ij}x_j, \\ &d_{\mathrm{R}_i}\left(\sum \beta_{ij}x_j - \delta_i\right) \leqslant q_i - \sum n_{ij}x_j, \\ &i = 1, \ldots, m, \end{aligned}$$

where

$$e_{H_i} = \sup\{u \mid H_i(u) = H_i(0) = 1\} \quad \text{and} \quad d_{H_i} = \left\{u \mid H_i(u) = \lim_{s \to \infty} H(s)\right\}.$$

Additional insights may be gained by considering other comparison criteria (Dubois and Prade [20], Freeling [27].

Although the above described approach is attractive from a computational point of view, it turns out that the requirement that $\hat{a}_{ij}$ and $\hat{b}_i$ must be fuzzy numbers of the same type severely limits the applicability of the method.

Much energy has been devoted recently to allow the incorporation of fuzzy parameters characterized by not too restrictive possibility distributions in the scope of a linear programming model. A device which suggests itself is the replacement of involved possibility variables by some judiciously chosen fixed values (Luhandjula [49]). Nevertheless such an approach may be criticized in that only some particular values on the support of involved possibility distributions are taken into account. Some remedies have been proposed to cure this weakness.

Definition. $x \in X$ is α-possibly feasible (α-necessarily feasible) for (P6) if $\mathrm{Poss}(\hat{A}_i x \leqslant \hat{b}_i) \geqslant \alpha$ $(\mathrm{Nec}(\hat{A}_i x \leqslant \hat{b}_i) \geqslant \alpha)$, $i = 1, \ldots, m$, where Poss and Nec denote possibility and necessity respectively.

For some desired α, such alternatives are interesting in that they maintain the possibility (necessity) of achievement of each constraint to the desired level.

Let us now give some characterizations of α-possibly feasible alternatives.

Proposition 3.6. *$x \in X$ is α-possibly feasible for*

$$\max_{x \in X} \{cx, \hat{A}_i x = \hat{b}_i, i = 1, \ldots, m\}.$$

if and only if

$$u^{-1}_{[\hat{b}_i,\infty)}(\alpha) \leq u^{-1}_{(-\infty,\hat{A}_i x]}(\alpha) \quad \textit{and} \quad u^{-1}_{[\hat{A}_i x,\infty)}(\alpha) \leq u^{-1}_{(-\infty,\hat{b}_i)}(\alpha), \quad i = 1, \ldots, m,$$

where $u_{[\hat{m},\infty)}$ ($u_{(-\infty,\hat{m}]}$) is the membership function of the fuzzy set of numbers which are possibly greater than or equal to (less than or equal to) $\hat{m}$ (Dubois [18]).

Proposition 3.7. *$x \in X$ is α-possibly feasible for* (P6) *if and only if there are $V^i \in \hat{A}_i^\alpha$ and $s_i \in \hat{b}_i^\alpha$ such that $V^i x \leq s_i$, $i = 1, \ldots, m$, ($\hat{A}_i^\alpha$ and $\hat{b}_i^\alpha$ denoting α-cut of $\hat{A}_i$ and $\hat{b}_i$ respectively).*

Corollary. *A solution of the system $A_i^+ x \leq b_i^-$ ($A_i^- x \geq b_i^-$), $i = 1, \ldots, m$, where $A_i^+ = (a_{i1}^+, \ldots, a_{in}^+)$ and $A_i^- = (a_{i1}^-, \ldots, a_{in}^-)$ with $m^+ = \max_{t \in \hat{m}^\alpha} t$, $m^- = \min_{t \in \hat{m}^\alpha} t$, is α-possibly feasible for*

$$\max\{cx \mid \hat{A}_i x \leq \hat{b}_i, i = 1, \ldots, m\}$$
$$(\max\{cx \mid \hat{A}_i x \geq \hat{b}_i, i = 1, \ldots, m\}).$$

Characterizations of α-necessarily feasible actions may be found in Dubois [18] and Luhandjula [52].

In order to integrate simultaneously optimistic and pessimistic features, one may define the feasible set of the resulting deterministic program as the intersection of α-possibly feasible and β-necessarily feasible ones for some desired target levels α and β (Bukley [9]).

A further possibility is to define the feasible set on the basis of the joint possibility distribution (Orlovski [63]), i.e. to put

$$P = \{x \in X \mid \text{Poss}(\hat{A}_1 x_1 \leq \hat{b}_1, \ldots, \hat{A}_m x_m \leq \hat{b}_m) \geq \alpha\}.$$

The resulting problem in such an approach is unfortunately complex to analyze and hard to solve.

An entirely different approach consists in translating the original fuzzy problem into a semi-infinite program. Consider (P6) and assume that the corresponding technological matrix $\hat{A}$ and the second vector $\hat{b}$ are $m \times n$ and m-ary possibility distributions characterized by $\pi_{\hat{A}}$ and $\pi_{\hat{b}}$ respectively.

For convenience, let us introduce two parameter sets T^1 and T^2 which are in univocal correspondence with $\operatorname{supp} \hat{A}$ and $\operatorname{supp} \hat{b}$ respectively (supp denotes support).

Let $A(t_1)$ and $b(t_2)$ the corresponding bijections. Put $T = T^1 \times T^2$. For $t = (t_1, t_2) \in T^1 \times T^2$ put $u(t) = \min(\pi_{\hat{A}}(A(t_1)), \pi_{\hat{b}}(b(t_2)))$. This is nothing but the degree of compatibility of $A(t_1)$ and $b(t_2)$ with restrictions defined by $\pi_{\hat{A}}$ and $\pi_{\hat{b}}$. For simplicity $A(t_1)$ and $b(t_2)$ will be merely denoted $A(t)$ and $b(t)$ respectively.

Consider T_{α_i} $(i = 1, \ldots, p+1)$ where $T_{\alpha_i} = \{t \in T \mid u(t) \geqslant \alpha_i\}$ and α_i are real numbers such that $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_{p+1} \leqslant 1$. Let now $0 = \delta_{p+1} < \cdots < \delta_1$ be chosen to penalize $t \in T$ such that $u(t)$ is at a low level. Consider the following semi-infinite program:

$$
\begin{aligned}
&\max cx,\\
&A(t)x \leqslant b(t), \quad t \in T_{\alpha_{p+1}},\\
&0 \leqslant A(t)x - b(t) \leqslant \delta_p, \quad t \in T_{\alpha_p},\\
&\delta_p \leqslant A(t)x - b(t) \leqslant \delta_{p-1}, \quad t \in T_{\alpha_{p-1}},\\
&\quad\vdots\\
&\delta_2 \leqslant A(t)x - b(t) \leqslant \delta, \quad t \in T_{\alpha_1},\\
&x \geqslant 0.
\end{aligned}
\tag{P6'}
$$

A solution of (P6′) may be regarded as a satisfying solution of the fuzzy program (P6). As a matter of fact such a solution optimizes the objective and meets the realistic requirement of being feasible for most favourable circumstances ($t \in T_{\alpha_{p+1}}$), strongly violating the feasibility restriction for less favourable circumstances ($t \in T_{\alpha_1}$) and weakly violating the feasibility for intermediary situations.

For further details on these matters the reader is referred to Luhandjula [52].

Our next concern will be the case where coefficients of the objective function are fuzzy.

3.2. *Fuzzy objective function*

In order to draw the reader's attention to the problem of incorporating fuzzy components into the objective function of a linear program, we assume that constraints are deterministic. It is clear that ideas developed in 3.1 may be appropriately combined with those to be discussed here to get a satisfying solution for the general problem (P4).

For easy reference we state the problem at hand as follows:

$$\max \hat{c}x, \quad x \in Y = \{x \in R^n \mid Ax \leqslant b, x \geqslant 0\}. \tag{P7}$$

This fuzzy program may be converted in a deterministic one by exploring the link-up between imprecision (here fuzziness) and infinity.

Consider the following semi-infinite program:

$$
\begin{aligned}
&\max \alpha,\\
&sx \geqslant \alpha, \quad s \in S^1,\\
&\alpha - \delta \leqslant sx \leqslant \alpha, \quad s \in S^2,\\
&\alpha - 2\delta \leqslant sx \leqslant \alpha - \delta, \quad s \in S^3,\\
&\quad\vdots\\
&\alpha - (p-1)^{\delta} \leqslant sx \leqslant \alpha - (p-2), \quad s \in S^p,\\
&x \in Y,
\end{aligned}
\tag{P7'}
$$

where $S^i = S_1^i \times \cdots \times S_n^i$ with

$$S_j^1 = \hat{c}_j^{\beta_1}, \quad S_j^2 = \hat{c}_j^{\beta_2} - S_j^1, \quad \ldots, \quad S_j = \hat{c}_j^{\beta_p} - \bigcup_{k=1}^{p-1} S_j^k$$

(β_i are real numbers such that $0 < \beta_p < \cdots < \beta_1 < 1$ and $\hat{c}_j^{\beta_i}$ denotes the β_i-cut of $\hat{c}_j$).

A solution of (P7′) which may be obtained by semi-infinite programming techniques (Glashoff and Gustafson [28], Timsi and Kerri [94]) may be regarded as a satisfying solution for (P7). Such a solution realizes better values of sx for s in the support of $\hat{c}$ with high degree of compatibility with the restriction defined by $\hat{c}$, worse values for s with less degree of compatibility and average values for intermediary situations.

Furthermore we can state:

Proposition 3.8. *If* (α^0, x^0) *is an optimal solution of the semi-infinite program* (P7′) *then*

$$\operatorname{Poss}(\hat{c}x^0 \geqslant \alpha^0) = \max_{x \in Y} \operatorname{Poss}(\hat{c}x \geqslant \alpha^0)$$

where Poss *denotes possibility.*

Another approach in a similar vein has been put forward by Rommelfanger [70] who converts the original fuzzy program (P7) into a multiple objective deterministic program by substituting $\hat{c}_j$ by a finite number of level cuts.

The above mentioned approaches necessitate the elicitation of particular elements on the support of possibilistic variables $\hat{c}_j$. One can equally well proceed directly. To this end we generalize the concept of optimality in the following manner:

Definition. $x^0 \in Y$ is β-possibly optimal for (P7) if there is no $x \in Y$ such that $\operatorname{Poss}(\hat{c}x > \hat{c}x^0) \geqslant \beta$.

The reader can easily verify that this concept coincides with that of optimality in the deterministic case. Thus it is consistent with our identification of a real number with a degenerate possibility distribution of R.

A β-possibly optimal solution ($\beta \approx 1$) is interesting for (P7) in that it is not dominated to a great extent.

We now turn to the characterizations of β-possibly optimal alternatives (Luhandjula [51]).

Proposition 3.9. $x^0 \in Y$ *is β-possibly optimal for* (P7) *if and only if* x^0 *is optimal for the following infinite family of linear programs*:

$$\begin{aligned} &\max cx, \\ &x \in Y, \quad c \in S_\beta = \{c \in R^n \mid c_j \in \hat{c}_j^\beta\}. \end{aligned} \tag{P7″}$$

For practical purposes the following result can be used.

Proposition 3.10. (P7″) *is equivalent to*

$$\max cx,$$

$$x \in Y, \quad c \in S'_\beta = \left\{ c \in S_\beta \mid c_j \in \left\{ \max_{t \in \hat{c}_j^\beta} t, \min_{t \in \hat{c}_j^\beta} t \right\} \right\}.$$

Corollary. *$x^0 \in X$ is β-possibly optimal for* (P7) *if and only if $S'_\beta \subseteq T^+(Y, x^0)$ where $T^+(Y, x^0)$ is the polar to the cone of tangents of Y at x^0.*

3.3. *Discussion*

The methodological style described in Section 1 focuses flexible and mathematical programming with fuzzy parameters in a unified framework.

Equivalent deterministic problems of linear programs with fuzzy parameters are generally linear mathematical programs for which there exist a lot of user friendly packages. This feature makes the fuzzy approach computationally attractive and strongly departs from probabilistic ones.

Let me at this point make some first steps toward a comparative assessment of fuzzy and stochastic programming. The approach consisting in replacing fuzzy quantities by their more possible values looks like the stochastic programming procedure where random variables are replaced by their expectations or fairly good estimates of them.

The method based on the concept of possibilistic dominance (α-possibly feasibility) is the fuzzy counterpart of the well known chance constrained programming (Kall [39], Vaja [96]).

Furthermore the semi-infinite programming approach conveys the original fuzzy problem into a control configuration framework (T being the state space and steps being described by $t \in T_{\alpha_{p+1}}, \ldots, t \in T_{\alpha_1}$). Therefore this approach may be regarded as an analog to multistage stochastic programming [39].

In the stochastic programming literature some severe criticisms have been raised against the procedure consisting in replacing random variables by some estimates of them and against chance constrained programming techniques (Kall [39], Hogan et al. [34]). It is worth noting that these objections may be transferred to their fuzzy counterparts. Nevertheless these criticisms are not valid for the multistage programming approach which lends itself better for situations involving imprecision and/or uncertainty. So it is our opinion that the semi-infinite approach is a promising direction for mathematical programming with fuzzy parameters.

Throughout this section, emphasis has been placed upon possibilistic dominance for converting the fuzzy program into a deterministic one. It would be also interesting to consider dominance criteria based on necessity measures and to combine the two type of dominance criteria in order to yield more credible solutions.

Such theoretical questions as what is the dual of a mathematical program with fuzzy parameters and what is its associated possibility distribution (the distribution problem) are still to be fully answered in a way to get more insights. In connection with the latter problem, early results may be found in Buckley [8].

In contrast to flexible programming, applications of mathematical programming with fuzzy parameters are very scarce (ÓhÉigeartaigh [58]: Transportation problem, Tanaka et al. [91]: Value of information problem). Efforts must be invested along this line in order to test advantages and drawbacks of existing methods.

4. Extensions

Ideas developed in previous sections have also been extended in three important directions. Attempts have been made to incorporate fuzziness and randomness simultaneously in an optimization framework, to come to grips with complexity inherent to the presence of several deterministic or fuzzy objectives and to handle multistage processes (dynamic programming). For lack of space we will in this section briefly survey the first two of the above mentioned topics. Readers interested in issues regarding multistage optimization in a fuzzy environment are referred to Kacprzyk [37, 38].

4.1. *Fuzzy stochastic programming*

Combining fuzziness and randomness in the scope of a mathematical program is an important issue which has not received the attention it merits.

Consider the mathematical program

$$\begin{aligned} &\min c(w)x, \\ &A_i(w)x \lesssim b_i(w), \quad i = 1, \dots, m, \\ &x \in X = \{x \in R^n \mid x \geqslant 0\}, \end{aligned} \tag{P8}$$

where $b_i(w)$ are random variables and $A_i(w)$, $c(w)$ random vectors on (Ω, F, P).

After having converted (as in flexible programming) the objective function into the form $c(w)x \lesssim c^0$ (c^0 being some desired target level), this constraint as well as those of (P8) are represented by probabilistic sets in $X \times \Omega$ (Hirota [33]) where the membership functions are $u_0(x, w), \dots, u_m(x, w)$ respectively.

By virtue of the Bellman and Zadeh confluence principle [5], the decision is defined as the probabilistic set D defined by

$$u_D(x, w) = \min_i u_i(x, w).$$

The resulting problem is then

$$\max_{x \in X} u_D(x, w). \tag{P8'}$$

For x fixed, $u_D(x, w)$ is nothing but a random variable on (Ω, F, P) denoted in the sequel by $u_D(x)$. Consequently, criteria used in stochastic programming may be used to translate (P8′) into a deterministic program, i.e. the corresponding deterministic program for (P8′) may be either

$$\max_{x\in X} E(u_D(x)) \quad \text{or} \quad \min_{x\in X} \frac{E(u_D(x))}{V(u_D(x))}$$

or the bicriterion program:

$$\left(\max_{x\in X} E(u_D(x)), \min_{x\in X} V(u_D(x))\right),$$

where E and V denote the expectation and the variance respectively. An optimal alternative for (P8′) may also be obtained via concepts of stochastic dominance (Pearman and Kmietowitcz [64]).

Immediately the question arises of the analytical representation of the distribution of $u_D(x)$ in symbol $F_{u_D(x)}$. The following result gives an answer to this question.

Proposition 4.1.

$$F_{\min_i(u_i(x))}(z) = \sum_{i=0} F_{u_i(x)}(z) - \sum_{j,k,m} F_{u_j(x),u_k(x)}(z, z) + \cdots + (-1)^{m+2} F_{u_0(x),\ldots,u_m(x)}(z, \ldots, z)$$

where $F_{u_i(x)}$ is the distribution function of the random variable $u_i(x, w)$ and $F_{u_0(x),\ldots,u_m(x)}$ is the joint distribution of $(u_0(x, w), \ldots, u_m(x, w))$.

Further,

$$F_{\gamma \min_i u_i(x)+(1-\gamma)\min(1,\sum_i u_i(x))}(t) = L^{-1}(g_1(s)\cdot g_2(s))$$

where

$$g_1(s) = \int_0^{\infty} e^{-(s/\gamma)t} F_{\min_i u_i(x)}(t)\, dt,$$

$$g_2(s) = \int_0^{+\infty} e^{-(s/1-\gamma)} F_{\min\{1,\sum_i u_i(x)\}}(t)\, dt$$

and L^{-1} is the inverse of the Laplace transform.

Corollary. *If $u_i(x, w)$ $(i = 0, \ldots, m)$ are independent, then*

$$F_{\min_i u_i(x)}(z) = 1 - \prod_i (1 - F_{u_i(x)}(z)).$$

An asymmetrical approach similar to that discussed in Subsection 2.3 and based on the concept of probability of a fuzzy event is described in Luhandjula [47].

A lot of work remains to be done in fuzzy stochastic programming. It would be interesting, for instance, to envisage the incorporation of fuzzy random variables in a mathematical program. Readers interested in directing efforts to these matters may find prerequisite material in Hirota [33], Czogała [16], Luhandjula [47], Yazenin [102] and Kwakernaak [42].

4.2. *Multiple objective programming problems*

Recall that a multiple objective programming program is a problem of the type

$$\begin{aligned}&\max(f_1(x), \ldots, f_k(x)),\\ &x \in Z = \{x \in R^n \mid g_i(x) \leq b_i, i = 1, \ldots, m\},\end{aligned} \tag{P9}$$

where f_i $(i = 1, \ldots, k)$ and g_j $(j = 1, \ldots, m)$ are continuous real functions of n variables.

Fuzzy sets theory lends itself better for dealing with (P9). While papers coping with the non-linear case are seldom seen (Sakawa et al. [75, 76], Dumitru and Luban [21], Baptistella [3]) there is plethoric literature devoted to fuzzy approaches for multiple objective linear programming problems [105, 10, 11, . . .].

Consider the mathematical program

$$\begin{aligned}&\max(c^1x, \ldots, c^kx),\\ &x \in Y = \{x \in R^n \mid Ax \leq b, x \geq 0\}.\end{aligned} \tag{P9$'$}$$

The basic idea behind fuzzy methodologies for (P9′) is to consider this problem as a flexible program of the form [105]

$$c^ix \lessapprox L_i, \quad i = 1, \ldots, k, \quad x \in Y,$$

where L_i are some prescribed thresholds (for instance $L_i = \max_{x \in Y} c^ix$).

Representing the above fuzzy constraints by fuzzy sets of X characterized by piecewise linear membership function $u_i(x)$ and considering the min operator as appropriate for intersection of fuzzy sets, the resulting program is

$$\max_{x \in Y} \min_i u_i(x),$$

which is equivalent to a linear program as stipulated by the following:

Proposition 4.2. *$x^0 \in X$ is optimal for* (P9″) *if and only if* (x^0, λ^0) *where* $\lambda^0 = \min_i u_i(x^0)$ *is optimal for the linear program*

$$\begin{aligned}&\max \lambda,\\ &\lambda \leq u_i(x), \quad i = 1, \ldots, k,\\ &x \in Y.\end{aligned}$$

A need for more realism has led Leberling [44] and Luhandjula [46] to consider non-linear membership functions and compensatory operators. A fascinating fact is that the resulting problems remain linear programs. In a multiple objective program context, a non-efficient solution is less attractive since it is dominated by other alternatives. So it is interesting to get a solution which is efficient (Pareto optimal). Zimmermann's proposal and its variants Feng [25], Leberling [44], Luhandjula [46] offer a weakly efficient action which is efficient when the unicity is guaranteed.

A discussion on conditions for the existence of Pareto optimal alternatives via the Hausdorff maximality principle may be found in Buckley [7] where an approach for generating the set of efficient alternatives is also proposed. This set may be very large and filtering techniques are needed.

Let me also mention that the efficiency of an alternative may be tested via Kuhn and Tucher conditions for efficiency (Mekaouche and Rezzik [53]). Among other attempts of underlying principles of fuzzy mathematical programming with multiple objective problems are the methodological extension of goal programming (Hannan [31], Rudin and Narasimhan [74]), of cone of dominance (Takeda and Nishida [93]) and of multiple objective linear and non-linear fractional programs (Sakawa and Yumine [75], Luhandjula [48]).

In an attempt to reduce the complexity inherent to the presence of several objectives, a fuzzy solution which is richer from an informational point of view may be desirable. Such a solution may be obtained via techniques of parametric programming (Chanas [14]).

Advantages of using a fuzzy approach for finding a solution of a multiple objective programming problem are flexibility, easiness to be adapted for interactive use and the fact that such a methodology meets the main demands for operational models: simplicity, robustness adaptivity (Little [45]).

The ideas outlined above have also been extended to the case when relevant data are fuzzy parameters. Slowinski [80] has considered the case where components of the objective function are fuzzy numbers of the same type and obtained via possibility grade of dominance a deterministic equivalent problem which is a multicriteria linear fractional program. See also Roubens and Thegem [72] for a similar approach with flat fuzzy numbers and for a comparison with multiple objective stochastic programming problems.

Luhandjula [50] generalizes the concept of efficiency within the Simonian philosophy of satisfying alternatives, establishes necessary and sufficient conditions for a satisfying solution (β-possibly efficient action) and proposes some ways for singling out such an alternative.

5. Conclusion

The young and multifacet field of fuzzy mathematical programming provides some ways for taking into account the effects of inaccuraties in information flow, making decision under uncertainty or simulating the stochastic nature of some elements in a mathematical model. From time to time we must look back on what could be accomplished and could not, what we have learned and what remains to be done. This paper has been written in this spirit.

We have highlighted the unifying principle governing fuzzy optimization methods – eliciting a set of feasible actions and a criterion which induces a preferential scheme on this set in order to get a deterministic, satisfying or fuzzy solution.

It is a shame that only a small fraction of the works cited in this overview can actually be used in a routine manner. Reasons for this as well as ways for improvement are briefly discussed in the sequel.

Existing works rely mainly upon the following postulate: preferences of the Decider are representable by fuzzy sets which share nice properties in order to get tractable deterministic problems. The postulate allows to tackle the problem in an operational way but such questions as how to obtain these fuzzy sets (membership function elicitation) and to what extent these fuzzy sets match the Decider's preferences are to be answered. Hence before using a fuzzy mathematical programming method, one must help the Decider to define membership functions, thresholds and this can be a huge task.

Very few researches have studied these model building aspects in detail. It is our opinion that fuzzy expert systems methodologies (Baldwin [2]) may be helpful for this problem. Furthermore, fuzzy sets representing the Decider's preferences generally do not have nice properties required to apply the methods discussed here (linearity, covexity, LR form, etc.). Extensions to situations implying less restrictive assumptions on relevant sets are needed. A deep exploration of link-ups between fuzzy and semi-infinite optimization (Hettich [32], Glashoff and Gustafson [28]) may give insightful features in connection with this problem.

In order to solve concrete problems most effectively, progress must be made in both quality and availability of fuzzy optimization software. The work by Mekaouche and Rezzik [53] is along this line but much more is needed.

Extremely useful would be the increase of high quality case studies in order to demonstrate the usefulness of fuzzy optimization approaches and to meet their advantages and drawbacks.

A better understanding of theoretical issues: duality, distribution problems, . . . , may also contribute to most advancing the state of the fields.

Let us hope that successful developments in the above mentioned directions will proceed in the near future, thus bridging the gap between the language used for fuzzy optimization techniques and the language used by potential users of these techniques.

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