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ScienceDirect
Fuzzy Sets and Systems 160 (2009) 3263–3289
FUZZY sets and systems
www.elsevier.com/locate/fss

Application of fuzzy minimum cost flow problems to network design under uncertainty☆

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Received 21 April 2008; received in revised form 6 April 2009; accepted 13 April 2009
Available online 24 April 2009

Abstract

This paper deals with fuzzy quantities and relations in multi-objective minimum cost flow problem. When t-norms and t-conorms are available, the goal programming is applied to minimize the deviation among the multiple costs of fuzzy flows and the given targets when the fuzzy supplies and demands are satisfied. To obtain the most optimistic and the most pessimistic satisficing solutions of this problem, two polynomial time algorithms are introduced applying some network transformations. To demonstrate the performance of this approach in actual substances, network design under fuzziness is considered and an efficient scheme is proposed including genetic algorithm together with fuzzy minimum cost flow problem. This scheme is applied on a pilot network for more description.

Keywords: Fuzzy network design; Fuzzy relations; Optimistic and pessimistic attitudes

1. Introduction

Network flow models provide a rich and powerful framework to formulate and solve many engineering and management problems. A variety of applications, including the analysis and design of computer networks, cable television networks, transportation systems, communication networks, project schedules, queuing systems, inventory systems, and manpower allocation have been reported in the literature [2,28]. The minimum cost flow problem (MCFP) is a general structure in these models which provides a unified approach to many applications. This paper studies uncertainty and impreciseness, multiple objectives and optimistic and pessimistic attitudes, dealing with a traditional minimum cost flow problem and its application in network design. This problem is a reasonable generalization of fuzzy transportation problem which was introduced by many researchers, see e.g., [32,43,21,25, 7,39,27,1,26,8].

☆ The authors were partially supported by Road Maintenance and Transportation Organization, Ministry of Roads and Transportation of Iran.
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doi:10.1016/j.fss.2009.04.004

1.1. Uncertainty and impreciseness

Actual problems include uncertain data. Ramik [36] discussed on the origin of uncertainty and mentioned to the errors in measuring physical quantities and representing the data in a computer. Uncertainty can be captured applying fuzzy quantities. Keeping a fuzzy optimization problem in mind, one can apply satisficing approach in which the fuzzy target values are used in order to evaluate solutions absolutely [20,3]. Applying the possibility and the necessity measures [10], the traditional optimal solution set is extended to a pair of fuzzy sets, namely, the possibly optimal and the necessarily optimal solution sets [41,42]. Inuiguchi and Sakawa [18] considered different extensions of relations utilizing t-norms and t-conorms treating with fuzzy linear programming. Some extensions of satisficing approach have been presented in [19]. It is also demonstrated that the different combination of possibility and necessity measures yields various solutions depending on the decision maker's intention [10,20,19,36]. To see the application of this approach in cellular manufacturing systems, we refer to [37].

1.2. Multiple objectives

Multi-objective models have crucial role in the mathematical programming and applications [17,24,30]. For example, Kothari and Dhillon [23] collected some industrial applications of this viewpoint in power system optimization and Brar et al. [5] investigated fuzzy multiple objectives in this problem. An important approach dealing with such problems is the goal programming in which the deviation between each objective function and its *target value* should be minimized [16]. The linguistic statements as target values are qualified by Ramik [33] eliciting membership functions of fuzzy sets.

1.3. Optimistic and pessimistic attitudes

In the fuzzy linear programming, the necessity and the possibility measures are used to evaluate to what extent it is necessary and it is possible that a solution satisfies a constraint, respectively. Analogues, a feasible and optimal solution for at least one real scenario with membership degree not less than h is called a α-possibly optimal solution. An α-necessarily optimal solution is a feasible and optimal solution of any real scenario with membership degree more than $1-\alpha$. Julien [22] transformed the fuzzy linear programming problem with the best and the worst linear programming problems at different α-cuts and Liu [29] measured the fulfillment of the constraints when the constraints are tight or loose based on own pessimistic or optimistic attitudes.

1.4. MCFP related works

Multi-objective MCFP has been significantly studied in the literature [15]. Fuzzy multi-level MCFP with fuzzy costs was also introduced by Shih and Lee considering different certainty degrees and applying linear programming algorithms. Liu and Kao [28] solved MCFP with fuzzy costs using Yager's ranking function which maintains the network structure of the problem which permits to use network simplex method. Ghatee and Hashemi [13] presented some different cases of fuzzy MCFP utilizing a total order and nominal flows. Then, the authors in [12] investigated fully fuzzified MCFP considering a large variety of ranking functions. Duality concepts were addressed in [14], too. The application of fuzzy MCFP in internet transmission [28], petroleum industry [12] and bus network planning [13], have been also presented in recent literature.

1.5. The contribution of the present paper

This paper treats with a multi-objective fully fuzzified MCFP with fuzzy quantities and fuzzy relations. Considering fuzzy goals, the satisficing approach is pursued. Taking this scheme in mind, the solution of how many units should be transported between each pair of supplier and demander nodes, may be obtained in which the amount of supplies, demands and transportation costs are exhibited with fuzzy numbers. Then the application of this scheme in network design under fuzziness is discussed. For this aim a genetic algorithm whose utility function is evaluated by the proposed algorithms is implemented. On a real network this idea is illustrated. The rest of paper is organized as follows. Some basic definitions are given in next section. In Section 3, the fully fuzzified multi-objective MCFP is discussed.

The optimistic and pessimistic satisficing flows are introduced in Sections 4 and 5, respectively. Fuzzy network design and numerical experiments are presented in Section 6. The last section ends this paper with a brief conclusion and future directions.

2. Fuzzy sets and relations

The class of fuzzy sets on X is denoted with $\mathcal{F}(X)$. Let $\widetilde{a} \in \mathcal{F}(X)$ with membership function $\mu_{\widetilde{a}}(x)$, for each $x \in X$. The α-cut or α-level of $\widetilde{a}$ is defined as an ordinary set $[\widetilde{a}]_\alpha$ whose members satisfy $\mu_{\widetilde{a}}(x) \geq 0$, see [6]. The support and the height are given by $Supp(\widetilde{a}) = closure\{x \in X|\mu_{\widetilde{a}}(x) > 0\}$ and $Hgt(\widetilde{a}) = \{x \in X|\mu_{\widetilde{a}}(x) = 1\}$, respectively. Each element of $Supp(\widetilde{a})$ is said a realization or *a scenario*. $a \in \widetilde{a}$ means that $a \in Supp(\widetilde{a})$ or equivalently $\mu_{\widetilde{a}}(a) > 0$. When $\widetilde{A}$ is a fuzzy matrix including fuzzy sets as elements, $A \in \widetilde{A}$ implies that each element of A belongs to the corresponding element of $\widetilde{A}$. $\widetilde{a} \in \mathcal{F}(\mathbb{R})$ is a fuzzy number when $\widetilde{a}$ is a convex normalized fuzzy set whose membership function is piecewise continuous. It is well defined denoting the α-level set of a fuzzy number $\widetilde{a}$ with a closed interval $[a^L(\alpha), a^R(\alpha)]$. In what follows, we focus on triangular norms whose best results have been introduced by Dubois and Prade [9], Ramik [33,35,36], Ramik and Vlach [34] and Inuiguchi et al. [19].

Definition 2.1. Let $T : [0, 1]^2 \longrightarrow [0, 1]$ and $S : [0, 1]^2 \longrightarrow [0, 1]$ are commutative, associative, non-decreasing in every variable and satisfy the following boundary conditions:

$$T(a, 1) = a \quad \text{for all } a \in [0, 1],$$

$$S(a, 0) = a \quad \text{for all } a \in [0, 1].$$

Then T and S are called the triangular norm (t-norm) and triangular conorm (t-conorm), respectively.

Definition 2.2. Let a_i belongs to $[0, 1]$ for $i = 1, \ldots, n$. When $m \in \{1, \ldots, n\}$, one can denote $T_{1,\ldots,m}$, $T_{i\in\{1,\ldots,m\}}$ or $T(a_i|i \in \{1, \ldots, m\})$ for the following induction definition:

$$T_{1,2} = T(a_1, a_2),$$

and for $m > 2$,

$$T_{1,\ldots,m} = T(T_{1,\ldots,(m-1)}, a_m).$$

A particular fuzzy extension of relation P is often denoted by tilde, i.e., $\widetilde{P}$. When t-norm T or t-conorm S is used to extend a particular fuzzy set $\widetilde{a}$, the membership function is denoted with $\mu_{\widetilde{a}^T}$ or $\mu_{\widetilde{a}_S}$, respectively.

Definition 2.3 (*Ramik [33]*). Let X, Y be non-empty sets, T and S be a t-norm and t-conorm, respectively. Let P be a binary relation on $X \times Y$. Then, two fuzzy extended relations $\widetilde{P}^T$ and $\widetilde{P}_S$ may be defined for all fuzzy sets $\widetilde{a}, \widetilde{b}$, that $\widetilde{a} \in \mathcal{F}(X)$ and $\widetilde{b} \in \mathcal{F}(Y)$ with the following membership functions, respectively,

$$\mu_{\widetilde{P}^T}(\widetilde{a}, \widetilde{b}) = \sup\{T(\mu_P(x, y), T(\mu_{\widetilde{a}}(x), \mu_{\widetilde{b}}(y)))|x \in \widetilde{a}, y \in \widetilde{b}\},$$

$$\mu_{\widetilde{P}_S}(\widetilde{a}, \widetilde{b}) = \inf\{S(\mu_P(x, y), S(1 - \mu_{\widetilde{a}}(x), 1 - \mu_{\widetilde{b}}(y)))|x \in \widetilde{a}, y \in \widetilde{b}\},$$

where $\sup(\phi) = 0$ and $\inf(\phi) = 1$.

Commonly $Pos(\widetilde{a} \widetilde{\leqq} \widetilde{b}) = \mu_{\widetilde{\leqq}^T}(\widetilde{a}, \widetilde{b})$ and $Nes(\widetilde{a} \widetilde{\leqq} \widetilde{b}) = \mu_{\widetilde{\leqq}_S}(\widetilde{a}, \widetilde{b})$ are referred to possibility and necessity measures. Note that $Pos(\widetilde{a} \widetilde{\leqq} \widetilde{b})$ evaluates to what extent it is possible that the variable $\widetilde{a}$ is in the fuzzy set $\widetilde{b}$. On the other hand, $Nes(\widetilde{a} \widetilde{\leqq} \widetilde{b})$ evaluates to what extent it is certain that the variable $\widetilde{a}$ is in the fuzzy set $\widetilde{b}$. Also, $\widetilde{a} \widetilde{\leqq}^T \widetilde{b}$ and $\widetilde{a} \widetilde{\leqq}_S \widetilde{b}$ imply that $\widetilde{a}$ is less than $\widetilde{b}$ with possibility degree $\mu_{\widetilde{\leqq}^T}(\widetilde{a}, \widetilde{b})$ and necessity degree $\mu_{\widetilde{\leqq}_S}(\widetilde{a}, \widetilde{b})$, respectively. In fact, if one says x is a possible solution for a fuzzy programming problem with positive membership value α, it means that x is optimal for at least one realization of this fuzzy programming problem with a certainty degree greater than α. On the other hand, a necessary solution of fuzzy programming problem with positive membership value α is an optimal solution for all of the realizations whose membership degrees are greater than $1 - \alpha$, see [18].

Proposition 2.4 (*Inuiguchi et al.* [19]). *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{F}(X)$ *be normal and compact fuzzy sets, with* $T = \min$, $S = \max$ *and* $\alpha \in (0, 1)$, *we have*,

(i) $\mu_{\leqq_T}(\widetilde{a}, \widetilde{b}) \geq \alpha \;\Leftrightarrow\; \inf[\widetilde{a}]_\alpha \leq \sup[\widetilde{b}]_\alpha$,

(ii) $\mu_{\lesseqgtr_S}(\widetilde{a}, \widetilde{b}) \geq \alpha \;\Leftrightarrow\; \sup[\widetilde{a}]_{1-\alpha} \leq \inf[\widetilde{b}]_{1-\alpha}$,

(iii) $\mu_{\geqq_T}(\widetilde{a}, \widetilde{b}) \geq \alpha \;\Leftrightarrow\; \sup[\widetilde{a}]_\alpha \geq \inf[\widetilde{b}]_\alpha$,

(iv) $\mu_{\gtreqless_S}(\widetilde{a}, \widetilde{b}) \geq \alpha \;\Leftrightarrow\; \inf[\widetilde{a}]_{1-\alpha} \geq \sup[\widetilde{b}]_{1-\alpha}$.

3. Fully fuzzified multi-objective MCFP

MCFP arises naturally in engineering and economics contexts; it appears in problems involving equilibrium models such as urban transportation systems [40], resistive electrical networks [23] designing and production–distribution problems [2]. The central concept in all of these problems is to find the least transportation cost of a commodity through a capacitated network in order to satisfy demands at certain nodes using available supplies at other nodes. Let $G = (N, A)$ be a directed network where N and A denote sets of nodes and links, respectively. For each link $(i, j) \in A$, multiple costs $c^k_{i,j}$, $k = 1, \ldots, K$, an upper bound capacity $u_{i,j}$ and a lower bound capacity $l_{i,j}$ are considered. The multiple costs including e.g., the economic, the shortness, the environmental and the security indices may be simultaneously considered in engineering problems [23,5]. Assume $l_{i,j} = 0$. Multi-objective MCFP requirements can be expressed as follows when the network is bipartite:

$$\min f^k(y) = \sum_{k=1}^{K} \sum_{(i,j)\in A} c^k_{i,j} u_{i,j} y_{i,j} \tag{1}$$

s.t.

$$\begin{cases} \text{(a)} \displaystyle\sum_{\{j:(i,j)\in A\}} u_{i,j} y_{i,j} \leq s_i, \quad i \in \mathfrak{S}, \\ \text{(b)} \displaystyle\sum_{\{j:(j,i)\in A\}} u_{j,i} y_{j,i} \geq d_i, \quad i \in \mathfrak{D}, \\ \text{(c)}\ 0 \leq y_{i,j} \leq 1, \end{cases} \tag{2}$$

where $y_{i,j}$ is the rate of usage of link $(i, j) \in A$ and $x_{i,j} = y_{i,j} u_{i,j}$ is the corresponding flow through link (i, j). $\mathfrak{S}$ and $\mathfrak{D}$ are the sets of supplier and demander nodes. Eq. (2a) ensures that the output of supplier node i is at most s_i, and Eq. (2b) implies that the input to demander node $i \in \mathfrak{D}$ is at least d_i. It seems that this formulation is different from traditional MCFP formulation [2,28,38,12]; however, the following shows the equivalency.

Remark 3.1. The optimization problem (1) subject to (2) can be transformed into the following traditional multi-objective MCFP:

$$\min f^k(y) = \sum_{k=1}^{K} \sum_{(i,j)\in A} c^k_{i,j} x_{i,j} \tag{3}$$

s.t.

$$\sum_{\{j:(i,j)\in A\}} x_{i,j} - \sum_{\{j:(j,i)\in A\}} x_{j,i} = b_i, \quad \forall i \in N, \quad 0 \leq x_{i,j} \leq u_{i,j}, \quad \forall (i, j) \in A. \tag{4}$$

The converse statement is also true.

Proof. Consider a new node p which can play the role of an inventory or a dummy customer which is dependent on the greatness of $\sum_{i\in\mathfrak{S}} s_i$ and $\sum_{i\in\mathfrak{D}} d_i$. When $\sum_{i\in\mathfrak{S}} s_i \geq \sum_{i\in\mathfrak{D}} d_i$, node p is a dummy costumer otherwise p is an inventory node. Then (2a) and (2b) can be rewritten as

$$\begin{cases} (\mathrm{a}') \displaystyle\sum_{\{j:(i,j)\in A\}} x_{i,j} + x_{i,p} = s_i, & i \in \mathfrak{S}, \\ (\mathrm{b}') -\displaystyle\sum_{\{j:(j,i)\in A\}} x_{j,i} - x_{p,i} = -d_i, & i \in \mathfrak{D}. \end{cases} \tag{5}$$

For node p one can write

$$\sum_{\{j:(p,j)\in A\}} x_{p,j} - \sum_{\{j:(j,i)\in A\}} x_{j,p} = -\left(\sum_{i\in\mathfrak{S}} s_i - \sum_{i\in\mathfrak{D}} d_i\right). \tag{6}$$

Thus, connecting each node $i \in \mathfrak{S}$ to node p with link (i, p) and node p to node $i \in \mathfrak{D}$ with link (p, i) produces a network with balance flow constraints analogues to (4) in which a new $(n + 1)$-vector b is defined that for each $i \in \mathfrak{S}$ $(i \in \mathfrak{D})$, $b_i = s_i$ $(b_i = -d_i)$ and $b_i = 0$ otherwise. Also set $b_{n+1} \triangleq b_p = -(\sum_{i\in\mathfrak{S}} s_i - \sum_{i\in\mathfrak{D}} d_i)$. To ensure that such dummy links are chosen when there is no other possibility, the cost, also, the capacity of such links are assumed to be sufficiently large. As a simple exercise, it is possible to show that each feasible solution of the original problem is feasible for this model and conversely. □

When one treats with problem (3) subjected to (4), it is possible to define another equivalent bipartite network. For this aim all of the supplier nodes are included in $\mathfrak{S}$ and the demander nodes are included in $\mathfrak{D}$. Other nodes can be augmented to $\mathfrak{S}$ or $\mathfrak{D}$ arbitrary whose supplies and demands are zero. Now define

$$\mathfrak{S}' = \{\, i' \mid i \in \mathfrak{S} \cup \mathfrak{D}\},$$

$$\mathfrak{D}' = \{\, i'' \mid i \in \mathfrak{S} \cup \mathfrak{D}\},$$

$$A' = \{\, (i', i'') \mid i' \in \mathfrak{S}' \,\&\, i'' \in \mathfrak{D}'\} \cup \{(i', j'') \mid (i, j) \in A\}.$$

For each link (i', i'') the cost and upper bound capacity are defined as zero and infinity, respectively. Also for each link (i', j'') the cost and the upper bound capacity are equal to $c_{i,j}$ and $u_{i,j}$, respectively. Let Q be the sum of supplies or the sum of demands. For each node $i \in \mathfrak{S}$ set $s_{i'} = s_i + Q$ and $s_{i''} = Q$. For each node $i \in \mathfrak{D}$ set $d_{i''} = d_i + Q$ and $d_{i'} = Q$. Applying these modifications, the network depicted in Fig. 2c is a transformed network of the network of Fig. 2a. The interested reader can show the equivalency of the results of flow transmission through both of the networks.

In what follows, we investigate a more reasonable variant of this transportation problems in which the links costs, the nodes supplies and the node demands are not precisely known. Due to granular information about human, social, economic, and political interactions, the expert system may state the following fully fuzzified multi-objective MCFP capturing with uncertainty:

$$\min \widetilde{f^k}(y) = \sum_{(i,j)\in A} \widetilde{c}^k_{i,j} \widetilde{u}_{i,j} y_{i,j}, \quad \forall k = 1, \ldots, K \tag{7}$$

s.t.

$$\begin{cases} \widetilde{g}_{s,i}(y) = \displaystyle\sum_{\{j:(i,j)\in A\}} \widetilde{u}_{i,j} y_{i,j} \mathrel{\widetilde{\leqq}} \widetilde{s}_i, & i \in \mathfrak{S}, \\ \widetilde{g}_{d,i}(y) = \displaystyle\sum_{\{j:(j,i)\in A\}} \widetilde{u}_{j,i} y_{j,i} \mathrel{\widetilde{\geqq}} \widetilde{d}_i, & i \in \mathfrak{D}, \\ \displaystyle\sum_{\{j:(i,j)\in A\}} \widetilde{u}_{i,j} . y_{i,j} = \sum_{\{j:(j,i)\in A\}} \widetilde{u}_{j,i} . y_{j,i}, & i \in N \backslash (\mathfrak{S} \cup \mathfrak{D}), \\ 0 \leq y_{i,j} \leq 1, \end{cases} \tag{8}$$

where $\widetilde{c}^k_{i,j}$, $\widetilde{u}_{i,j}$, $\widetilde{s}_i$ and $\widetilde{d}_i$ are fuzzy numbers. Note that this problem is a special variant of fuzzy linear programming which was introduced e.g., by Inuiguchi et al. [19] and Ramik [31,33,36].

Analogues to the crisp case, the fuzzy flow is defined as $\widetilde{x}_{i,j} = \widetilde{u}_{i,j} y_{i,j}$. Hereafter, for notation convenience denote $\widetilde{\gamma}^k_{i,j} \triangleq \widetilde{c}^k_{i,j} \widetilde{u}_{i,j}$. A matrix $y = (y_{i,j}) \in [0,1]^{n\times n}$ such that $\sum_{\{j:(i,j)\in A\}} \widetilde{x}_{i,j} = \sum_{\{j:(i,j)\in A\}} \widetilde{u}_{i,j} y_{i,j} = \sum_{\{j:(j,i)\in A\}} \widetilde{u}_{j,i} y_{j,i} = \sum_{\{j:(j,i)\in A\}} \widetilde{x}_{j,i}$ for all $i \in N\backslash(\mathfrak{S} \cup \mathfrak{D})$ is said a *semi-feasible rate matrix*. Y_{SF} is a set including semi-feasible rate matrices.

Since, $\widetilde{f^k}(y)$ for $k = 1, \ldots, K$, $\widetilde{g}_{s,i}(y)$ for $i \in \mathfrak{S}$ and $\widetilde{g}_{d,i}(y)$ for $i \in \mathfrak{D}$ are parametric fuzzy sets with respect to crisp vector $y \in Y_{SF}$, their membership functions may be stated as below using t-norm T,

$$\mu_{\widetilde{f^k(y)}^T}(\overline{z}) = \sup \left\{ T(\mu_{\widetilde{\gamma}^k_{i,j}}(\gamma^k_{i,j}) | (i,j) \in A) \,|\, \gamma^k \in \widetilde{\gamma}^k,\ y \in Y_{SF},\ \sum_{(i,j)\in A} \gamma^k_{i,j} y_{i,j} = \overline{z} \right\}, \tag{9}$$

$$\mu_{\widetilde{g}_{s,i}(y)^T}(\overline{s}) = \sup \left\{ T(\mu_{\widetilde{u}_{i,j}}(u_{i,j}) | (i,j) \in A) \,|\, u \in \widetilde{u},\ y \in Y_{SF},\ \sum_{(i,j)\in A} u_{i,j} y_{i,j} = \overline{s} \right\}, \tag{10}$$

$$\mu_{\widetilde{g}_{d,i}(y)^T}(\overline{d}) = \sup \left\{ T(\mu_{\widetilde{u}_{j,i}}(u_{j,i}) | (j,i) \in A) \,|\, u \in \widetilde{u},\ y \in Y_{SF},\ \sum_{(j,i)\in A} u_{j,i} y_{j,i} = \overline{d} \right\}. \tag{11}$$

Similarly one can express the following membership functions applying t-conorm S:

$$\mu_{\widetilde{f^k(y)}_S}(\overline{z}) = \inf \left\{ S(1 - \mu_{\widetilde{\gamma}^k_{i,j}}(\gamma^k_{i,j}) | (i,j) \in A) \,|\, \gamma^k \in \widetilde{\gamma}^k,\ y \in Y_{SF},\ \sum_{(i,j)\in A} \gamma^k_{i,j} y_{i,j} = \overline{z} \right\}, \tag{12}$$

$$\mu_{\widetilde{g}_{s,i}(y)_S}(\overline{s}) = \inf \left\{ S(1 - \mu_{\widetilde{u}_{i,j}}(u_{i,j}) | (i,j) \in A) \,|\, u \in \widetilde{u},\ y \in Y_{SF},\ \sum_{(i,j)\in A} u_{i,j} y_{i,j} = \overline{s} \right\}, \tag{13}$$

$$\mu_{\widetilde{g}_{d,i}(y)_S}(\overline{d}) = \inf \left\{ S(1 - \mu_{\widetilde{u}_{j,i}}(u_{j,i}) | (j,i) \in A) \,|\, u \in \widetilde{u},\ y \in Y_{SF},\ \sum_{(j,i)\in A} u_{j,i} y_{j,i} = \overline{d} \right\}. \tag{14}$$

Definition 3.2. Let T be a t-norm. A fuzzy set $\widetilde{Y}^T$, whose membership function, $\mu_{\widetilde{Y}^T}(.)$, is defined for each $y \in Y_{SF}$ by

$$\mu_{\widetilde{Y}^T}(y) = T(T(\mu_{\gtrsim^T}(\widetilde{g}_{s,i}(y), \widetilde{s_i}) | i \in \mathfrak{S}), T(\mu_{\gtrsim^T}(\widetilde{g}_{d,i}(y), \widetilde{d_i}) | i \in \mathfrak{D})), \tag{15}$$

and zero otherwise, is called a T-rate matrix set. Analogues, for t-conorm S, S-rate matrix set $\widetilde{Y}_S$, is defined with

$$\mu_{\widetilde{Y}_S}(y) = S(S(1 - \mu_{\gtrsim_S}(\widetilde{g}_{s,i}(y), \widetilde{s_i}) | i \in \mathfrak{S}), S(1 - \mu_{\gtrsim_S}(\widetilde{g}_{d,i}(y), \widetilde{d_i}) | i \in \mathfrak{D})), \tag{16}$$

and zero otherwise.

Definition 3.3. A fuzzy feasible flow for the fully fuzzified MCFP (7)–(8) can be obtained considering each feasible rate matrix $y \in \widetilde{Y}^T$, or $y \in \widetilde{Y}_S$, as below:

$$\widetilde{x} = (\widetilde{x}_{i,j}) = (\widetilde{u}_{i,j} y_{i,j}).$$

When $y \in \widetilde{Y}^T$ or $y \in \widetilde{Y}_S$, $\widetilde{x}$ is called a T-feasible or S-feasible fuzzy flow.

Proposition 3.4. *Let $\widetilde{u}_{i,j}$, $\widetilde{s}_i$ and $\widetilde{d}_i$ be fuzzy numbers and denote $[\widetilde{u}_{i,j}]_\alpha = [u^L_{i,j}(\alpha), u^R_{i,j}(\alpha)]$, $[\widetilde{s}_i]_\alpha = [s^L_i(\alpha), s^R_i(\alpha)]$, and $[\widetilde{d}_i]_\alpha = [d^L_i(\alpha), d^R_i(\alpha)]$. Assume $T = \min$ and $S = \max$. For each $\alpha \in (0, 1)$, the followings are true*:

$$\begin{cases} \mu_{\widetilde{\leqq}_T}(\widetilde{g}_{s,i}(y), \widetilde{s}_i) \geq \alpha \Leftrightarrow \sum\limits_{(i,j)\in A} u^L_{i,j}(\alpha) y_{i,j} \leq s^R_i(\alpha), & i \in \mathfrak{S}, \\ \mu_{\widetilde{\geqq}_T}(\widetilde{g}_{d,i}(y), \widetilde{d}_i) \geq \alpha \Leftrightarrow \sum\limits_{(j,i)\in A} u^R_{j,i}(\alpha) y_{j,i} \geq d^L_i(\alpha), & i \in \mathfrak{D}, \end{cases} \tag{17}$$

for each $y \in \widetilde{Y}^T$, and

$$\begin{cases} \mu_{\widetilde{\leqq}_S}(\widetilde{g}_{s,i}(y), \widetilde{s}_i) \geq \alpha \Leftrightarrow \sum\limits_{(i,j)\in A} u^R_{i,j}(1-\alpha) y_{i,j} \leq s^L_i(1-\alpha), & i \in \mathfrak{S}, \\ \mu_{\widetilde{\geqq}_S}(\widetilde{g}_{d,i}(y), \widetilde{d}_i) \geq \alpha \Leftrightarrow \sum\limits_{(j,i)\in A} u^L_{j,i}(1-\alpha) y_{j,i} \geq d^R_i(1-\alpha), & i \in \mathfrak{D}, \end{cases} \tag{18}$$

for each $y \in \widetilde{Y}_S$. Moreover, constraint (17) is an optimistic interpretation of constraints (8) and (18) is a pessimistic one.

Proof. Referring to Proposition 2.4, the first claim is trivial. For the second, note that the constraint (17) and (18) can be rewritten as follows (see also [33]):

$$\begin{cases} \sum\limits_{(i,j)\in A} u^L_{i,j}(\alpha) y_{i,j} \leq s^R_i(\alpha), & i \in \mathfrak{S}, \\ \sum\limits_{(j,i)\in A} u^R_{j,i}(\alpha) y_{j,i} \geq d^L_i(\alpha), & i \in \mathfrak{D} \end{cases} \tag{19}$$

and

$$\begin{cases} \sum\limits_{(i,j)\in A} u^R_{i,j}(1-\alpha) y_{i,j} \leq s^L_i(1-\alpha), & i \in \mathfrak{S}, \\ \sum\limits_{(j,i)\in A} u^L_{j,i}(1-\alpha) y_{j,i} \geq d^R_i(1-\alpha), & i \in \mathfrak{D}. \end{cases} \tag{20}$$

Equivalently for each $(i, j), (j, i) \in A$

$$\begin{cases} \exists\, u_{i,j} \in [\widetilde{u}_{i,j}]_\alpha,\ \exists s_i \in [\widetilde{s}_i]_\alpha : \sum\limits_{(i,j)\in A} u_{i,j} y_{i,j} \leq s_i, & \forall i \in \mathfrak{S}, \\ \exists\, u_{j,i} \in [\widetilde{u}_{j,i}]_\alpha,\ \exists d_i \in [\widetilde{d}_i]_\alpha : \sum\limits_{(j,i)\in A} u_{j,i} y_{j,i} \geq d_i, & \forall i \in \mathfrak{D}, \end{cases} \tag{21}$$

$$\begin{cases} \forall\, u_{i,j} \in [\widetilde{u}_{i,j}]_{1-\alpha},\ \forall s_i \in [\widetilde{s}_i]_{1-\alpha} : \sum\limits_{(i,j)\in A} u_{i,j} y_{i,j} \leq s_i, & \forall i \in \mathfrak{S}, \\ \forall\, u_{j,i} \in [\widetilde{u}_{j,i}]_{1-\alpha}, \forall d_i \in [\widetilde{d}_i]_{1-\alpha} : \sum\limits_{(j,i)\in A} u_{j,i} y_{j,i} \geq d_i, & \forall i \in \mathfrak{D}, \end{cases} \tag{22}$$

which shows the attitude toward the optimistic and pessimistic decisions in the first and second groups of constraints.

Proposition 3.5. *Let $T = \min$ and $S = \max$. The set $\widetilde{Y}^T$ in (15) is width, while the set $\widetilde{Y}^S$ in (16) is narrow. Furthermore, for each set of feasible rate matrices, say $\widetilde{Y}$, applying an order extended by Zadeh's extension principle [44], the following is true*:

$$\widetilde{Y}^S \subseteq \widetilde{Y} \subseteq \widetilde{Y}^T.$$

Proof. Noting to Eqs. (19)–(22), the result is straightforward. □

Definition 3.6 (*Fuzzy goal*). Let $\widetilde{P^k} \in \{\widetilde{\leq_T}, \widetilde{\leq_S}\}$, $k = 1, \ldots, K$. $\widetilde{\zeta}^k \in \mathcal{F}(\mathcal{R})$ is called a fuzzy goal for kth objective function of the fully fuzzified MCFP (7) if

$$\widetilde{f}^k(y) = \sum_{(i,j)\in A} \widetilde{\gamma}^k_{i,j} y_{i,j} \widetilde{P^k} \widetilde{\zeta}^k.$$

Definition 3.7 (*Desirable solution*). Two fuzzy sets $\widetilde{Y}^{T,G}$ and $\widetilde{Y}^{S,G}$ with the following membership functions for each $y \in Y_{SF}$, are called desirable rate matrices for fully fuzzified multi-objective MCFP (7) with respect to T and S, respectively,

$$\mu_{\widetilde{Y}^{T,G}}(y) = T(\mu_{\widetilde{P^k}^T}(\widetilde{f}^k(y), \widetilde{\zeta}^k) \mid k = 1, \ldots, K),$$

$$\mu_{\widetilde{Y}^{S,G}}(y) = S(1 - \mu_{\widetilde{P^k}_S}(\widetilde{f}^k(y), \widetilde{\zeta}^k) \mid k = 1, \ldots, K).$$

Proposition 3.8. *Let $\widetilde{\gamma}^k_{i,j}$ and $\widetilde{\zeta}^k$ for each $(i, j) \in A$ and $k = 1, \ldots, K$ be fuzzy numbers. For $T = \min$, $S = \max$, $\alpha \in (0, 1)$ and each $y \in Y_{SF}$, the following statements are true*:

$$\mu_{\widetilde{\leq}^T}(\widetilde{f}^k(y), \widetilde{\zeta}^k) \geq \alpha \;\Leftrightarrow\; \sum_{(i,j)\in A} \gamma^{k,L}_{i,j}(\alpha) y_{i,j} \leq \zeta^{k,R}(\alpha), \tag{23}$$

$$\mu_{\widetilde{\leq}_S}(\widetilde{f}^k(y), \widetilde{\zeta}^k) \geq \alpha \;\Leftrightarrow\; \sum_{(i,j)\in A} \gamma^{k,R}_{i,j}(1-\alpha) y_{i,j} \leq \zeta^{k,L}(1-\alpha). \tag{24}$$

Definition 3.9 (*Optimistic and pessimistic solutions*). Keeping t-norm T and t-conorm S in mind, the fuzzy sets $\widetilde{Y}^{T,*}$ and $\widetilde{Y}^*_S$ with the following membership functions for each $y \in Y_{SF}$, are called the most optimistic and the most pessimistic satisficing sets of rate matrices of fully fuzzified multi-objective MCFP (7)–(8),

$$\mu_{\widetilde{Y}^{T,*}}(y) = T(\mu_{\widetilde{Y}^{T,G}}(y), \mu_{\widetilde{Y}^T}(y)),$$

$$\mu_{\widetilde{Y}^*_S}(y) = S(\mu_{\widetilde{Y}^{S,G}}(y), \mu_{\widetilde{Y}_S}(y)).$$

Definition 3.10. Let $\widetilde{Y}^{T,*}$ and $\widetilde{Y}^*_S$ be fuzzy numbers with singleton cores. A vector $\widetilde{x} = \widetilde{u} y^{T,*}$ in which $\mu_{\widetilde{Y}^{T,*}}(y^{T,*}) = Hgt(\widetilde{Y}^{T,*})$ is called the most optimistic max-satisficing flow and $\widetilde{x} = \widetilde{u} y^*_S \in \Re^n$ in which $\mu_{\widetilde{Y}^*_S}(y^*_S) = Hgt(\widetilde{Y}^*_S)$, is called the most pessimistic max-satisficing flow. Usually, the decision makers wish to find such max-satisficing flows.

Proposition 3.11. (i) *The certainty degree α^* and feasible rate matrix y^* are corresponding to optimal state of the following problem*:

$$\max \alpha \tag{25}$$

s.t.

$$\mu_{\widetilde{Y}^{T,*}}(y) \geq \alpha,$$
$$y \in Y_{SF}, \tag{26}$$

if and only if $\widetilde{x}^* = \widetilde{u} y^*$ is the most optimistic max-satisficing flow of the fully fuzzified multi-objective MCFP (7)–(8).

(ii) *The certainty degree α^* and feasible rate matrix y^* are corresponding to the optimal state of the following problem*:

$$\max \alpha \tag{27}$$

s.t.

$$\mu_{\widetilde{Y}^*_S}(y) \geq \alpha,$$
$$y \in Y_{SF}, \tag{28}$$

if and only if $\widetilde{x}^* = \widetilde{u} y^*$ is the most pessimistic max-satisficing flow of the fully fuzzified multi-objective MCFP (7)–(8).

Proof. Straightforward. □

Proposition 3.11 provides a direct method for finding the most optimistic and the most pessimistic max-satisficing solutions. In conclusion of the presented discussion, T-fuzzy extension is applied for an optimistic decision maker while S-fuzzy extension is used for a pessimistic one. In what follows, these concepts will be introduced.

4. The most optimistic flow

Throughout this section $T = \min$ is used for extending the fuzzy extensions of relations. One can simplify the constraint (28) as follows:

$$\mu_{\gtrapprox^T}(\widetilde{f}^k(y), \widetilde{\zeta}^k) \geq \alpha, \quad k = 1, \ldots, K,$$
$$\mu_{\gtrapprox^T}(\widetilde{g}_{s,i}(y), \widetilde{s}_i) \geq \alpha, \quad i \in \mathfrak{S},$$
$$\mu_{\gtrapprox^T}(\widetilde{g}_{d,i}(y), \widetilde{d}_i) \geq \alpha, \quad i \in \mathfrak{D}, \tag{29}$$

for each $y \in Y_{SF}$. Therefore, the following problem may be solved, instead of model (27)–(28)

$$\max \alpha \tag{30}$$

s.t.

$$\sum_{(i,j)\in A} \gamma_{i,j}^{k,L}(\alpha) y_{i,j} \leq \zeta^{k,R}(\alpha), \quad k = 1, \ldots, K,$$
$$\sum_{(i,j)\in A} u_{i,j}^{L}(\alpha) y_{i,j} \leq s_i^{R}(\alpha), \quad i \in \mathfrak{S},$$
$$\sum_{(j,i)\in A} u_{j,i}^{R}(\alpha) y_{j,i} \geq d_i^{L}(\alpha), \quad i \in \mathfrak{D},$$
$$y \in Y_{SF}. \tag{31}$$

Note that $y \in Y_{SF}$ means that for all $i \in N\backslash(\mathfrak{S} \cup \mathfrak{D})$, $\sum_{\{j:(i,j)\in A\}} \widetilde{u}_{i,j} y_{i,j} = \sum_{\{j:(j,i)\in A\}} \widetilde{u}_{j,i} y_{j,i}$, which implies that for each $\alpha \in [0, 1]$,

$$\sum_{\{j:(i,j)\in A\}} [\widetilde{u}_{i,j}]_\alpha y_{i,j} = \sum_{\{j:(j,i)\in A\}} [\widetilde{u}_{j,i}]_\alpha y_{j,i},$$

that reduces as below,

$$\sum_{\{j:(i,j)\in A\}} u_{i,j}^{L}(\alpha) y_{i,j} = \sum_{\{j:(j,i)\in A\}} u_{j,i}^{L}(\alpha) y_{j,i},$$
$$\sum_{\{j:(i,j)\in A\}} u_{i,j}^{R}(\alpha) y_{i,j} = \sum_{\{j:(j,i)\in A\}} u_{j,i}^{R}(\alpha) y_{j,i}.$$

Substituting in model (30)–(31) provides below,

$$\max \alpha \tag{32}$$

s.t.

$$\sum_{\{j:(i,j)\in A\}} \gamma_{i,j}^{k,L}(\alpha) y_{i,j} \leq \zeta^{k,R}(\alpha), \quad k = 1, \ldots, K,$$
$$\sum_{(i,j)\in A} u_{i,j}^{L}(\alpha) y_{i,j} \leq s_i^{R}(\alpha), \quad i \in \mathfrak{S},$$
$$\sum_{(j,i)\in A} u_{j,i}^{R} y_{j,i} \geq d_i^{L}(\alpha), \quad i \in \mathfrak{D},$$

$$\sum_{\{j:(i,j)\in A\}} u^L_{i,j}(\alpha)y_{i,j} = \sum_{\{j:(j,i)\in A\}} u^L_{j,i}(\alpha)y_{j,i}, \quad i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}),$$

$$\sum_{\{j:(i,j)\in A\}} u^R_{i,j}(\alpha)y_{i,j} = \sum_{\{j:(j,i)\in A\}} u^R_{j,i}(\alpha)y_{j,i}, \quad i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}). \tag{33}$$

Since, $x^L_{i,j}(\alpha) = u^L_{i,j}(\alpha)y_{i,j}$, $x^R_{i,j}(\alpha) = u^R_{i,j}(\alpha)y_{i,j}$, and also $0 \le y_{i,j} \le 1$, it is clear that

$$x^L_{i,j}(\alpha) \le u^L_{i,j}(\alpha),$$

$$x^R_{i,j}(\alpha) \le u^R_{i,j}(\alpha).$$

Thus, the following model may be considered:

$$\max \alpha \tag{34}$$

s.t.

$$\text{(a)} \sum_{\{j:(i,j)\in A\}} c^{k,L}_{i,j}(\alpha)x^L_{i,j}(\alpha) \le \zeta^{k,R}(\alpha), \quad k = 1,\ldots,K,$$

$$\text{(b)} \sum_{(i,j)\in A} x^L_{i,j}(\alpha) \le s^R_i(\alpha), \quad i \in \mathfrak{S},$$

$$\text{(c)} \sum_{(j,i)\in A} x^R_{j,i}(\alpha) \ge d^L_i(\alpha), \quad i \in \mathfrak{D},$$

$$\text{(d)} \sum_{\{j:(i,j)\in A\}} x^L_{i,j}(\alpha) = \sum_{\{j:(j,i)\in A\}} x^L_{j,i}(\alpha), \quad i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}),$$

$$\text{(e)} \sum_{\{j:(i,j)\in A\}} x^R_{i,j}(\alpha) = \sum_{\{j:(j,i)\in A\}} x^R_{j,i}(\alpha), \quad i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}),$$

$$\text{(f) } x^L_{i,j}(\alpha) \le u^L_{i,j}(\alpha), \quad (i,j) \in A,$$

$$\text{(g) } x^R_{i,j}(\alpha) \le u^R_{i,j}(\alpha), \quad (i,j) \in A. \tag{35}$$

Because $x^L_{i,j}(\alpha) \le x^R_{i,j}(\alpha)$ one can assume that

$$x^R_{i,j}(\alpha) = x^L_{i,j}(\alpha) + z_{i,j}(\alpha), \quad \forall (i,j) \in A,$$

where $z_{i,j}(\alpha) \ge 0$, is a slack variable. Notice that, only for $i \in \mathfrak{D}$, $z_{i,j}(\alpha)$ has a role in model (35) and for other nodes, $z_{i,j}(\alpha)$ is assumed to be zero. In other words, model (35) produces optimal lower flow and only Eq. (35c) depends on the upper flow. From (35c) $z_{j,i}(\alpha)$ means an excess flow transported to node $i \in \mathfrak{D}$ from node j. One can use a new link (j,i) corresponding to flow $z_{j,i}(\alpha)$. In order to prevent from parallel links, insert a new node τ_i and add the links (j,τ_i) and (τ_i, i) for each node $i \in \mathfrak{D}$, see Figs. 1a and b.

Since $z_{j,i}(\alpha)$ only gets positive value, when $x^L_{i,j}(\alpha)$ obtains its upper bound, the following capacities and costs may be taken into account for new inserted links,

$$\begin{cases} c^{k,L}_{j,\tau_i}(\alpha) = c^{k,L}_{j,i}(\alpha) + \varepsilon, \\ u^L_{j,\tau_i}(\alpha) = u^R_{j,i}(\alpha) - u^L_{j,i}(\alpha), \\ c^{k,L}_{\tau_i,i}(\alpha) = 0, \\ u^L_{\tau_i,i}(\alpha) = u^R_{j,i}(\alpha) - u^L_{j,i}(\alpha), \end{cases} \quad \forall i \in \mathfrak{D},\ \forall (j,i) \in A,$$

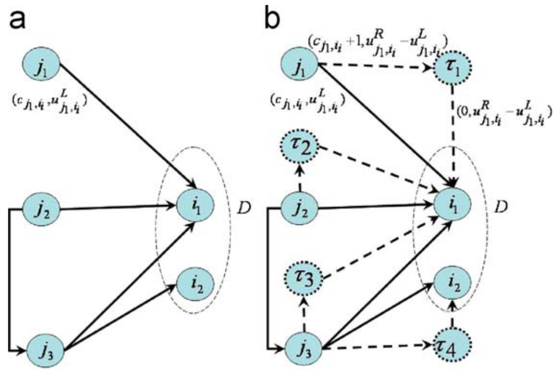

Fig. 1. Inserting a new node and two new links to transport excess flow $z_{i,j}$, (a) before transformation, (b) after first transformation.

where $\varepsilon > 0$ is an arbitrary constant. Needless to say, the cost for path $j \to \tau_i \to i$ is greater than that of link (j,i) which ensures that $z_{j,i}(\alpha)$ gets positive value after $x^L_{i,j}(\alpha)$ filling. The optimal value of the objective function will be corrected easily in final step by losing the effect of ε. Considering these inserted links and nodes, instead of (35c) one can state,

$$\sum_{(j,i)\in A} x^L_{j,i}(\alpha) \geq d^L_i(\alpha).$$

Due to $\sum_{\{j:(i,j)\in A\}} x^L_{i,j}(\alpha) = \sum_{\{j:(j,i)\in A\}} x^L_{j,i}(\alpha)$, it is derived that

$$\sum_{\{j:(i,j)\in A\}} x^R_{i,j}(\alpha) = \sum_{\{j:(j,i)\in A\}} x^R_{j,i}(\alpha).$$

It is necessary to check whether or not $x^R_{j,i}(\alpha) \leq u^R_{j,i}(\alpha)$ or equivalently $x^L_{j,i}(\alpha)+z_{j,i}(\alpha) \leq u^R_{j,i}(\alpha)$. Since $x^L_{j,i}(\alpha) \leq u^L_{j,i}(\alpha)$ and $z_{j,i}$ is amount of flow through a path whose upper bound capacity is $u^R_{j,i}(\alpha) - u^L_{j,i}(\alpha)$, this inequality satisfies. In fact, a traditional constraint of MCFP yields. In order to fulfill

$$\sum_{\{j:(i,j)\in A\}} c^{k,L}_{i,j}(\alpha) x^L_{i,j}(\alpha) \leq \zeta^{k,R}(\alpha),$$

for each $k = 1, \ldots, K$, the following objective function can be expressed:

$$\phi^k = \min \sum_{\{j:(i,j)\in A\}} c^{k,L}_{i,j}(\alpha) x^L_{i,j}(\alpha).$$

By the fact that $\phi^k \leq \zeta^{k,R}(\alpha)$, the original model can be outlined as follows:

$$\max \alpha \tag{36}$$

s.t.

$$\phi^k \leq \zeta^{k,R}(\alpha), \quad \forall k, 1, \ldots, K, \tag{37}$$

where

$$\phi^k = \min \sum_{\{j:(i,j)\in A\}} c^{k,L}_{i,j}(\alpha) x^L_{i,j}(\alpha) \tag{38}$$

s.t.

$$\text{(a)} \ \sum_{(i,j)\in A} x^L_{i,j}(\alpha) \leq s^R_i(\alpha), \quad i \in \mathfrak{S},$$

$$\text{(b)} \sum_{(j,i)\in A} x^L_{j,i}(\alpha) \geq d^L_i(\alpha), \quad i \in \mathfrak{D},$$

$$\text{(c)} \sum_{\{j:(i,j)\in A\}} x^L_{i,j}(\alpha) = \sum_{\{j:(j,i)\in A\}} x^L_{j,i}(\alpha), \quad i \in N\backslash(\mathfrak{S} \cup \mathfrak{D}),$$

$$\text{(d)}\ x^L_{i,j}(\alpha) \leq u^L_{i,j}(\alpha), \quad \forall (i,j) \in A. \tag{39}$$

Note that the inner program is a traditional MCFP which is solvable efficiently.

Proposition 4.1. *Let the fuzzy parameters be LR positive numbers. The programming problem* (38) *subject to* (39) *is increasing with respect to certainty degree* α.

Proof. Denote the feasible set of constraint (39) with respect to the certainty degree α with $\mathfrak{F}_o(\alpha)$. Let $\alpha_1 \leq \alpha_2$. It is easy to derive that

$$s^R_i(\alpha_1) \leq s^R_i(\alpha_2),$$

$$d^L_i(\alpha_1) \geq d^L_i(\alpha_2),$$

thus, $\mathfrak{F}_o(\alpha_1) \supseteq \mathfrak{F}_o(\alpha_2)$, which shows that feasible set of the problem with respect to α_1 is more extended in comparison with that of α_2. Also in light of $c^L_{i,j}(\alpha_1), c^L_{i,j}(\alpha_2) \geq 0$, and $c^L_{i,j}(\alpha_1) \leq c^L_{i,j}(\alpha_2)$, the objective function of α_1 problem is less than that of α_2 problem. Thus, the optimal value of α_1 problem is less than that of α_2 problem, and this proves the assertion. □

Remark 4.2. By assumption of Proposition 4.1 if for one α_1 the feasible set provided by (39) is empty, for each $\alpha \geq \alpha_1$ the feasible set of (39) remains empty.

Proof. It can be derived from proof of Proposition 4.1. □

This remark permits to pursue the following bisection algorithm in order to find the lower optimal flow of fuzzy multi-objective MCFP (7)–(8) with optimistic attitude. Because of finding the most optimistic max-satisficing solution by this algorithm, it can be named as "MOMS" algorithm. Before we formally present the algorithm, note that checking the feasibility of constraints (39) is a famous problem. For this aim in [2, p. 169] a transformation is presented in which the feasible flow problem is replaced by a maximum flow problem. For the latter problem a lot of polynomial algorithms are introduced. In Sections 7 and 8 in [2] one can see a review on such algorithms. For example FIFO preflow-push algorithm in the presented reference can be utilized to check the feasibility of constraints (39) in $O(n^3)$ where n is the number of nodes of network.

On the other hand, model (7) subjected to (8) is a traditional minimum cost flow problem. This problem is solvable in polynomial time. Some of such algorithms are introduced in Sections 10 and 11 of [2], e.g., enhanced capacity scaling algorithm can be used to solve MCFP in $O(m\log(n)(m+n\log(n)))$ where m and n are the number of links and nodes, respectively [2, p. 387]. Applying these schemes, we can propose the following algorithm. In the first part of this algorithm, a transformation is done to create a new network with respect to optimistic attitude. Then in step 8, the feasibility of problem is examined. The loop including steps 3–9 is terminated after finding maximum certainty degree with respect to mathematical supply–demand constraints. Then another loop consisting of steps 11–17 is loaded to find a flow satisfying target values. The result of latter loop is maximum certainty degree considering the constraints and targets. In step 18, all of the optimal solutions of fuzzy multi-objective MCFP can be obtained. Then the efficient solutions among these solutions are selected through step 19. The formal description of algorithm is as below.

Algorithm 4.3 (*MOMS*).
Input: *A network with fuzzy multiple costs, fuzzy capacities, fuzzy supplies and fuzzy demands.*
Output: *Maximal certainty degree and maximal-satisficing optimistic solutions.*
1. *Define* $\mathfrak{S}$ *and* $\mathfrak{D}$.
2. *Set* $\overline{\alpha} = 1$ *and* $\underline{\alpha} = 0$

3. *While* $(\overline{\alpha} - \underline{\alpha}) > \varepsilon$
4. *Set* $\alpha = (\overline{\alpha} + \underline{\alpha})/2$

/ ∗ *Comment*:
Transformation on the network are presented as follows to pursue the later steps.
End of Comment. ∗ /

5. *For each link* (i, j) *and the objective function* k *find* $c_{i,j}^{k,L}(\alpha)$ *and* $u_{i,j}^{L}(\alpha)$.
6. *For each node* $i \in \mathfrak{S}$ *obtain* $s_i^R(\alpha)$ *and for each node* $i \in \mathfrak{D}$ *obtain* $d_i^L(\alpha)$.
7. *For each node* $k \in \mathfrak{D}$, *if link* (k, i) *exists*, *define new node* $\tau_{k,i}$ *with imbalance* 0 *and augment two links* $(k, \tau_{k,i})$ *and* $(\tau_{k,i}, i)$ *with costs* $c_{k,i}^{k,L}(\alpha)$ *and* 0, *respectively. The capacities of these links are equal to* $u_{k,i}^{R}(\alpha) - u_{k,i}^{L}(\alpha)$.
8. *Check the feasibility of constraints* (39) *applying a maximum flow algorithm.*
9. *End of while*.

/ ∗ *Comment* :
Since $\underline{\alpha}$ *is the greatest certainty degree which yields feasible flow*, *it is sufficient to solve the original problem* (7)–(8) *when* $\alpha \in [0, \underline{\alpha}]$.
End of Comment. ∗ /

10. *Set* $\overline{\alpha} = \underline{\alpha}$ *and* $\underline{\alpha} = 0$.
11. *While* $(\overline{\alpha} - \underline{\alpha}) > \varepsilon$
12. *Set* $\alpha = (\overline{\alpha} + \underline{\alpha})/2$
13. *Applying a minimum cost flow algorithm*, *solve model* (38) *subject to* (39) *with respect to* α *to obtain* ϕ^k *for* $k = 1, \ldots, K$.
14. *For each* $k = 1, \ldots, K$, *obtain* $\zeta^{k,R}(\alpha)$.
15. *If* $\phi^k \leq \zeta^{k,R}(\alpha)$, *for each* $k = 1, \ldots, K$ *set* $\underline{\alpha} = \alpha$.
16. *Else set* $\overline{\alpha} = \alpha$.
17. *End of while*.

/ ∗ *Comment* :
Finding satisficing solutions with respect to maximal certainty degree.
End of Comment. ∗ /

18. *Set* $\alpha_{opt} = \underline{\alpha}$. *For* $k = 1 : K$ *solve minimum cost flow problem* (47) *subject to* (48) *with respect to* α_{opt}. *Denote the optimal solutions of these problems with* $x_1, x_2, \ldots, x_K$, *respectively*.
19. *Between* $x_1, x_2, \ldots, x_K$ *select efficient solution*(*s*).
20. *Return* α_{opt} *as maximal certainty degree and efficient solution*(*s*) *as maximal-satisficing solutions*.
21. *End of algorithm*.

To select the efficient solutions, there are some methods in literature e.g., Zeleny's test, refer to [45].

Proposition 4.4. *Algorithm* 4.3 *can find the most optimistic max-satisficing solution of model* (7)–(8) *in*

$$\mathcal{O}(|log(1/\varepsilon)| \max\{\mathcal{C}_{MFP}(n, m) + \mathcal{C}_{TP}(n, m), \mathcal{C}_{MCFP}(n, m) + \mathcal{C}_{TP}(n, m))\},$$

where $\mathcal{C}_{TP}(n, m)$, $\mathcal{C}_{MFP}(n, m)$ *and* $\mathcal{C}_{MCFP}(n, m)$ *are the complexity of algorithms of transformation process*, *maximum flow problem and traditional MCFP*, *respectively*.

Remark 4.5. The transformation process in optimistic attitude increases the number of nodes and links with $\mathcal{O}(m)$.

Proof. Since the transformation is done on demander nodes and for each demander node, one node and two links are augmented, the number of nodes and links after transformation can be calculated as below which are denoted with n' and m', respectively,

$$n' = n + \sum_{i \in \mathfrak{D}} |A(i)|,$$

$$m' = m + 2 \sum_{i \in \mathfrak{D}} |A(i)|,$$

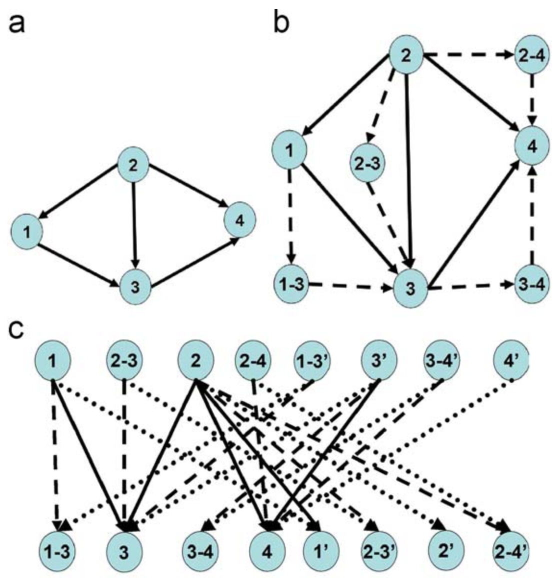

Fig. 2. A simple network with one supplier and two demanders, (a) is original network, (b) is its transformed network and (c) is a bipartite network which is equivalent to the network depicted in (b).

where $|A(i)|$ is the number of adjacent nodes of node i. Since $\sum_{i\in\mathfrak{D}}|A(i)| \leq \sum_{i\in N}|A(i)| = 2m$, it can be derived that

$$n' \leq n + \mathcal{O}(m)$$

and

$$m' \leq m + \mathcal{O}(m). \qquad \square$$

Theorem 4.6. *Using $\varepsilon \geq 10^{-n}$ the complexity of Algorithm 4.3 dominates with $\mathcal{O}(n\mathcal{C}_{MCFP}(n,m))$ and so the complexity of this algorithm is linear regarding to n and m.*

Proof. By the assumption, $\mathcal{O}(|log(1/\varepsilon)) \leq \mathcal{O}(n)$ and noting to Remark 4.5, $\mathcal{C}_{TP}(n,m) \leq \mathcal{O}(m)$. Applying also combinatorial algorithms to solve maximum flow and MCFP, implies that $\mathcal{C}_{MFP}(n,m) \leq \mathcal{C}_{MCFP}(n,m)$, see e.g., [2] for details. Thus, the results are obvious. $\square$

To present how the methods work and to interpret the results, the following example is useful.

Example 4.7. Consider a network with four nodes including one supplier and two demanders. The network is represented in Fig. 2a. The supplier node 2, produces (150, 250, 300) while two nodes 3 and 4 demand (100, 160, 200) and (50, 90, 100) units, respectively. Two costs are considered, denoted with Cost 1 and Cost 2. These costs may be considered as traverse time and risk corresponding to each link. The details of data are presented in Table 1.

The optimistic transformation is depicted in Fig. 2b. The transformation of this network into a bipartite network is also depicted in Fig. 2c. Let all of the parameters be triangular numbers with linear left and right shape functions.

MOMS algorithm investigates some real scenarios iteratively by creating transformed networks. For example for $\alpha = 0.75$ as certainty degree, the data of transformed network is presented in Table 2. *Crisp Cost* 1 and *Crisp Cost* 2 are used as traverse time and risk of each link when the certainty degree is fixed. Node 2 has 475 supplies and nodes 3 and 4 have 40 and 22.5 units demands, respectively.

Keeping Remark 3.1 in mind, one new node denoted with p is considered as an inventory with demand 412.5. By applying the first part of Algorithm 4.3, the maximal certainty degree for this problem is $\alpha = 0.48$. The trend of

Table 1
The fuzzy capacities and costs of network links.

Link	Capacity	Cost 1	Cost 2
(2, 1)	(10, 15, 20)	(10, 15, 20)	(5, 10, 12)
(2, 3)	(7, 11, 30)	(7, 11, 30)	(9, 13, 20)
(2, 4)	(8, 19, 23)	(8, 19, 23)	(20, 22, 26)
(1, 3)	(7, 12, 19)	(7, 12, 19)	(17, 32, 40)
(3, 4)	(5, 10, 20)	(5, 10, 20)	(16, 18, 20)

Table 2
The crisp data of transformed network with respect to $\alpha = 0.75$.

Link	Crisp capacity	Crisp Cost 1	Crisp Cost 2
(2, 1)	7.5	7.5	6.25
(2, 3)	5.75	5.75	6.25
(2, 4)	13	13	7
(1, 3)	6.75	6.75	19.25
(3, 4)	6.25	6.25	6
(1, 1 − 3)	19.5	7.75	20.25
(1 − 3, 3)	19.5	0	0
(2, 2 − 3)	27.75	6.75	7.25
(2 − 3, 3)	27.75	0	0
(2, 2 − 4)	23.25	14	8
(2 − 4, 4)	23.25	0	0
(3, 3 − 4)	18.75	7.25	7
(3 − 4, 4)	18.75	0	0

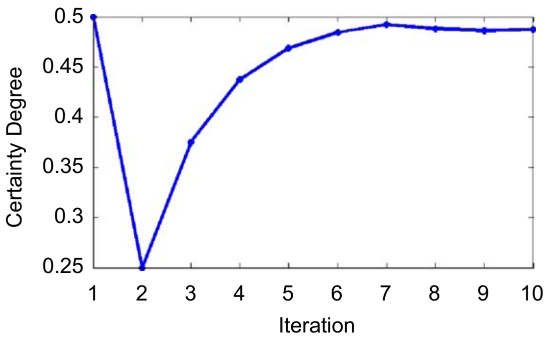

Fig. 3. The trend of finding maximal certainty degree with respect to optimistic attitude when the network of Fig. 2 is considered.

receiving this certainty degree is represented in Fig. 3. When α decreases, the values of $u^L_{i,j}(\alpha)$ increases for each link (i, j) and $s^R_i(\alpha)$ decreases for each $i \in \mathfrak{S}$. Similarly, $u^R_{i,j}(\alpha)$ decreases and $d^L_i(\alpha)$ increases for each $i \in \mathfrak{D}$. These cause to find a feasible solution if it exists. This means that when the certainty degree increases, the pessimistic attitude of link capacities, denoted with $u^L_{i,j}(\alpha)$, decreases while optimistically this parameter increases. Demand also appears in this model with the left shape function which is decreasing with respect to the certainty degree. On the other hand, the supplies whose right shape function is considered is increasing regarding to α.

Now we consider $\zeta^{1,R} = (1000, 4000, 6000)$ and $\zeta^{2,R} = (2000, 3000, 7000)$. Implementing the second part of Algorithm 4.3 we find the maximal satisficing solution for this problem. In Fig. 4 the trend of both of the objective functions and their goal values are presented separately. The details of values of the objective functions and the goal values are presented in Table 3. These results show that from the second step, the second objective function satisfies its goal value, while the first objective function tries to satisfy the goal value by maximal certainty degree, i.e., it is

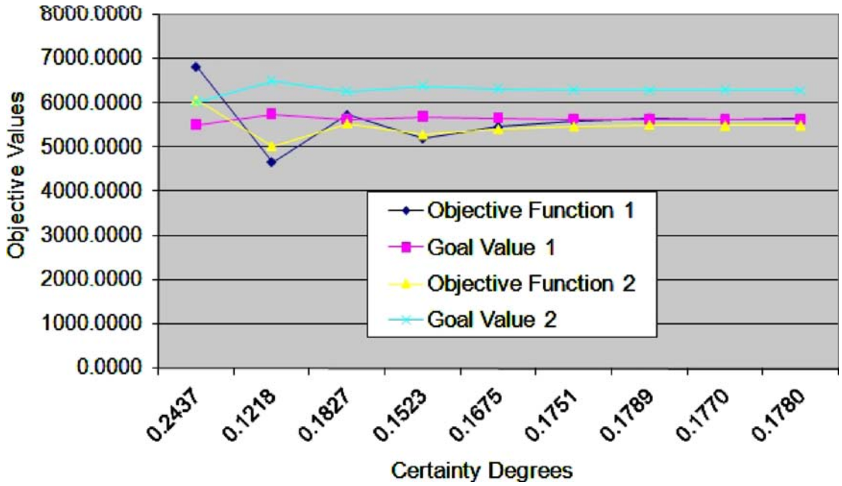

Fig. 4. The trend of objective functions (1) and (2) and their goal values when the certainty degrees are varying through Algorithm 4.3.

Table 3
The values of the objective functions and the goals when the optimistic attitude on Example 4.7 is pursued.

α	ϕ^1	$\zeta^{1,R}$	ϕ^2	$\zeta^{2,R}$
0.2437	6825.7612	5512.6953	6077.7792	6025.3906
0.1218	4668.8160	5756.3477	5030.0869	6512.6953
0.1827	5736.2724	5634.5215	5550.4824	6269.0430
0.1523	5199.7902	5695.4346	5289.4219	6390.8691
0.1675	5467.3428	5664.9780	5419.7365	6329.9561
0.1751	5601.6355	5649.7498	5485.0555	6299.4995
0.1789	5668.9109	5642.1356	5517.7554	6284.2712
0.1770	5635.2624	5645.9427	5501.4021	6291.8854
0.1780	5652.0840	5644.0392	5509.5779	6288.0783

possible to present a solution with a higher certainty degree when only the second objective function considered into account.

As the final point, note that with respect to the customer's perspective, this problem is pessimistic while it is optimistic for manager, because the demand increases without increasing the supplies. It is an important concept which can be followed in real problems to reach equilibrium status. We retain to this concept in next sections.

5. The most pessimistic flow

In this section $S = \max$ is utilized. Constraints (28) yield the below inequalities:

$$\sum_{(i,j)\in A} \gamma_{i,j}^{k,R}(\alpha) y_{i,j} \le \zeta^{k,L}(\alpha), \quad k = 1, \ldots, K,$$

$$\sum_{(i,j)\in A} u_{i,j}^{R}(\alpha) y_{i,j} \le s_i^{L}(\alpha), \quad i \in \mathfrak{S},$$

$$\sum_{(j,i)\in A} u_{j,i}^{L}(\alpha) y_{j,i} \ge d_i^{R}(\alpha), \quad i \in \mathfrak{D}, \tag{40}$$

so model (27)–(28) reduces to the following problem:

$$\max \alpha \tag{41}$$

s.t.

$$\sum_{\{j:(i,j)\in A\}} \gamma_{i,j}^{k,R}(\alpha) y_{i,j} \le \zeta^{k,L}(\alpha), \quad k = 1, \ldots, K,$$

$$\sum_{(i,j)\in A} u_{i,j}^{R}(\alpha) y_{i,j} \le s_i^L(\alpha), \quad i \in \mathfrak{S},$$

$$\sum_{(j,i)\in A} u_{j,i}^{L}(\alpha) y_{j,i} \ge d_i^R(\alpha), \quad i \in \mathfrak{D},$$

$$\sum_{\{j:(i,j)\in A\}} u_{i,j}^{R}(\alpha) y_{i,j} = \sum_{\{j:(j,i)\in A\}} u_{j,i}^{R}(\alpha) y_{j,i}, \quad i \in N \backslash (\mathfrak{S} \cup \mathfrak{D}),$$

$$\sum_{\{j:(i,j)\in A\}} u_{i,j}^{L}(\alpha) y_{i,j} = \sum_{\{j:(j,i)\in A\}} u_{j,i}^{L}(\alpha) y_{j,i}, \quad i \in N \backslash (\mathfrak{S} \cup \mathfrak{D}). \tag{42}$$

Thank to $x_{i,j}^R(\alpha) = u_{i,j}^R(\alpha) y_{i,j}$, $x_{i,j}^L(\alpha) = u_{i,j}^L(\alpha) y_{i,j}$, and $0 \le y_{i,j} \le 1$, the following statements are true:

$$x_{i,j}^R(\alpha) \le u_{i,j}^R(\alpha),$$

$$x_{i,j}^L(\alpha) \le u_{i,j}^L(\alpha).$$

Substituting $x_{i,j}^L(\alpha)$ and $x_{i,j}^R(\alpha)$ in constraints (42) with noting that $x_{i,j}^L(\alpha) \le x_{i,j}^R(\alpha)$, one can consider,

$$x_{i,j}^R(\alpha) = x_{i,j}^L(\alpha) + z_{i,j}(\alpha), \quad \forall i \in \mathfrak{S}\ \forall (i,j) \in A,$$

where $z_{i,j}(\alpha) \ge 0$ is a slack variable. Thus, the following model is stated in order to find the pessimistic max-satisficing flow.

$$\max \alpha \tag{43}$$

s.t.

$$\begin{aligned}
&\sum_{\{j:(i,j)\in A\}} c_{i,j}^{k,R}(\alpha)(x_{i,j}^L(\alpha) + z_{i,j}(\alpha)) \le \zeta^{k,L}(\alpha), \quad k = 1, \ldots, K,\\
&\sum_{(i,j)\in A} x_{i,j}^L(\alpha) + \sum_{(i,j)\in A} z_{i,j}(\alpha) \le s_i^L(\alpha), \quad i \in \mathfrak{S},\\
&\sum_{(j,i)\in A} x_{j,i}^L(\alpha) \ge d_i^R(\alpha), \quad i \in \mathfrak{D},\\
&\sum_{\{j:(i,j)\in A\}} x_{i,j}^R(\alpha) = \sum_{\{j:(j,i)\in A\}} x_{j,i}^R(\alpha), \quad i \in N \backslash (\mathfrak{S} \cup \mathfrak{D}),\\
&\sum_{\{j:(i,j)\in A\}} x_{i,j}^L(\alpha) = \sum_{\{j:(j,i)\in A\}} x_{j,i}^L(\alpha), \quad i \in N \backslash (\mathfrak{S} \cup \mathfrak{D}),\\
&x_{i,j}^R(\alpha) \le u_{i,j}^R(\alpha), (i,j) \in A,\\
&x_{i,j}^L(\alpha) \le u_{i,j}^L(\alpha), (i,j) \in A.
\end{aligned} \tag{44}$$

Note that, only for $i \in \mathfrak{S}$, $z_{i,j}(\alpha)$ effects in model and for other cases $z_{i,j}(\alpha)$ is supposed to be zero. Remember that $z_{i,j}(\alpha)$ is excess flow on link (i, j) and is used for enhancing the level of certainty in applied problems for increasing the reliability of commodity supplies. Utilizing the magnitude cost $c_{i,j}^{k,R}(\alpha)$ for $z_{i,j}(\alpha)$, reveals that getting a positive value for $z_{i,j}(\alpha)$ is a disadvantage according to the decision maker's perspective.

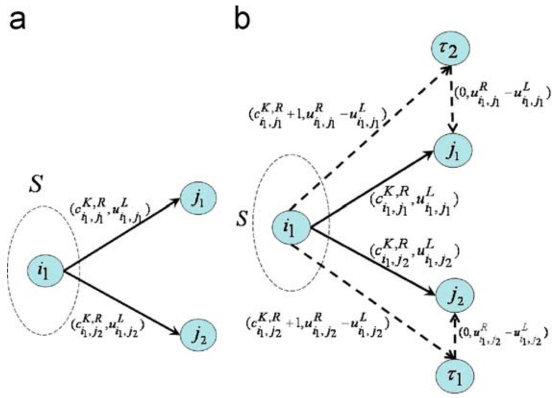

Fig. 5. Inserting a new node and two new links to transport excess flow $z_{i,j}$, (a) before transformation, (b) after first transformation.

In what follows, some nodes and arcs are considered to transform this problem into a network problem. For each $i \in \mathfrak{S}$, and each adjacent node j, consider dummy node τ_i and two links (i, τ_i) and (τ_i, j) with the following costs and capacities, see also Fig. 5:

$$\begin{cases} c_{i,\tau_i}^{k,R}(\alpha) = c_{i,j}^{k,R}(\alpha) + \varepsilon, \\ u_{i,\tau_i}^{L}(\alpha) = u_{i,j}^{R}(\alpha) - u_{i,j}^{L}(\alpha), \\ c_{\tau_i,j}^{k,R}(\alpha) = 0, \\ u_{\tau_i,j}^{L}(\alpha) = u_{i,j}^{R}(\alpha) - u_{i,j}^{L}(\alpha), \end{cases} \quad \forall i \in \mathfrak{S}\ \forall (j,i) \in A,$$

where $\varepsilon > 0$ is a positive constant. Since the per unit cost of path $i \to \tau_i \to j$ is greater than that of link (i, j), the flow first transport through link (i, j) and after filling (i, j) the flow leads through path $i \to \tau_i \to j$.

To satisfy,

$$\sum_{\{j:(i,j)\in A\}} c_{i,j}^{k,R}(\alpha) x_{i,j}^{L}(\alpha) \le \zeta^{k,L}(\alpha),$$

for each $k = 1, \ldots, K$, the following objective function can be taken into account:

$$\phi^k = \min \sum_{\{j:(i,j)\in A\}} c_{i,j}^{k,R}(\alpha) x_{i,j}^{L}(\alpha).$$

As conclusion of these observations, the following model should be solved in transformed network:

$$\max \alpha \tag{45}$$

s.t.

$$\phi^k \le \zeta^{k,L}(\alpha), \quad \forall k, 1, \ldots, K, \tag{46}$$

where

$$\phi^k = \min \sum_{\{j:(i,j)\in A\}} c_{i,j}^{k,R}(\alpha) x_{i,j}^{L}(\alpha) \tag{47}$$

s.t.

$$\text{(a)} \sum_{(i,j)\in A} x_{i,j}^{L}(\alpha) \le s_i^{L}(\alpha), \quad i \in \mathfrak{S},$$

$$\text{(b)} \sum_{(j,i)\in A} x^L_{j,i}(\alpha) \geq d^R_i(\alpha), \quad i \in \mathfrak{D},$$

$$\text{(c)} \sum_{\{j:(i,j)\in A\}} x^L_{i,j}(\alpha) = \sum_{\{j:(j,i)\in A\}} x^L_{j,i}(\alpha), \quad i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}),$$

$$\text{(d)}\ x^L_{i,j}(\alpha) \leq u^L_{i,j}(\alpha), \quad \forall (i,j) \in A. \tag{48}$$

It is clear that the inner submodel is a traditional MCFP and can be solved in polynomial time.

Proposition 5.1. *Let the fuzzy parameters be LR positive numbers. The MCFP* (47) *subject to* (48) *is decreasing with respect to certainty degree* α.

Proof. Let $\mathfrak{F}_p(\alpha)$ shows the feasible set of constraint (48) with respect to certainty degree α. Let $\alpha_1 \leq \alpha_2$. One can see,

$$s^L_i(\alpha_1) \leq s^L_i(\alpha_2),$$

$$d^R_i(\alpha_1) \geq d^R_i(\alpha_2),$$

thus, $\mathfrak{F}_p(\alpha_1) \subseteq \mathfrak{F}_p(\alpha_2)$. On the other hand, $c^R_{i,j}(\alpha_1), c^R_{i,j}(\alpha_2) \geq 0$, and $c^R_{i,j}(\alpha_1) \geq c^R_{i,j}(\alpha_2)$. Therefore, the problem with respect to α_2 has an extended feasible set with the less objective function values in comparison with those of α_1 problem and so the optimal value of α_2 problem is less than that of α_1 problem. $\square$

Remark 5.2. By assumption of Proposition 5.1, if for one α_1 the feasible set produced by (48) is empty, for each $\alpha \leq \alpha_1$ the feasible set of these constraints is empty, too.

As similar as Algorithm 4.3, the following bisection algorithm can be implemented in order to obtain the most pessimistic max-satisficing solution. We entitle this algorithm "MPMS" referring to the most pessimistic max-satisficing solution which can be found by this algorithm. After transforming the network through the first part of this algorithm, two loops are implemented to find the maximum certainty degrees with respect to the constraints and targets, respectively. Such certainty degrees are dependent on the pessimistic perspective. The efficient solutions among the optimal solutions of the created MCFP are the result of algorithm.

Algorithm 5.3 (*MPMS*).
Input: *A network with fuzzy multiple costs, fuzzy capacities, fuzzy supplies and fuzzy demands.*
Output: *Maximal certainty degree and the most pessimistic max-satisficing solutions.*
1. *Define* $\mathfrak{S}$ *and* $\mathfrak{D}$.
2. *Set* $\overline{\alpha} = 1$ *and* $\underline{\alpha} = 0$
3. *While* $(\overline{\alpha} - \underline{\alpha}) > \varepsilon$
4. *Set* $\alpha = (\overline{\alpha} + \underline{\alpha})/2$
/ ∗ *Comment* :
Transformation on the network is presented as follows to pursue the later steps.
End of Comment. ∗ /
5. *For each link* (i, j) *and the objective function k find* $c^{k,R}_{i,j}(\alpha)$ *and* $u^L_{i,j}(\alpha)$.
6. *For each node* $i \in \mathfrak{S}$ *obtain* $s^L_i(\alpha)$ *and for each node* $i \in \mathfrak{D}$ *obtain* $d^R_i(\alpha)$.
7. *For each node* $i \in \mathfrak{S}$, *if link* (i, j) *exists, define new node* $\tau_{i,j}$ *with imbalance* 0 *and augment two links* $(i, \tau_{i,j})$ *and* $(\tau_{i,j}, j)$ *with costs* $c^{k,R}_{i,j}(\alpha) + \varepsilon$ *and* 0, *respectively.* (*ε is a small number close to zero predefined by programmer.*) *The capacities of these links are equal to* $u^R_{i,j}(\alpha) - u^L_{i,j}(\alpha)$.
8. *Check the feasibility of constraints* (48) *applying a maximum flow algorithm.*
9. *End of while.*
/ ∗ *Comment* :
Since $\underline{\alpha}$ *is the greatest certainty degree which yields feasible flow, it is sufficient to solve the original problem* (7)–(8) *when* $\alpha \in [0, \underline{\alpha}]$.
End of Comment. ∗ /

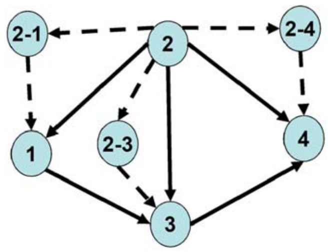

Fig. 6. The transformation of the network depicted in Fig. 2a with respect to the pessimistic attitude.

Table 4
The values of objective function (1) and (2) and their goals for Example 5.4 when the decision maker is pessimistic.

α	ϕ^1	$\zeta^{1,L}$	ϕ^2	$\zeta^{2,L}$
0.2939	9670.6892	10 940.9180	9464.0848	9587.8906
0.4409	12 446.7621	11 161.3770	10 578.1081	9881.8359
0.3674	11 069.9150	11 051.1475	10 041.8063	9734.8633
0.3307	10 373.0994	10 996.0327	9758.1230	9661.3770
0.3123	10 022.5937	10 968.4753	9612.3983	9624.6338
0.3215	10 198.0214	10 982.2540	9685.5842	9643.0054
0.3169	10 110.3512	10 975.3647	9649.0722	9633.8196
0.3146	10 066.4834	10 971.9200	9630.7554	9629.2267
0.3135	10 044.5412	10 970.1977	9621.5819	9626.9302
0.3140	10 055.5130	10 971.0588	9626.1699	9628.0785

10. *Set* $\overline{\alpha} = \underline{\alpha}$ *and* $\underline{\alpha} = 0$.
11. *While* $(\overline{\alpha} - \underline{\alpha}) > \varepsilon$
12. *Set* $\alpha = (\overline{\alpha} + \underline{\alpha})/2$
13. *Applying a minimum cost flow algorithm, solve model* (47) *subject to* (48) *with respect to* α *to obtain* ϕ^k *for* $k = 1, \ldots, K$.
14. *For each* $k = 1, \ldots, K$, *obtain* $\zeta^{k,L}(\alpha)$.
15. *If* $\phi^k \leq \zeta^{k,L}(\alpha)$, *for each* $k = 1, \ldots, K$ *set* $\underline{\alpha} = \alpha$.
16. *Else set* $\overline{\alpha} = \alpha$.
17. *End of while*.

/ ∗ *Comment* :
Finding satisficing solutions with respect to maximal certainty degree.
End of Comment. ∗ /

18. *Set* $\alpha_{opt} = \underline{\alpha}$. *For* $k = 1 : K$ *solve minimum cost flow problem* (47) *subject to* (48) *with respect to* α_{opt}. *Denote the optimal solutions of these problems with* $x_1, x_2, \ldots, x_K$, *respectively*.
19. *Between* $x_1, x_2, \ldots, x_K$ *select efficient solution(s)*.
20. *Return* α_{opt} *as maximal certainty degree and efficient solution(s) as maximal-satisficing solutions*.
21. *End of algorithm*.

Example 5.4. Again consider the network represented in Fig. 2a with the data of Example 4.7. The pessimistic transformation is presented in Fig. 6. As previous, considering $\tilde{\zeta}^1 = (1000, 4000, 6000)$ and $\tilde{\zeta}^2 = (2000, 3000, 7000)$ and applying Algorithm 5.3 there is no solution, i.e., there is no solution satisfying these goals with respect to the pessimistic attitude. Table 4 illustrates the results of finding the maximal certainty degree $\alpha_{opt} = 0.3140$ from Algorithm 5.3 when the goal values are assumed as $\tilde{\zeta}^1 = (10\,500, 12\,000, 14\,000)$ and $\tilde{\zeta}^2 = (9000, 11\,000, 13\,000)$.

It is interesting to note that with respect to customer's perspective, this problem is optimistic while it is pessimistic for manager, because the supplies increase without increasing the demand. When the government wish to reach equilibrium status, the optimistic and pessimistic problems should be considered into account to fulfill the requirement of both of the boards.

6. Application in network design

Network design is an important problem in theory and practice. This problem was considered in uncertain environment by many researchers e.g., [4]. The contribution of this paper may be pursued to design a network satisfying uncertain costs and demands. To present how the methods work and to interpret the results, the following example is useful. Consider the network of Great Khorasan State depicted in Fig. 7. The nodes 32, 30, 58 and 60 are the Khorasan border terminals for abroad export. While 38, 23, 52 and 55 are supplier nodes. To extend this network to meet the demand for transportation in long-term period, one can consider the volume of commodities which are necessary to transit between these supplier and demander nodes. But note that, the extension of this network can be considered with interchange objectives such as minimum cost, maximal reliability, maximal mobility and maximal accessibility. It is worth reporting the sources of uncertainty in measurement of data in this problem:

- The demand for transportation of each node which is dependent on many items such as time, season and weather conditions.
- Incomplete definition of the important travels.
- Imperfect realization of the definition of the important travels.
- Non-representative sampling (the sample measured may not represent the defined travels).
- Inadequate knowledge of the effects of environmental conditions on the measurement, or imperfect measurement of environmental conditions.
- Personal bias in reading analogue instruments.
- Finite instrument resolution or discrimination threshold.
- Linguistic measurement standards.
- Imprecise traffic constants and other parameters obtained from external sources and used in the data-reduction algorithm.
- Approximations and assumptions incorporated in the measurement method and procedure.
- Variations in repeated observations of the travels under apparently identical conditions.

Thus, a multi-objective fully fuzzified MCFP can be applied to understand the fuzzy flow in long-term period. With respect to such flows, the manager can extend the network to meet the considered objectives. For this aim, the following

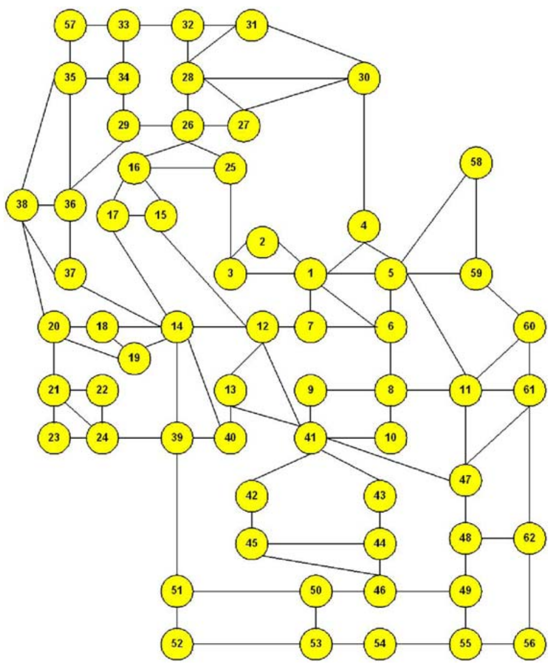

Fig. 7. The network including vital nodes and roads of Great Khorasan State.

model can be stated:

$$\min \sum_{(i,j)\in A} \widetilde{c}^k_{i,j}\widetilde{x}_{i,j} + \sum_{(i,j)\in A} \widetilde{\mu}_{i,j} z_{i,j}, \quad \forall k = 1,\ldots,K \tag{49}$$

s.t.

$$\begin{cases} \sum_{\{j:(i,j)\in A\}} \widetilde{x}_{i,j} \widetilde{\leqq} \widetilde{s}_i, & i \in \mathfrak{S}, \\ \sum_{\{j:(j,i)\in A\}} \widetilde{x}_{j,i} \widetilde{\geqq} \widetilde{d}_i, & i \in \mathfrak{D}, \\ \sum_{\{j:(i,j)\in A\}} \widetilde{x}_{i,j} = \sum_{\{j:(j,i)\in A\}} \widetilde{x}_{j,i}, & i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}), \\ 0 \leq \widetilde{x}_{i,j} \leq \widetilde{u}_{i,j} z_{i,j}, \end{cases} \tag{50}$$

where $\widetilde{x}_{i,j} = \widetilde{u}_{i,j} y_{i,j}$ is fuzzy flow and variables $z_{i,j} \in \{0, 1\}$ are associated with the construction of link $(i, j) \in A$: $z_{i,j} = 1$ if link (i, j) belongs to the final solution of network design problem, otherwise $z_{i,j} = 0$. The objective function (49) is the sum of variable and fixed fuzzy costs. In this function, $\widetilde{c}^k_{i,j}$ is the kth linear fuzzy cost associated with fuzzy flow through link (i, j) and $\widetilde{\mu}_{i,j}$ is the fixed fuzzy cost associated with the selection of link (i, j) in the final solution. Other parameters are analogues to before. When the priority of construction costs is greater than the priority of traveling costs, a weighting parameter w may be considered together with the following bi-level model:

$$\min \sum_{(i,j)\in A} \widetilde{\mu}_{i,j} z_{i,j} + w \sum_{k=1,\ldots,K} \widetilde{f^k}(y) \tag{51}$$

s.t.

$$\widetilde{f^k}(y) = \min \sum_{(i,j)\in A} \widetilde{c}^k_{i,j}\widetilde{x}_{i,j}, \quad \forall k = 1,\ldots,K,$$

$$\text{s.t.} \begin{cases} \sum_{\{j:(i,j)\in A\}} \widetilde{x}_{i,j} \widetilde{\leqq} \widetilde{s}_i, & i \in \mathfrak{S}, \\ \sum_{\{j:(j,i)\in A\}} \widetilde{x}_{j,i} \widetilde{\geqq} \widetilde{d}_i, & i \in \mathfrak{D}, \\ \sum_{\{j:(i,j)\in A\}} \widetilde{x}_{i,j} = \sum_{\{j:(j,i)\in A\}} \widetilde{x}_{j,i}, & i \in N\backslash(\mathfrak{S}\cup\mathfrak{D}), \\ 0 \leq \widetilde{x}_{i,j} \leq \widetilde{u}_{i,j} z_{i,j}. \end{cases} \tag{52}$$

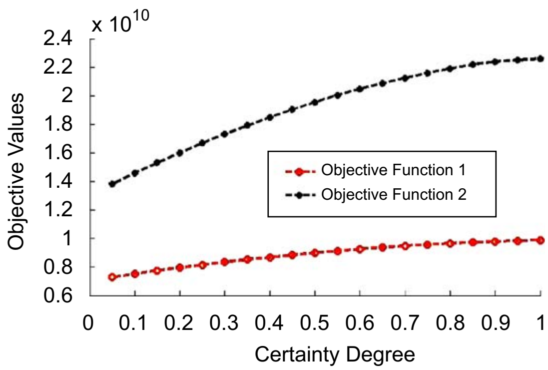

Fig. 8. The graph of objective functions (1) and (2) in great Khorasan network with respect to the different certainty degrees.

Table 5
The objective functions (1) and (2) and their goal values when Algorithm MOMS is used.

α	ϕ^1	$\zeta^{1,L}$	ϕ^2	$\zeta^{2,L}$
0.243900299	8 143 063 042	14 087 804 985	23 724 303.31	85 287 809.37
0.365850449	8 588 121 398	14 081 707 478	25 873 921.43	85 031 714.06
0.426825523	8 787 071 943	14 078 658 724	26 845 431.35	84 903 666.4
0.457313061	8 880 652 558	14 077 134 347	27 305 361.54	84 839 642.57
0.472556829	8 925 969 201	14 076 372 159	27 528 870.43	84 807 630.66
0.480178714	8 948 259 106	14 075 991 064	27 639 010.84	84 791 624.7
0.483989656	8 959 311 955	14 075 800 517	27 693 677.52	84 783 621.72
0.485895127	8 964 815 353	14 075 705 244	27 720 909.99	84 779 620.23
0.486847863	8 967 561 296	14 075 657 607	27 734 501	84 777 619.49
0.48732423	8 968 932 828	14 075 633 788	27 741 290.2	84 776 619.12
0.487562414	8 969 618 234	14 075 621 879	27 744 683.23	84 776 118.93
0.487681506	8 969 960 847	14 075 615 925	27 746 379.35	84 775 868.84
0.487741052	8 970 132 131	14 075 612 947	27 747 227.31	84 775 743.79
0.487770825	8 970 217 768	14 075 611 459	27 747 651.26	84 775 681.27
0.487785712	8 970 260 585	14 075 610 714	27 747 863.23	84 775 650.01
0.487793155	8 970 281 993	14 075 610 342	27 747 969.22	84 775 634.37

Table 6
Genetic algorithm parameters.

Parameter	Value
Population size	45
Crossover rate	0.51
Mutation rate	0.9
Threshold converge	0.001
Maximum iteration	100

It is easy to follow an enumeration scheme such as branch and bound or genetic algorithm on the upper-level model and use the MOMS and MPMS algorithms in lower-level.

In what follows we take two objective function into account: minimum cost and maximal mobility. The latter objective is also transformed into a minimization objective function. Fig. 8 illustrates the optimistic values of these two objective functions with respect to the different certainty degrees.

To design a network satisfying the future demand, we note that number of links which can be augmented to the great Khorasan network to create a complete graph is 3782. Because no exact prediction of construction costs is in hand, we produce random numbers as construction costs of these links considering our granular information. Although we consider crisp numbers as construction costs, it is easy to follow same strategy when the costs are captured as fuzzy numbers. The corresponding evaluation shows that the minimum, the maximum and the average costs of construction are 30 470, 99 928 104 and 49 461 134 units, respectively. To eliminate those of the links which cannot be constructed because of cost restrictions, we consider only links whose construction costs are less than 49 461 134 units. The number of such links is 204. The values of two objective functions for these links are constructed as random numbers with average and standard deviation close to current statues.

To find a reasonable value for certainty degree, considering current network and supplies and demands, we implement Algorithm MOMS. The results are presented in Table 5. These results show that maximal certainty degree is $\alpha_{opt} = 0.487793155$ and for each certainty degree less than α_{opt}, it is possible to satisfy the goals, supplies and demands.

We use a binary genetic algorithm [11] on this problem to find which of links may be constructed considering two objective functions. The parameters in Table 6 are considered through genetic algorithm:

To evaluate the utility function of genetic algorithm, we examine two cases considering the optimistic attitude and pessimistic attitude. We assume $\alpha = 0.25$ in calculations which is less than α_{opt}. It is a reasonable certainty degree with

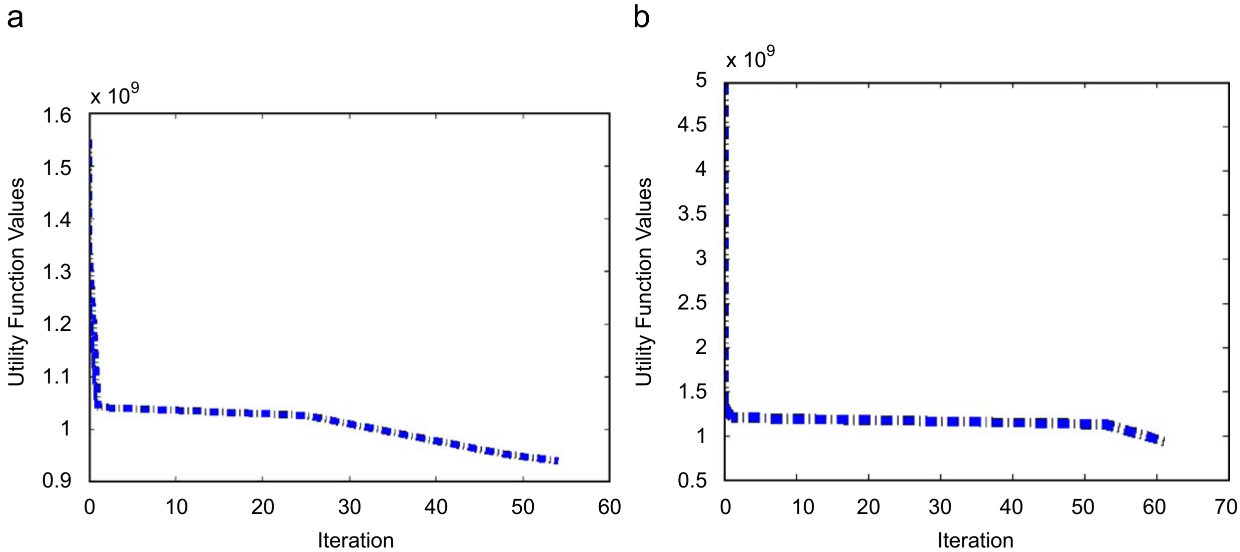

Fig. 9. (a) The convergence trend through genetic algorithm when MOMS is used in lower level of the proposed scheme of network design. (b) shows same results when MPMS algorithm is used in lower level.

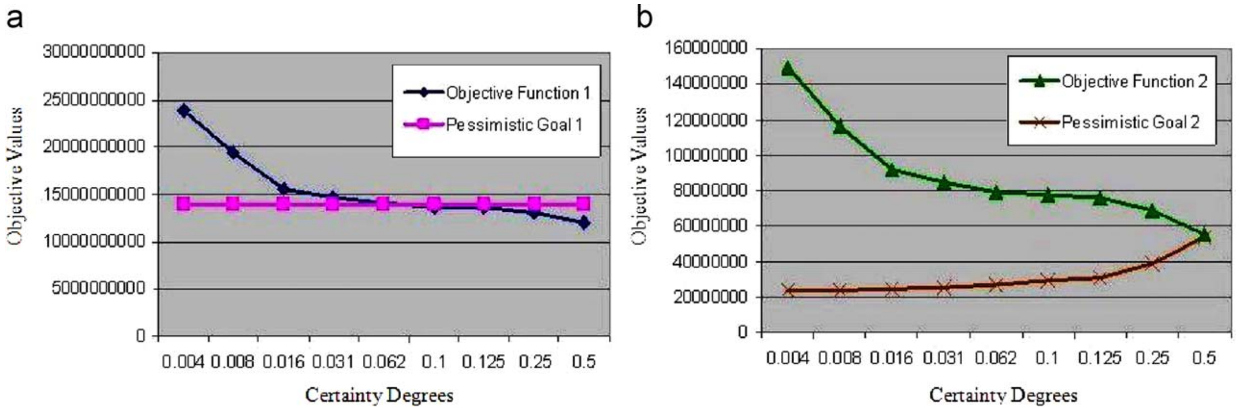

Fig. 10. The values of objective functions and their goals, (a) is referred to the first objective function and (b) is referred to the second one.

respect to industrial experts, but it can be similarly done for each level of certainty degree. When MOMS algorithm is used, the proposed genetic algorithm consumes 1716.188000 s in which 12 690 times MOMS is loaded. The trend of convergence of genetic algorithm under optimistic attitude is depicted in Fig. 9a.

In the case of using MPMS algorithm in network design algorithm for great Khorasan state network, according to Figs. 10 and 11, the first objective function satisfies its goal before satisfying the second objective function by the corresponding goal. It shows that by a higher possibility we can satisfy the first goal in comparison with the second goal. The networks corresponding to the optimistic and pessimistic attitudes are also depicted in Fig. 11a and b. The dotted links of Fig. 11 are those links which should be inserted to the current network to satisfy the supply–demand constraints when MOMS algorithm and MPMS algorithm are used. Note that since the data are random not real values, we cannot present a judgment about the reasonability of solution, however, this implementation shows that it is possible to find solutions for large-scale network design under fuzziness by applying the algorithms MOMS and MPMS.

As a final point, it is worthwhile to note that, in some cases, when the network has been constructed and the manager needs to extend this network, a sensitivity analysis scheme can be utilized on the proposed algorithms of this paper to provide appropriate solution(s) which are referred to next works. Also when there is conflicting between objective functions, it is more reasonable to take a weighted sum of objective functions and goal values into account.

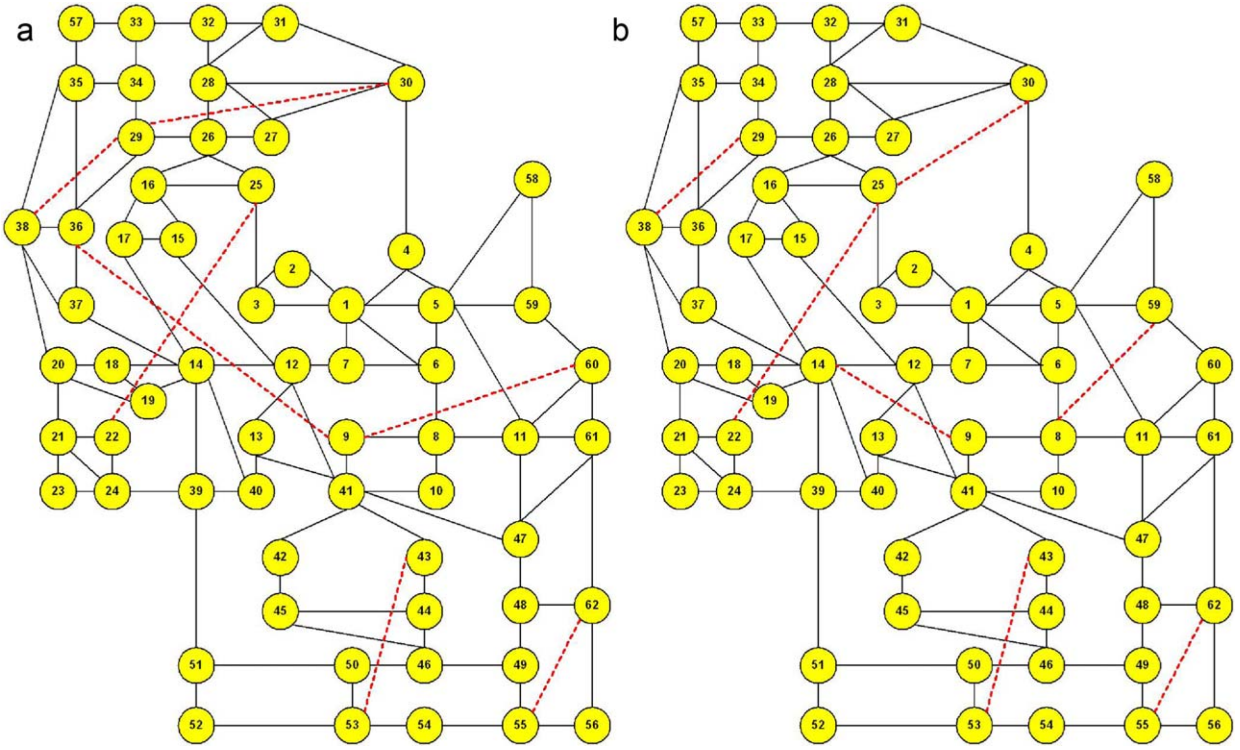

Fig. 11. Dotted links are the links which should be inserted to the network to satisfy the supply–demand constraints. (a) Network design result when MOMS algorithm is used, (b) network design result when MPMS algorithm is used.

7. Conclusion and future directions

The optimistic and pessimistic attitudes are important concepts in management. They are important for both of the suppliers and the customers. In fact, the optimistic decision for supplier is a pessimistic result for customer and a pessimistic decision for supplier causes an optimistic result for customer. In this paper, such attitudes are considered in the solving process of a fully fuzzified multi-objective MCFP. The presented model in this paper can be applied to understand the fuzzy flow through a network when multiple fuzzy costs, fuzzy capacities, fuzzy supplies and fuzzy costs are captured by decision maker. Based on the fuzzy goal programming and t-norms and t-conorms, two algorithms are also proposed to find the maximal satisficing solutions. The application of these algorithms for network design under fuzziness is also investigated. Since the network optimization is one of the new theories which has a lot of applications in real engineering problems especially in transportation, it is necessary to investigate fuzzy network optimization with a higher attention. Such investigation may be covered by the modeling and solution processes. In order to show the reason of this necessity, we mention two points. When the amount of supplies and demands are defined imprecisely, the flow should be fuzzy. In these cases, some reasonable defuzzifier should be employed to interpret the fuzzy flow in real network problems. Also the constraint of a lot of network optimization problems consists of a multiplication of a crisp matrix including -1, 1 and 0 into a fuzzy flow. We need to find a promising interpretation about this matrix multiplication coinciding with reality. On the other hand, a lot of problems of network optimization are discrete and some of them cannot be solved in polynomial time. These problems under fuzziness which are able to capture granular information, cannot usually be solved with traditional fuzzy optimization algorithms. These concepts prove that we should focus on fuzzy network optimization independent from fuzzy linear and nonlinear programming problems. Such investigations reveal that which of the network optimization can be solved efficiently under fuzziness and which of them provides a better interpretation of real environment.

In the next works, the following directions may be followed:

- Finding customized tests to find all of the efficient solutions in the last part of Algorithm MOMS and MPMS.

- Using a combination of multiple fuzzy costs when there is conflicting between objectives.
- Introducing more efficient heuristics and meta-heuristics which can be applied to solve real large-scale fuzzy network design.

Acknowledgment

The authors would like to express particular thanks to the Area Editor, Editors-in-Chief and honorable reviewers for their useful comments which improve the quality of paper.

References

[1] W.F. Abd El-Wahed, S.M. Lee, Interactive fuzzy goal programming for multi-objective transportation problems, Omega 34 (2006) 158–166.
[2] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network Flows, Prentice-Hall, Englewood Cliffs, 1993.
[3] S.K. Bath, J.S. Dhillon, D.P. Kothari, Fuzzy satisfying stochastic multi-objective generation scheduling by weightage pattern search methods, Electric Power Systems Research 69 (2004) 311–320.
[4] P.R. Bhave, R. Gupta, Optimal design of water distribution networks for fuzzy demands, Civil Engineering and Environmental Systems 21 (2004) 229–245.
[5] Y.S. Brar, J.S. Dhillon, D.P. Kothari, Fuzzy satisfying multi-objective generation scheduling based on simplex weightage pattern search, Electrical Power and Energy Systems 27 (2005) 518–527.
[6] P.T. Chang, K.C. Hung, α-Cut fuzzy arithmetic: simplifying rules and a fuzzy function optimization with a decision variable, IEEE Transactions on Fuzzy Systems 14 (2006) 496–510.
[7] S. Chanas, D. Kuchta, Fuzzy integer transportation problem, Fuzzy Sets and Systems 98 (1998) 291–298.
[8] M. Chen, H. Ishii, C. Wu, Transportation problems on a fuzzy network, International Journal of Innovative Computing, Information and Control 4 (2008) 1105–1109.
[9] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[10] D. Dubois, H. Prade, Possibility Theory, Plenum Press, New York, 1988.
[11] M. Gen, R. Cheng, Genetic Algorithms and Engineering Optimization, Wiley, New York, 2000.
[12] M. Ghatee, S.M. Hashemi, Ranking function-based solutions of fully fuzzified minimal cost flow problem, Information Sciences 177 (2007) 4271–4294.
[13] M. Ghatee, S.M. Hashemi, Generalized minimal cost flow problem in fuzzy nature: an application in bus network planning problem, Applied Mathematical Modelling 32 (2008) 2490–2508.
[14] M. Ghatee, S.M. Hashemi, B. Hashemi, M. Dehghan, Solution and duality of imprecise network problems, Computers and Mathematics with Applications 55 (2008) 2767–2790.
[15] H.W. Hamacher, C.R. Pedersen, S. Ruzika, Multiple objective minimum cost flow problems: a review, European Journal of Operational Research 176 (2007) 1404–1422.
[16] S.M. Hashemi, M. Ghatee, B. Hashemi, Fuzzy goal programming: complementary slackness conditions and computational schemes, Applied Mathematics and Computation 179 (2006) 506–522.
[17] F. Hernandes, M.T. Lamata, J.L. Verdegay, A. Yamakami, The shortest path problem on networks with fuzzy parameters, Fuzzy Sets and Systems 158 (2007) 1561–1570.
[18] M. Inuiguchi, M. Sakawa, Possible and necessary optimality tests in possibilistic linear programming problems, Fuzzy Sets and Systems 67 (1994) 29–46.
[19] M. Inuiguchi, J. Ramik, T. Tanino, M. Vlach, Satisficing solutions and duality in interval and fuzzy linear programming, Fuzzy Sets and Systems 135 (2003) 151–177.
[20] M. Inuiguchi, Necessity measure optimization in linear programming problems with fuzzy polytopes, Fuzzy Sets and Systems 158 (2007) 1882–1891.
[21] F. Jimenez, J.L. Verdegay, Solving fuzzy solid transportation problems by an evolutionary algorithm based parametric approach, European Journal of Operational Research 117 (1999) 485–510.
[22] B. Julien, An extension to possibilistic linear programming, Fuzzy Sets and Systems 64 (1994) 195–206.
[23] D.P. Kothari, J.S. Dhillon, Power System Optimization, Prentice-Hall, India, 2004.
[24] S. Li, C. Hu, Two-step interactive satisfactory method for fuzzy multiple objective optimization with preemptive priorities, IEEE Transactions on Fuzzy Systems 15 (2007) 417–425.
[25] Y. Li, K. Ida, M. Gen, Improved genetic algorithm for solving multiobjective solid transportation problem with fuzzy numbers, Computers & Industrial Engineering 33 (1997) 589–592.
[26] T.F. Liang, Interactive multi-objective transportation planning decisions using fuzzy linear programming, Asia-Pacific Journal of Operational Research 25 (2008) 11–31.
[27] S.T. Liu, C. Kao, Solving fuzzy transportation problems based on extension principle, European Journal of Operational Research 153 (2003) 661–674.
[28] S.T. Liu, C. Kao, Network flow problems with fuzzy arc lengths, IEEE Transactions on Systems, Man and Cybernetics–Part B: Cybernetics 34 (2004) 765–769.
[29] X. Liu, Measuring the satisfaction of constraints in fuzzy linear programming, Fuzzy Sets and Systems 122 (2001) 263–275.

[30] J.A.M. Perez, J.M.M. Vega, J.L. Verdegay, Fuzzy location problems on networks, Fuzzy Sets and Systems 142 (2004) 393–405.
[31] J. Ramik, H. Rommelfanger, Fuzzy mathematical programming based on some new inequality relations, Fuzzy Sets and Systems 81 (1996) 77–88.
[32] J. Ramik, Multi-index transportation problem with fuzzy coefficients and fuzzy decision variables, Tatra Mountains Mathematical Publications 12 (1997) 151–168.
[33] J. Ramik, Fuzzy goals and fuzzy alternatives in goal programming problems, Fuzzy Sets and Systems 111 (2000) 81–86.
[34] J. Ramik, M. Vlach, Generalized Concavity in Fuzzy Optimization and Decision Analysis, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
[35] J. Ramik, Duality in fuzzy linear programming with possibility and necessity relations, Fuzzy Sets and Systems 157 (2006) 1283–1302.
[36] J. Ramik, Optimal solutions in optimization problem with objective function depending on fuzzy parameters, Fuzzy Sets and Systems 158 (2007) 1873–1881.
[37] N. Safaei, M. Saidi-Mehrabad, R. Tavakkoli-Moghaddam, F. Sassani, A fuzzy programming approach for a cell formation problem with dynamic and uncertain conditions, Fuzzy Sets and Systems 159 (2008) 215–236.
[38] H.S. Shih, E.S. Lee, Fuzzy multi-level minimum cost flow problems, Fuzzy Sets and Systems 107 (1999) 159–176.
[39] L.H. Shih, Cement transportation planning via fuzzy linear programming, International Journal of Production Economics 58 (1999) 277–287.
[40] B.L. Smith, W.T. Scherer, J.H. Conklin, Exploring imputation techniques for missing data in transportation management systems, in: 82nd Annu. Meeting of the Transportation Research Board, 2002.
[41] H. Tanaka, P. Guo, H.J. Zimmermann, Possibility distributions of fuzzy decision variables obtained from possibilistic linear programming problems, Fuzzy Sets and Systems 113 (2000) 323–332.
[42] S.A. Torabi, E. Hassini, An interactive possibilistic programming approach for multiple objective supply chain master planning, Fuzzy Sets and Systems 159 (2008) 193–214.
[43] G.-H. Tzeng, D. Teodorovi, M.-J. Hwang, Fuzzy bicriteria multi-index transportation problems for coal allocation planning of Taipower, European Journal of Operational Research 95 (1996) 62–72.
[44] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3–28.
[45] M. Zeleny, Multiple Criteria Decision Making, McGraw-Hill, New York, 1982.