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Fuzzy linear programming and applications

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Abstract

This paper presents a survey on methods for solving fuzzy linear programs. First LP models with soft constraints are discussed. Then LP problems in which coefficients of constraints and/or of the objective function may be fuzzy are outlined. Pivotal questions are the interpretation of the inequality relation in fuzzy constraints and the meaning of fuzzy objectives. In addition to the commonly applied extended addition, based on the min-operator and used for the aggregation of the left-hand sides of fuzzy constraints and fuzzy objectives, a more flexible procedure, based on Yager's parametrized t-norm T_p, is presented. Finally practical applications of fuzzy linear programs are listed.

Keywords: Fuzzy sets; Mathematical Programming; Extended addition of fuzzy intervals; Compromise solution; Inequality relation in fuzzy constraints

1. Introduction

Empirical surveys reveal that Linear Programming is one of the most frequently applied OR techniques in real-world problems, see, e.g. Kivijärvi, Korhonen and Wallenius (1986), Lilien (1987), Tingley (1987) and Meyer zu Selhausen (1989). However, given the power of LP one could have expected even more applications. This might be due to the fact that LP requires much well-defined and precise data which involves high information costs. In real-world applications certainty, reliability and precision of data is often illusory. Furthermore the optimal solution of an LP only depends on a limited number of constraints and, thus, much of the information collected has little impact on the solution. Being able to deal with vague and imprecise data may greatly contribute to the diffusion and application of LP. The use of probability distributions has not proved very useful in doing so. However, since the seminal paper "Fuzzy sets" by Lofti A. Zadeh in 1965, there exists a convenient and powerful way of modeling vague data without having recourse to stochastic concepts. The subject of this paper is to review how fuzzy data can be integrated into LP systems.

In order to reduce information costs and at the same time avoid unrealistic modeling, the use of fuzzy linear programs can be recommended. Their application implies that the problems will be solved in an interactive way. In the first step the fuzzy system is modeled by using only the information which the decision maker can provide without any expensive additional information acquisition. Knowing a first 'compromise solution' the decision maker can perceive which further information should be obtained and he is able to justify the decision

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by comparing carefully additional advantages and arising costs. In doing so, step by step the compromise solutions are improved. This procedure obviously offers the possibility to limit the acquisition and processing of information to the relevant components and therefore information costs will be distinctly reduced.

A general model of a fuzzy linear programming problem (FLP-problem) is presented by the following system: [1]

$$\tilde{C}_1 x_1 \oplus \tilde{C}_2 x_2 \oplus \cdots \oplus \tilde{C}_n x_n \;\to\; \tilde{\text{Max}} \tag{1}$$

$$\text{subject to} \quad \tilde{A}_{i1} x_1 \oplus \tilde{A}_{i2} x_2 \oplus \cdots \oplus \tilde{A}_{in} x_n \tilde{\leqslant} \tilde{B}_i, \quad i = 1, \ldots, m,$$
$$x_1, x_2, \ldots, x_n \geqslant 0.$$

$\tilde{A}_{ij}$, $\tilde{B}_i$, $\tilde{C}_j$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, are *fuzzy sets* in $\mathbb{R}$. The symbol $\oplus$ represents the *extended addition* explained in Section 4. The interpretation of the *inequality relation* $\tilde{\leqslant}$ is discussed in Sections 2 and 5.

As each real number a can be modeled as a fuzzy number

$$\tilde{A} = \{(x, f_A(x)) \mid x \in \mathbb{R}\} \quad \text{with } f_A(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{else,} \end{cases}$$

the general system (1) includes the special cases where:

1. The objective function is crisp, i.e.

$$z(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \;\to\; \text{Max}. \tag{2}$$

2. Some or all constraints are crisp, i.e.

$$g_i(\boldsymbol{x}) = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leqslant b_i. \tag{3}$$

3. Some or all constraints have the soft form

$$g_i(\boldsymbol{x}) = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \tilde{\leqslant} \tilde{B}_i. \tag{4}$$

These special cases may be combined.

The application of FLP-systems offers the advantage that the decision maker can model his problem in accordance to his current state of information. At the same time he is no longer able to use the well known simplex algorithms for computing a solution of his problem. Therefore various procedures for calculating a compromise solution of an FLP-system (1) have been developed. They mainly differ in the assumptions made in order to reduce the FLP to a classical mathematical optimization problem.

In this paper we present a survey on procedures for solving FLP-problems. First we deal with the simplest case, LP-models with soft constraints, for getting an idea of the handling of fuzzy optimization problems. We then tackle the essential problems using FLP-systems:

- modeling of fuzzy data;
- extended addition for aggregating fuzzy objectives and left-hand sides of fuzzy constraints;
- inequality relations between fuzzy sets in constraints;
- treatment of fuzzy objectives;
- extended addition based on Yager's t-norm T_p; and
- computing of a compromise solution.

Subsequently we give a survey of applications of fuzzy linear programs published in the literature.

[1] Basics of fuzzy set theory are presented in the Appendix, in order to assist in understanding the main issues of this paper.

2. Linear Programming with soft constraints

We get the simplest form of FLP-models if the decision maker is able to specify all coefficients, but not all right-hand sides of the constraints by crisp numbers. Such systems with soft constraints of the type

$$g_i(\boldsymbol{x}) = g_i(x_1, x_2, \dots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \lessapprox \tilde{B}_i \tag{4}$$

were discussed for the first time by Zimmermann (1975), who described the imprecise right-hand side $\tilde{B}_i$ by a fuzzy set with the support $[b_i, b_i + d_i] \subseteq \mathbb{R}$, $d_i \geqslant 0$, and a monotone decreasing membership function μ_{B_i}.

Moreover the membership function μ_{B_i} must be specified so that the function

$$\mu_{D_i}(g_i) = \begin{cases} 1 & \text{if } g_i < b_i, \\ \mu_{B_i}(g_i) & \text{if } b_i \leqslant g_i \leqslant b_i + d_i, \\ 0 & \text{if } b_i + d_i < g_i, \end{cases} \tag{5}$$

expresses the individual satisfaction of the decision maker in relation to $g_i = g_i(x_1, \dots, x_n)$.

The composition of the functions $\mu_{D_i}(g_i)$ and $g_i = g_i(\boldsymbol{x}) = a_{i1}x_1 + \cdots + a_{in}x_n$ to $\mu_i(\boldsymbol{x}) = \mu_{D_i} \circ g_i(\boldsymbol{x}) = \mu_{D_i}(g_i(\boldsymbol{x}))$ directly assigns a measure of the satisfaction of the i-th constraint to the solution $\boldsymbol{x} = (x_1, \dots, x_n)$.

According to Zimmermann and other authors, the inequality relation '$\lessapprox$' in soft constraints (4) may be interpreted as

$$g_i(\boldsymbol{x}) \lessapprox \tilde{B}_i \Leftrightarrow \begin{cases} g_i(\boldsymbol{x}) \leqslant b_i + d_i, \\ \mu_{D_i}(\boldsymbol{x}) \to \text{Max}, \end{cases} \tag{6}$$

i.e. each soft constraint adds an additional objective to the decision problem, called fuzzy objective in the literature.

There exist various propositions for modeling the function $\mu_{D_i}(g_i)$ on the interval $[b_i, b_i + d_i]$, e.g.:

(i) *linear shape* (Zimmermann, 1975; Sommer, 1978; Werners, 1984);
(ii) *concave shape:*
   (a) by exponential functions (Sakawa, 1983; Zimmermann, 1978);
   (b) by piecewise linear functions (e.g. Hannan, 1981; Rommelfanger, 1984; Nakamura, 1984; Sakawa and Yano, 1990);
(iii) *s-shape:*
   (a) by piecewise linear functions (e.g. Hannan, 1981; Rommelfanger, 1984);
   (b) hyperbolic functions (Leberling, 1981, 1983; Sakawa and Yano, 1990);
   (c) by hyperbolic inverse functions (Sakawa and Yano, 1990);
   (d) by logistic functions (Zimmermann and Zysno, 1982);
   (e) by cubic functions (Schwab, 1983).

Therefore, a linear programming system of type

$$\begin{aligned} & z(\boldsymbol{x}) = c_1x_1 + \cdots + c_nx_n \to \text{Max} \\ & \text{subject to} \quad a_{i1}x_1 + \cdots + a_{in}x_n \lessapprox \tilde{B}_i, \quad i = 1, \dots, m_1, \\ & \qquad a_{i1}x_1 + \cdots + a_{in}x_n \leqslant b_i, \quad i = m_1 + 1, \dots, m, \\ & \qquad x_1, \dots, x_n \geqslant 0, \end{aligned} \tag{7}$$

with m_1 soft constraints and $m - m_1$ crisp constraints may be described more precisely by a multiobjective optimization system of the type

$$\underset{\boldsymbol{x} \in X_U}{\text{Max}} \left(z(\boldsymbol{x}), \mu_1(\boldsymbol{x}), \dots, \mu_{m_1}(\boldsymbol{x}) \right) \tag{8}$$

where

$$X_U = \{x \in \mathbb{R}_0^n \mid a_{i1}x_1 + \cdots + a_{in}x_n \leqslant b_i + d_i, \forall i = 1, 2, \ldots, m_1, \text{ and } a_{i1}x_1 + \cdots a_{in}x_n \leqslant b_i, \forall i = m_1 + 1, \ldots, m\}.$$

For comparing the given objective function $z(x)$ with the fuzzy objective functions $\mu_i(x)$, it is usually proposed to substitute $z(x)$ with a function $\mu_Z(x)$, where the quantification of $\mu_Z(x)$ is obtained by specifying the membership function $\hat{\mu}_Z(z)$ of $\tilde{Z} = \{(z, \hat{\mu}_Z(z)) \mid z \in \mathbb{R}\}$ – the fuzzy set of the satisfying values, i.e. $\mu_Z(x) = \hat{\mu}_Z(z(x))$.

Basic data of $\hat{\mu}_Z(z)$ are

$$\bar{z} = \operatorname*{Max}_{x \in X_U} z(x) \quad \text{and} \quad \underline{z} = \operatorname*{Max}_{x \in X_L} z(x),$$

with $X_L = \{x \in \mathbb{R}_0^n \mid a_{i1}x_1 + \cdots + a_{in}x_n \leqslant b_i, \forall i = 1,2, \ldots, m\}$, and the basic shape of $\hat{\mu}_Z(z)$ is given by

$$\hat{\mu}_Z(z) = \begin{cases} 0 & \text{if } z < \underline{z}, \\ \bar{\mu}_Z(z) & \text{if } \underline{z} \leqslant z \leqslant \bar{z}, \\ 1 & \text{if } \bar{z} < z, \end{cases}$$

where $\bar{\mu}_Z(z)$ is a monotone increasing function of z, which may be modeled in analogy to $\mu_{D_i}(g_i)$.

In practical applications a DM is not interested in finding the complete solution of the multiobjective system (8), i.e. the set of all Pareto optimal solutions, but he needs a procedure which generates a so-called 'compromise solution'. To determine a compromise solution, it is usually assumed in the literature that the total satisfaction of a decision maker may be described by

$$\lambda(x) = \min(\mu_Z(x), \mu_1(x), \ldots, \mu_{m_1}(x)). \tag{9}$$

Empirical researches reveal that the min-operator is often too pessimistic, see, e.g. Zimmermann and Zysno (1979). However, not only the simple mathematical handling supports the use of this operator, but also the fact that the subjective specified membership values are only on an ordinal scale level, which means that only simple data comparisons are possible. Therefore the use of the mean value, as proposed by Sommer (1978), the añd-operator (see Werners, 1984), or a combination of min-operator and bounded sum (see Oder and Rentz, 1993), raises serious measurement problems.

In contrast to conventional mathematical programs, a not necessarily linear optimization system

$$\operatorname*{Max}_{x \in X_U} \operatorname{Min}(\mu_Z(x), \mu_1(x), \ldots, \mu_{m_1}(x)) \tag{10}$$

treats the objective in the same manner as the soft constraints, which explains why this approach is called *symmetric model* in the literature.

Following Negoita and Sularia (1976), the optimization system (10) is clearly equivalent to

$$\begin{aligned} &\lambda \to \text{Max} \\ &\text{subject to} \quad \lambda \leqslant \mu_Z(x), \\ &\qquad \lambda \leqslant \mu_i(x), \quad i = 1, \ldots, m_1, \\ &\qquad x \in X_U, \quad \lambda \leqslant 1. \end{aligned} \tag{11}$$

Using linear membership functions (see, e.g. Zimmermann, 1975; Werners, 1984) or piecewise linear, concave membership functions (see Rommelfanger, 1984), the system (11) matches a classical LP-model and can easily be solved by well-known algorithms.

The structure of this solution process suggests that it may easily be extended to multicriteria LP-systems with crisp or soft constraints, see, e.g. Zimmermann (1978), Leberling (1981, 1983), Werners (1984) and Rommelfanger (1984, 1988).

Some authors recommend to organize the solution process as an interactive process. Werners (1984) for example intends to ensure by this procedure that the calculated solution is a Pareto-optimal solution of (8), because this attribute is not necessarily given by using the min-operator as preference function. The interactive solution process MOLPAL of Rommelfanger (1988) is based on the idea that, at the beginning, a decision maker is just able to roughly approximate the membership functions, whereas he can later on specify the functions in detail referring to additional information of the solution process.

3. Modeling fuzzy data

In the literature the flexible right-hand side $\tilde{B}_i$ are modeled by L-R-type fuzzy numbers, where the left spread is zero and b_i is the largest value, which is accepted with certainty as the right-hand side of the constraint i, i.e. $\tilde{B}_i = (b_i, 0, \beta_i)_{RR}$; see the definitions and Figs. A.1 and A.2 in the Appendix. Obviously, the meaning of $\tilde{B}_i$ as the greatest value for the right-hand side does not permit to model $\tilde{B}_i$ as a trapezoid fuzzy number or as a fuzzy number with a positive left spread, as can be found in early papers on the subject.

Suitable reference functions are the functions presented above in Section 1, which are confirmed by utility theory. Yet the essential problems are neglected: satisfaction values can be measured only on an ordinal scale and interpersonal comparisons of membership values can only be attained with great difficulties. Furthermore the interpretation and evaluation of the compromise solution λ_{Max} calculated by system (10) are to be questioned. It is a pity that the problem of specifying the membership functions is often ignored in the literature because this point is an essential criterion for the approval and application of fuzzy models. In our opinion there exist only few cases when the decision maker is capable of describing the precise form of a fuzzy number $\tilde{B}_i = (b_i, 0, \beta_i)_{RR}$. In practice we will find only more or less successful approximations of the 'true' shape of membership functions. In particular it is very difficult to model realistically the part of a membership function belonging to small membership values.

Therefore we propose the following procedure as a practical way of getting suitable membership functions:

First the DM specifies some prominent membership values and associates them with special meanings. This procedure shall be explained for a *right-hand side* $\tilde{B}_i = \{(y, \mu_{B_i}(y)) \mid y \in \mathbb{R}\}$, which is interpreted as the maximal quantity of stock at the DM's disposal.

- $\alpha = 1$: $\mu_{B_i}(y) = 1$ means that y belongs with certainty to the set of available values.
- $\alpha = \lambda_A$: $\mu_{B_i}(y) \geqslant \lambda_A$ means that the decision maker is willing to accept y as an available value for the time being. A value y with $\mu_{B_i}(y) \geqslant \lambda_A$ has a good chance of belonging to the set of available values. Corresponding values of y are relevant to the decision. Obviously, a value y with $\mu_{B_i}(y) = \lambda_A$ is a sort of aspiration level.
- $\alpha = \varepsilon$: $\mu_{B_i}(y) < \varepsilon$ means that y has only very little chance of belonging to the set of available values. The decision maker is willing to neglect the values y with $\mu_{B_i}(y) < \varepsilon$.

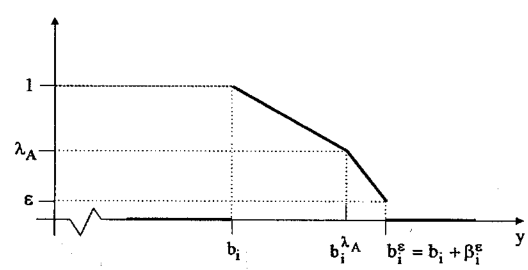


Fig. 1. Membership function of $\tilde{B}_i$.

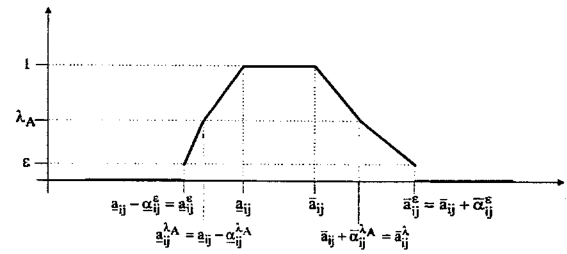


Fig. 2. $\tilde{A}_{ij} = (\underline{a}_{ij};\ \overline{a}_{ij};\ \underline{\alpha}^{\lambda_A}_{ij},\ \alpha^{\varepsilon}_{ij};\ \overline{\alpha}^{\lambda_A}_{ij},\ \overline{\alpha}^{\varepsilon}_{ij})^{\lambda_A, \varepsilon}$.

Subsequently the decision maker has to fix values $\overline{b}_i^{\lambda_A}$ and $\overline{b}_i^{\varepsilon}$ such that $\mu_{B_i}(\overline{b}_i^{\lambda_A}) = \lambda_A$ and $\mu_{B_i}(\overline{b}_i^{\varepsilon}) = \varepsilon$ Then the polygon line from $(b_i, 1)$ over $(\overline{b}_i^{\lambda_A}, \lambda_A)$ to $(\overline{b}_i^{\varepsilon}, \varepsilon)$ is a suitable approach to μ_{B_i} on the interval $[b_i, \overline{b}_i^{\varepsilon}]$. For all $y \notin [b_i, \overline{b}_i^{\varepsilon}]$ we set $\mu_{B_i}(y) = 0$; see Fig. 1.

Taking the pattern from L-R-type fuzzy numbers we symbolize a fuzzy number with this special membership function by $\tilde{B}_i = (b_i;\ 0,\ 0;\ \overline{\beta}_i^{\lambda_A},\ \overline{\beta}_i^{\varepsilon})^{\lambda_A, \varepsilon}$, where $\overline{\beta}_i^{\lambda_A} = \overline{b}_i^{\lambda_A} - b_i$ and $\overline{\beta}_i^{\varepsilon} = \overline{b}_i^{\varepsilon} - b_i$. If required the DM can specify additional membership levels and additional points $(y, f_{B_i}(y))$ on the polygon line.

Fuzzy coefficients $\tilde{A}_{ij}$ or $\tilde{C}_{kj}$ do not usually include elements that may be realized with certainty. An appropriate point of reference for a fuzzy set $\tilde{A}_{ij}$ is the subset $[\underline{a}_{ij}, \overline{a}_{ij}] \in \mathbb{R}$ consisting of the real numbers with the highest chance of realization, i.e.

$$\mu_{A_{ij}}(y) \begin{cases} = 1 & \text{if } y \in [\underline{a}_{ij}, \overline{a}_{ij}], \\ < 1 & \text{else.} \end{cases}$$

Accordingly the DM should specify numbers $\underline{a}^{\lambda_A}_{ij}$, $\overline{a}^{\lambda_A}_{ij}$, $\underline{a}^{\varepsilon}_{ij}$, $\overline{a}^{\varepsilon}_{ij}$, so that

$$\mu_{A_{ij}}(y) \begin{cases} \geqslant \lambda_A & \text{if } y \in [\underline{a}^{\lambda_A}_{ij}, \overline{a}^{\lambda_A}_{ij}], \\ < \lambda_A & \text{else,} \end{cases} \quad \text{and} \quad \mu_{A_{ij}}(y) \begin{cases} \geqslant \varepsilon & \text{if } y \in [\underline{a}^{\varepsilon}_{ij}, \overline{a}^{\varepsilon}_{ij}], \\ < \varepsilon & \text{else.} \end{cases}$$

The width of the intervals $[\underline{a}^{\alpha}_{ij}, \overline{a}^{\alpha}_{ij}]$, $\alpha = 1, \lambda_A, \varepsilon$, is inversely linked with the amount of information available to the decision maker. The special case in which $\underline{a}_{ij} = \overline{a}_{ij}$ is also imaginable, but in our opinion it is less realistic to assume that all coefficients $\tilde{A}_{ij}$ are fuzzy numbers as it was presumed by Ramik and Rimanek (1985) and Slowinski (1986).

Consequently the polygon line from $(\underline{a}^{\varepsilon}_{ij}, \varepsilon)$ over $(\underline{a}^{\lambda_A}_{ij}, \lambda_A)$, $(\underline{a}_{ij}, 1)$, $(\overline{a}_{ij}, 1)$, $(\overline{a}^{\lambda_A}_{ij}, \lambda_A)$ to $(\overline{a}^{\varepsilon}_{ij}, \varepsilon)$ is a suitable approach to the membership function of $\tilde{A}_{ij}$ on the support of $[\underline{a}^{\varepsilon}_{ij}, \overline{a}^{\varepsilon}_{ij}]$; see Fig. 2.

In comparison to the right-hand sides $\tilde{B}_i$, the spreads $\underline{\alpha}^{\varepsilon}_{ij} = \underline{a}_{ij} - \underline{a}^{\varepsilon}_{ij}$ and $\overline{\alpha}^{\varepsilon}_{ij} = \overline{a}^{\varepsilon}_{ij} - \overline{a}_{ij}$ of the coefficients $\tilde{A}_{ij}$ are relatively small, so one often skips level λ_A and uses coefficients of the simple type $\tilde{A}_{ij} = (\underline{a}_{ij};\ \overline{a}_{ij};\ \underline{\alpha}^{\varepsilon}_{ij};\ \overline{\alpha}^{\varepsilon}_{ij})^{\varepsilon}$ or $\tilde{C}_j = (\underline{c}_j;\ \overline{c}_j;\ \gamma_j^{\varepsilon};\ \overline{\gamma}_j^{\varepsilon})^{\varepsilon}$.

4. Aggregation of the left-hand sides of fuzzy constraints

The left-hand side of a fuzzy constraint

$$\tilde{A}_{i1} x_1 \oplus \tilde{A}_{i2} x_2 \oplus \cdots \oplus \tilde{A}_{in} x_n \lessapprox \tilde{B}_i \tag{12}$$

can be aggregated to a fuzzy set $\tilde{A}_i(x)$ by Zadeh's extension principle

$$f_{\tilde{A} * \tilde{B}}(z) = \sup_{z = x * y} T(f_A(x), f_B(y)), \quad z \in \mathbb{R}, \tag{13}$$

where $*$ is a real operation $*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $T: [0, 1] \times [0, 1] \to [0, 1]$ is any given T-norm; see the Appendix.

In the literature the min T-norm is generally applied. Then, if all coefficients $\tilde{A}_{ij}$ of the i-th constraint are fuzzy intervals of the same L-R-type, the left-hand side can be consolidated to a fuzzy interval with the same reference functions. Especially for coefficients of type $\tilde{A}_{ij} = (\underline{a}_{ij}; \bar{a}_{ij}; \underline{\alpha}^{\varepsilon}_{ij}; \bar{\alpha}^{\varepsilon}_{ij})^{\varepsilon}$, we get

$$\tilde{A}_i(x) = \tilde{A}_{i1}x_1 \oplus \tilde{A}_{i2}x_2 \oplus \cdots \oplus \tilde{A}_{in}x_n = \left(\underline{a}_i(x), \bar{a}_i(x); \underline{\alpha}^{\varepsilon}_i(x), \bar{\alpha}^{\varepsilon}_i(x)\right)^{\varepsilon},$$

with

$$\underline{a}_i(x) = \sum_{j=1}^{n} \underline{a}_{ij}x_j, \quad \bar{a}_i(x) = \sum_{j=1}^{n} \bar{a}_{ij}x_j, \quad \underline{\alpha}^{\varepsilon}_i(x) = \sum_{j=1}^{n} \underline{\alpha}^{\varepsilon}_{ij}x_j, \quad \bar{\alpha}^{\varepsilon}_i(x) = \sum_{j=1}^{n} \bar{\alpha}^{\varepsilon}_{ij}x_j.$$

Obviously the spreads $\underline{\alpha}^{\varepsilon}_i(x)$ and $\bar{\alpha}^{\varepsilon}_i(x)$ extend if number and size of the variables x_i increase. Thus the left-hand side $A_i(x)$ gets fuzzier and fuzzier. We will come back to this problem in Section 7.

5. Inequality relations

A pivotal question while determining a solution of an FLP-model is the interpretation of the inequality relation in fuzzy constraints, $\tilde{A}_i(x) \lessapprox \tilde{B}_i$. In the literature various concepts have been proposed for comparing fuzzy sets (see, e.g. Dubois and Prade, 1983; Bortolan and Degani, 1985; Rommelfanger, 1986), but all these techniques appear to be of little interest for fuzzy mathematical programming. Special interpretations of the inequality relation '$\lessapprox$' in fuzzy constraints $\tilde{A}_i(x) \lessapprox \tilde{B}_i$ are suggested for instance by Negoita and Sularia (1976), Tanaka and Asai (1984), Ramik and Rimanek (1985), Slowinski (1986), Carlson and Korhonen (1986), Luhandjula (1987), Rommelfanger (1988), Buckley (1988, 1989) and Sakawa and Yano (1989); see the survey in Rommelfanger (1989) and Lai and Hwang (1992).

In most of these approaches fuzzy constraints $\tilde{A}_i(x) \lessapprox \tilde{B}_i$ are replaced by one or two crisp linear constraints. For getting an impression of these crisp surrogates, some of them are formulated in the following as an LR-fuzzy interval $\tilde{A}_i(x) = (\underline{a}_i(x); \bar{a}_i(x); \underline{\alpha}_i(x); \bar{\alpha}_i(x))_{LR}$ and a fuzzy number $\tilde{B}_i = (b_i; 0; \beta_i)_{LL}$ or $\tilde{B}_i = (b_i; 0; \beta_i)_{RR}$:

- $\bar{a}_i(x) + \alpha_i(x)R^{-1}(\rho) \leq b_i$, $\rho \in [0, 1]$ (Tanaka and Asai, 1984; Buckley, 1988);
- $\bar{a}_i(x) + \alpha_i(x)R^{-1}(\mu) \leq b_i + \beta_i R^{-1}(\mu)$, $\mu \in [0, 1]$ (Carlson and Korhonen, 1986);
- $\bar{a}_i(x) \leq b_i$ and $\bar{a}_i(x) + \alpha_i(x)R^{-1}(\varepsilon) \leq b_i + \beta_i R^{-1}(\varepsilon)$, $\varepsilon \in [0, 1[$ (Ramik and Rimanek, 1985; Wolf, 1989);
- $\underline{a}_i - b_i \leq (\underline{\alpha}_i(x) + \beta_i)L^{-1}(\rho)$, $\rho \in]0, 1]$ (optimistic index) and $\bar{a}_i(x) + \alpha_i(x)R^{-1}(\varepsilon) \leq b_i + \beta_i L^{-1}(\varepsilon)$, $\varepsilon \in [0, 1]$ (pessimistic index) (Slowinski, 1986);
- $\bar{a}_i(x) - \alpha_i(x)L^{-1}(\alpha) \leq b_i + \beta_i R^{-1}(\alpha)$, $\alpha \in [0, 1]$ (Sakawa and Yano, 1989).

This procedure has the disadvantage that the DM has interpreted the fuzzy constraints in crisp LP-programs with no regard to the objectives. However, he can change the fixed parameters ρ, μ, ε or α in the next step of the interactive solution process.

A more flexible interpretation is proposed by Rommelfanger (1988):

$$\tilde{A}_i(x) \lessapprox_R \tilde{B}_i \iff \begin{cases} \sum_{j=1}^{n}\left(\bar{a}_{ij} + \bar{\alpha}^{\varepsilon}_{ij}\right)x_j \leq b_i + \beta^{\varepsilon}_i, & (14) \\ \mu_i(x) = \mu_{D_i}(\bar{a}_i(x)) \to \text{Max}. & (15) \end{cases}$$

It is composed of the 'pessimistic index' (14), which is used by Slowinski (1986) and other authors too, and the fuzzy goal (15). The membership function μ_{D_i} is defined according to (5) and may be interpreted as the subjective evaluation of the needed quantity $\bar{a}_i(x) = \sum_{j=1}^{n} \bar{a}_{ij}x_j$ with regard to the right-hand side $\tilde{B}_i$; see Fig. 3.

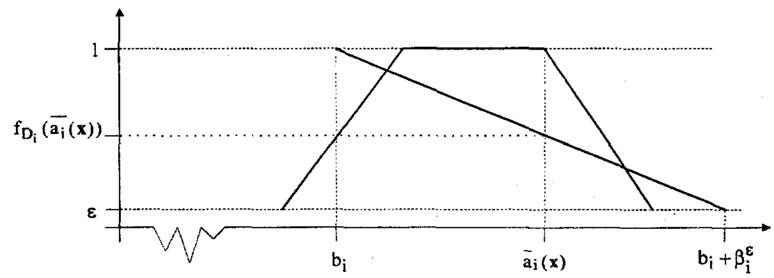


Fig. 3. Inequality relation '$\lesseqgtr_R$'.

The inequality relation '$\lesseqgtr_R$' has the advantage that a possible surplus $\bar{a}_i(x) - b_i$ directly influences the decision process. Moreover, '$\lesseqgtr_R$' coincides with the usual interpretation of the inequality relation in soft constraints (6); in the special case of deterministic inequalities it corresponds to the classical $\leqslant$ relation. Thus $\lesseqgtr_R$ is a general definition for inequality relations in optimization models.

When using the inequality relation $\lesseqgtr_R$, it is sufficient for all $\tilde{A}_{ij}$ to specify only the values $\bar{a}_{ij}$ and $\bar{a}^\varepsilon_{ij} = \bar{a}_{ij} + \bar{\alpha}^\varepsilon_{ij}$.

Obviously the influence of the extended addition, based on the min T-norm, on the feasible solution of the optimization system (1) depends on the interpretation of the inequality relation.

Using the pessimistic relation (4), as Slowinski (1986), Ramik and Rimanek (1985) (with $\varepsilon = 0$) and Rommelfanger (1988) propose, the set of feasible solutions of the FLP-problem (1) shrinks when the number and size of variables x_i increase, i.e. the pessimistic character of (4) is intensified by using the pessimistic min-norm.

On the other hand, the optimistic character of other interpretations are intensified too. This is true for the α-possible feasibility of Luhandjula and the G-α-Pareto optimal solution of Sakawa and Yano, but also the 'optimistic index' of Slowinski loses its restrictive character in case $\underline{\alpha}_i(x)$ increases.

6. Maximizing fuzzy objectives

It seems clear that a fuzzy objective function

$$\tilde{Z}(x) = \tilde{C}_1 x_1 \oplus \cdots \oplus \tilde{C}_n x_n \;\to\; \widetilde{\text{Max}} \tag{16}$$

should be interpreted as a multiobjective demand.

Even in the simple case in which the coefficients $\tilde{C}_j$ have the form $\tilde{C}_j = (\underline{c}_j;\ \bar{c}_j;\ \underline{\gamma}^\varepsilon_j;\ \bar{\gamma}^\varepsilon_j)^\varepsilon$ and $\tilde{Z}(x)$ can be written as

$$\tilde{Z}(x) = \left(\underline{c}(x);\ \bar{c}(x);\ \underline{\gamma}^\varepsilon(x);\ \bar{\gamma}^\varepsilon(x)\right)^\varepsilon, \tag{17}$$

with

$$\underline{c}(x) = \sum_{j=1}^n \underline{c}_j x_j,\quad \underline{\gamma}^\varepsilon(x) = \sum_{j=1}^n \underline{\gamma}_j x_j,\qquad \bar{c}(x) = \sum_{j=1}^n \bar{c}_j x_j,\quad \bar{\gamma}^\varepsilon(x) = \sum_{j=1}^n \bar{\gamma}_j x_j,$$

the fuzzy objective function (16) implies that the four goals

$$\begin{aligned} &\underline{c}(x) \to \text{Max}, \quad \underline{c}(x) - \underline{\gamma}^\varepsilon(x) \to \text{Max},\\ &\bar{c}(x) \to \text{Max}, \quad \bar{c}(x) + \bar{\gamma}^\varepsilon(x) \to \text{Max} \end{aligned} \tag{18}$$

should be satisfied simultaneously on the set of feasible solutions X. In general, an ideal solution to this problem (i.e. a solution maximizing all objectives at the same time) does not exist.

In the special case of triangular coefficients $\tilde{C}_j = (c_j;\ \underline{\gamma}_j;\ \overline{\gamma}_j)$, the set (18) corresponds to the three objectives

$$z_1(x) = c(x) \ \rightarrow\ \text{Max}, \quad z_2(x) = \underline{\gamma}(x) \ \rightarrow\ \text{Min}, \quad z_3(x) = \overline{\gamma}(x) \ \rightarrow\ \text{Max}.$$

Lai and Hwang (1992) proposed to substitute these objectives by fuzzy objective functions with linear membership functions, in which the basic values, the positive and the negative ideal solutions, should be calculated as

$$z_1^{\text{NIS}} = \underset{x\in X}{\text{Min}}\ c(x), \qquad z_2^{\text{NIS}} = \underset{x\in X}{\text{Max}}\ \underline{\gamma}(x), \qquad z_3^{\text{NIS}} = \underset{x\in X}{\text{Min}}\ \overline{\gamma}(x),$$

$$z_1^{\text{PIS}} = c(x_{\text{I}}) = \underset{x\in X}{\text{Max}}\ c(x), \qquad z_2^{\text{PIS}} = \underline{\gamma}(x_{\text{II}}) = \underset{x\in X}{\text{Min}}\ \underline{\gamma}(x), \qquad z_3^{\text{PIS}} = \overline{\gamma}(x_{\text{III}}) = \underset{x\in X}{\text{Max}}\ \overline{\gamma}(x).$$

These definitions, especially those of the values z_1^{NIS}, z_2^{NIS} and z_3^{NIS}, are not very convincing. In general, these values are too small and too big and therefore the linear membership functions do not seem to be adequately modeled. A better proposal would be

$$z_1^{\text{NIS}} = \underset{x\in X}{\text{Min}}\left[c(x_{\text{II}}),\ c(x_{\text{III}})\right], \quad z_2^{\text{NIS}} = \underset{x\in X}{\text{Max}}\left[\underline{\gamma}(x_{\text{I}}),\ \underline{\gamma}(x_{\text{III}})\right], \quad z_3^{\text{NIS}} = \underset{x\in X}{\text{Min}}\left[\overline{\gamma}(x_{\text{I}}),\ \overline{\gamma}(x_{\text{II}})\right].$$

The first method for getting a 'compromise solution' of (15) was proposed by Tanaka, Ichihashi and Asai (1984). They substitute the fuzzy objective with the crisp 'compromise objective'

$$z(x) = \frac{1}{6}\sum_{j=1}^{n}\left(2\underline{c}_j + 2\overline{c}_j + \underline{\gamma}_j + \overline{\gamma}_j\right)x_j.$$

Another procedure to compute a 'compromise solution' of (1) is the 'α-level related pair formation' which is based on a few crisp objective functions in analogy to (18) (see Rommelfanger, Hanuscheck and Wolf, 1989).

Sakawa and Yano (1989) propose to calculate an 'α-Pareto-optimal solution' by restricting the coefficients $\tilde{C}_j$ to α-level-sets $C_j^\alpha = [\underline{c}_j^\alpha,\ \overline{c}_j^\alpha]$. A similar concept is the 'β-possibility efficient solution' of Luhandjula (1987). As these authors do not explain the specification of the levels α or β and use several restrictive assumptions, we refrain from presenting here these proposals in detail. For example, Sakawa and Yano assume that the DM is able to specify the membership function of the coefficients of the objective function so precisely that it is possible to calculate the trade-off rates between the objectives by calculating derivatives of the corresponding membership functions.

As an ideal solution of (16) on a set of feasible solutions does not generally exist, Slowinski (1986) and Rommelfanger (1988) suggest to calculate a satisfying solution, a procedure which corresponds to the usual way of acting in practice.

In analogy to modeling a right-hand side $\tilde{B}_i$, a fuzzy aspiration level $\tilde{N}$ can be described as

$$\tilde{N} = (n;\ \nu^\varepsilon;\ 0)^\varepsilon. \tag{19}$$

Then, the satisfying condition

$$\tilde{N} \lessapprox \tilde{Z}(x) \tag{20}$$

is treated as an additional fuzzy constraint. In accordance to the chosen inequality-interpretation, (20) can be substituted by crisp inequalities or in case of '$\lessapprox_R$' by a crisp linear inequality and a fuzzy objective function. By changing the aspiration levels, the set of efficient solutions is restricted step by step (Slowinski, 1986; Rommelfanger, 1988, pp. 249–250). Obviously it is easy to extend this approach to multicriteria problems. For supporting the practical handling of both procedures, there exist PC-softwares, e.g. FLIP (see Slowinski, 1986) or FULP (see Rommelfanger, 1991).

7. Extended addition, based on Yager's t-norm T_p

To get a more realistic extended addition of the left-hand sides of fuzzy constraints and of fuzzy objectives, Rommelfanger and Keresztfalvi (1992) recommend the use of Yager's parametrized t-norm,

$$T_p(u, v) = \max\left\{0, 1 - \left((1-u)^p + (1-v)^p\right)^{1/p}\right\}, \quad u, v \in [0, 1], \quad p > 0, \tag{21}$$

which can be adapted to special situations. In case of n variables it has the form

$$T_p(t_1, \dots, t_n) = \max\left\{0, 1 - \left(\sum_{i=1}^{n} (1-t_i)^p\right)^{1/p}\right\}, \quad t_1, \dots, t_n \in [0, 1]. \tag{22}$$

In the special case that all coefficients $\tilde{A}_{ij}$ are trapezoid fuzzy intervals of type $\tilde{A}_{ij} = (\underline{a}_{ij}; \bar{a}_{ij}; \underline{\alpha}^{\varepsilon}_{ij}, \bar{\alpha}^{\varepsilon}_{ij})^{\varepsilon}$, the following theorem holds (see Rommelfanger and Keresztfalvi, 1992):

Theorem 1. *Suppose the coefficients $\tilde{A}_{ij}$ of the left-hand sides of the inequality constraints*

$$\tilde{A}_{i1}x_1 \oplus \tilde{A}_{i2}x_2 \oplus \cdots \oplus \tilde{A}_{in}x_n \lessapprox \tilde{B}_i, \quad i = 1, \dots, m, \tag{11}$$

are trapezoid fuzzy intervals of type $\tilde{A}_{ij} = (\underline{a}_{ij}, \bar{a}_{ij}, \underline{\alpha}^{\varepsilon}_{ij}, \bar{\alpha}^{\varepsilon}_{ij})^{\varepsilon}$. If the addition is extended by Yager's t-norm T_p with $p \geq 1$, then

$$\tilde{A}_i(x) = \tilde{A}_{i1}x_1 \oplus \tilde{A}_{i2}x_2 \oplus \cdots \oplus \tilde{A}_{in}x_n = \left(\underline{a}_i(x), \bar{a}_i(x), \underline{\alpha}^{\varepsilon}_i(x), \bar{\alpha}^{\varepsilon}_i(x)\right)^{\varepsilon} \tag{23}$$

is also a fuzzy interval with linear reference functions, such that

$$\underline{a}_i(x) = \sum_{j=1}^{n} \underline{a}_{ij}x_j, \qquad \bar{a}_i(x) = \sum_{j=1}^{n} \bar{a}_{ij}x_j,$$

$$\underline{\alpha}^{\varepsilon}_i(x, p) = \|(\underline{\alpha}^{\varepsilon}_{i1}x_1, \dots, \underline{\alpha}^{\varepsilon}_{in})\|_q = \left((\underline{\alpha}^{\varepsilon}_{i1}x_1)^q + \cdots + (\underline{\alpha}^{\varepsilon}_{in}x_n)^q\right)^{1/q}, \tag{24}$$

$$\bar{\alpha}^{\varepsilon}_i(x, p) = \|(\bar{\alpha}^{\varepsilon}_{i1}x_1, \dots, \bar{\alpha}^{\varepsilon}_{in}x_n)\|_q = \left((\bar{\alpha}^{\varepsilon}_{i1}x_1)^q + \cdots + (\bar{\alpha}^{\varepsilon}_{in}x_n)^q\right)^{1/q}, \tag{25}$$

where $q = p/(p-1) \geq 1$.

Looking for the consequences of using the extended addition based on t-norm T_p instead of the usually applied min-operator, we can state that the 1-level set $[a_i(x), \bar{a}_i(x)]$ does not change with the parameter p whereas the spreads $\underline{\alpha}^{\varepsilon}_i(x)$ and $\bar{\alpha}^{\varepsilon}_i(x)$ decrease if p decreases (q increases). The extent of change will be evident by looking at the extreme cases:

If p tends to infinity, then T_p tends to the min T-norm and we come back to the usual extended addition; see Section 4. Thus, if $p \to \infty$ and $q = 1$, then $\tilde{A}_i(x)$ has greatest spreads

$$\underline{\alpha}^{\varepsilon}_i(x, \infty) = \underline{\alpha}_{i1}x_1 + \cdots + \underline{\alpha}_{in}x_n \quad \text{and} \quad \bar{\alpha}^{\varepsilon}_i(x, \infty) = \bar{\alpha}_{i1}x_1 + \cdots + \bar{\alpha}_{in}x_n.$$

If $p = 1$ (and $q \to \infty$), then $T_p = T_L$ is the well known Lukasiewicz T-norm:

$$T_1(u, v) = T_L(u, v) = \text{Max}\{u, u + v - 1\}.$$

In this case $\tilde{A}_i(x)$ has smallest spreads

$$\underline{\alpha}^{\varepsilon}_i(x, 1) = \text{Max}\{\underline{\alpha}^{\varepsilon}_{i1}x_1, \dots, \underline{\alpha}^{\varepsilon}_{in}x_n\} \quad \text{and} \quad \bar{\alpha}^{\varepsilon}_i(x, 1) = \text{Max}\{\bar{\alpha}^{\varepsilon}_{i1}x_n, \dots, \bar{\alpha}^{\varepsilon}_{in}x_n\}.$$

If $p \in]1, +\infty[$, the spreads $\underline{\alpha}^{\varepsilon}_i(x, p)$ and $\bar{\alpha}^{\varepsilon}_i(x, p)$ are strictly monotone increasing functions of p. Therefore, if the set of feasible solutions of the inequality equation $\bar{a}_i(x) + \bar{\alpha}_i(x, p) \leq b_i + \beta^{\varepsilon}_i$ is denoted by

$X_i(p)$, we have $X_i(p) \subset X_i(p')$ if $p > p'$; $p, p' \in]1, +\infty[$. Moreover, as $\| \cdots \|_q$ satisfies the triangle inequality, the sets $X_i(p)$ are convex sets for all $p \in [1, \infty[$.

Using the T_p-norm based addition the inequality relation '$\lesssim_R$' should be modified to

$$\tilde{A}_i(x) \lesssim_{KR} \tilde{B}_i \Leftrightarrow \begin{cases} \bar{a}_i(x) + \bar{\alpha}_i(x, p) \leq b_i + \beta_i^\varepsilon, \\ \mu_i(x) = \mu_{D_i}(\bar{a}_i(x)) \to \text{Max}. \end{cases} \tag{26}$$

Obviously the inequality relation '$\lesssim_{KR}$' is identical with '$\lesssim_R$' for $p \to \infty$, but in general the inequality interpretation varies with p.

Now the decision maker has two ways of expressing his attitude towards risk:

- by specifying the values $\bar{\alpha}_{ij}^\varepsilon$, γ_{kj}^ε, β_i^ε, ν_k^ε;
- by choosing a value $p \in [1, +\infty[$ for each objective and for each constraint independently.

8. Solution process for getting a compromise solution

The discussion of the crucial points of fuzzy linear programming reveals that there exists an extensive offer of effective methods for reducing fuzzy linear programs in crisp systems. In order to make the best choice the decision maker has to consider the different assumptions of the suggested procedures and compare them with the actual decision problem. In any case the solution should be determined step by step in an interactive process, in which additional information out of the decision process itself or from outside should be used. In doing so, inadequate modeling of the real problem can be avoided and information costs will, in general, be decreased.

To date, the only procedure which offers a flexible extended addition of the left-hand sides of fuzzy constraints is the latest version of FULPAL, called FULPAL 2.0 (see Rommelfanger and Keresztfalvi, 1993). The basic characteristics of FULPAL 2.0 are:

- FULPAL is based on the inequality relation '$\lesssim_{KR}$'.
- The coefficients are modeled as $\tilde{A}_{ij} = (\underline{a}_{ij}; \bar{a}_{ij}; \underline{\alpha}_{ij}^\varepsilon; \bar{\alpha}_{ij}^\varepsilon)^\varepsilon$ or $\tilde{C}_{kj} = (\underline{c}_{kj}; \bar{c}_{kj}; \underline{\gamma}_{kj}^\varepsilon; \bar{\gamma}_{kj}^\varepsilon)^\varepsilon$.
- The right-hand sides of the constraints are modeled as $\tilde{B}_i = (b_i; 0, 0; \beta_i^{\lambda_A,\varepsilon}, \bar{\beta}_i^\varepsilon)^{\lambda_A,\varepsilon}$, where λ_A is the membership degree of the crisp aspiration level $b_i + \beta_i^{\lambda_A}$.
- FULPAL uses a satisfying approach, where the fuzzy aspiration levels $\tilde{N}_k = (n_k; \nu_k^{\lambda_A}, \nu_k^\varepsilon; 0, 0)^{\lambda_A,\varepsilon}$ for the objective functions are changed step by step by improving the crisp aspiration level $n_k - \nu_k^{\lambda_A}$.
- The total satisfaction of the decision maker with a solution x is expressed by the compromise objective function $\lambda(x) = \text{Min}(\mu_{Z_1}(\underline{c}_1(x)), \ldots, \mu_{Z_K}(\underline{c}_K(x)), \mu_{D_1}(\bar{a}_1(x)), \ldots, \mu_{D_{m_1}}(\bar{a}_{m_1}(x)))$.
- Regarding the calculation of the values $n_k - \nu_k^\varepsilon$ and n_k of $\tilde{N}_k$, in $\tilde{Z}_k = \tilde{C}_{k1}x_1 \oplus \cdots \oplus \tilde{C}_{kn}x_n \gtrsim \tilde{N}_k$ the inequality relation '$\lesssim_R$' is used.

Under these assumptions a compromise solution of a multiobjective fuzzy linear program of type

$$\left.\begin{array}{lcl} \tilde{Z}_1 & = & \tilde{C}_{11}x_1 \oplus \cdots \oplus \tilde{C}_{1n}x_n \\ \vdots & & \vdots \qquad\qquad\qquad \vdots \\ \tilde{Z}_K & = & \tilde{C}_{K1}x_1 \oplus \cdots \oplus \tilde{C}_{Kn}x_n \end{array}\right\} \to \text{Max} \tag{27}$$

subject to $\tilde{A}_{i1}x_1 \oplus \cdots \oplus \tilde{A}_{in}x_n \lesssim \tilde{B}_i, \quad i = 1, \ldots, m_1,$

$$x = (x_1, x_2, \ldots, x_2) \in X = \{x \in \mathbb{R}_0^n \mid a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, \forall i = m_1 + 1, \ldots, m\}$$

can be calculated by solving the crisp system

$$\lambda \rightarrow \text{Max} \tag{28}$$

$$\text{subject to} \quad \lambda \leqslant \mu_{Z_k}(\underline{c}_k(x)), \quad k=1,\dots,K,$$
$$\lambda \leqslant \mu_{D_i}(\bar{a}_i(x)), \quad i=1,\dots,m_1,$$
$$\underline{c}_k(x) - \underline{\gamma}^{\varepsilon}(x) \geqslant n_k - \nu_k^{\varepsilon}, \quad k=1,\dots,K,$$
$$\bar{a}_i(x) + \bar{\alpha}_i^{\varepsilon}(x, p) \leqslant b_i + \beta_i^{\varepsilon}, \quad i=1,\dots,m_1,$$
$$x=(x_1, x_2,\dots, x_2) \in X.$$

Following the algorithm FULPAL 2.0 we can assume that all the piecewise linear functions μ_{Z_k} and μ_{D_i} are concave membership functions. Then, if $p=1$ or $p \rightarrow +\infty$, the system (28) is a crisp linear program which can be solved by means of the well-known simplex algorithms. As far as the intermediate parameters $p \in]1, \infty[$ are concerned, it is sufficient in practical applications to work with a linear approximation, proposed by Rommelfanger and Keresztfalvi (1993). The advantage of this approximation procedure is that a compromise solution can be calculated by solving a crisp linear LP.

9. Applications of fuzzy linear programming

Fuzzy Linear Programs (FLP) were developed to tackle problems encountered in real-world applications. The following list shows that the applications of FLP are numerous and diverse.

Agricultural economics:

- analysis of water use in agriculture (Owsinski, Zadrozny and Kacprzyk, 1987);
- feed mix (Lai and Hwang, 1992);
- farm structure optimization problem (Czyzak, 1990);
- regional resource allocation (Leung, 1988; Mjelde, 1986);
- water supply planning (Slowinski, 1986, 1987).

Assignment problems:

- network location problem (Darzentas, 1987).

Banking and finance:

- capital asset pricing model (Ostermark, 1989);
- profit apportionment in concern (Ostermark, 1988);
- bank hedging decision (Lai and Hwang, 1992);
- project investment (Hanuscheck, 1986; Wolf, 1988; Lai and Hwang, 1992).

Environment management:

- air pollution regulation problem (Sommer and Pollatschek, 1978);
- energy emission models (Oder and Rentz, 1993).

Manufacturing and production:

- aggregate production planning problem (Verdegay, 1987);
- machine optimization problems (Trappey, Liu and Chang, 1988);
- magnetic tape production (Wagenknecht and Hartmann, 1987);
- optimal allocation of production of metal (Ramík and Rimanek, 1987);
- optimal system design (Zeleny, 1986);
- crude oil manufacturing (Wagenknecht and Hartmann, 1987);
- production-mix selection problem (Verdegay, 1987);
- production scheduling (Carlsson and Korhonen, 1986).

Personnel management:
- coordination of personnel demand and available personnel structure (Spengler, 1992).

Transportation:
- transportation problem (Verdegay, 1984);
- truck fleet (Zimmermann, 1976).

Appendix. A concise introduction to fuzzy set theory

Let X be a classical set of objects which should be evaluated with regard to a fuzzy statement. Then the set of ordered pairs

$$\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\}, \quad \text{where } \mu_A : X \to [0, 1],$$

is called a *fuzzy set in* X. The evaluation function $\mu_A(x)$ is called the *membership function* or the *grade of membership of* x in $\tilde{A}$.

A fuzzy set $\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\}$ is called *normalized* if $\text{Sup}_{x \in X} \mu_A(x) = 1$.

Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in [0, 1]$ a real number. Then a classical set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is called an *α-level set* or *α-cut of* $\tilde{A}$, and $A_{\overline{\alpha}} = \{x \in X \mid \mu_A(x) > \alpha\}$ is called a *strong α-level set* or *α-cut of* $\tilde{A}$.

A fuzzy set $\tilde{A}$ in a convex set X is called *convex* if

$$\mu_A(\lambda x_1 + (1 - \lambda) x_2) \geq \text{Min}(\mu_A(x_1), \mu_A(x_2)), \quad x_1, x_2 \in X, \quad \lambda \in [0, 1].$$

Obviously a fuzzy set $\tilde{A}$ is convex if and only if each α-level set of $\tilde{A}$ is convex.

A convex normalized fuzzy set $\tilde{A} = \{(x, \mu_A(x)) \mid x \in \mathbb{R}\}$ on the real line $\mathbb{R}$ such that
(i) there exist exactly one $x_0 \in \mathbb{R}$ with the membership degree $\mu_A(x_0) = 1$, and
(ii) $\mu_A(x)$ is piecewise continuous in $\mathbb{R}$,
is called a *fuzzy number*; see Fig. A.1.

A convex normalized fuzzy set $\tilde{A} = \{(x, \mu_A(x)) \mid x \in \mathbb{R}\}$ on the real line $\mathbb{R}$ is called a *fuzzy interval* if
(i) there exists more than one real number x with a membership degree $\mu_A(x) = 1$;
(ii) $\mu_A(x)$ is piecewise continuous in $\mathbb{R}$; see Fig. 2.

A binary operator $T: [0, 1] \times [0, 1] \to [0, 1]$ is called a *triangular-norm* or *t-norm* if, for all $a, b, c, d \in [0, 1]$:

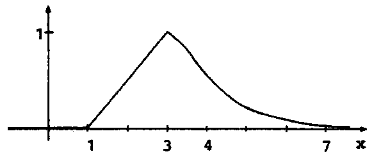
(T1) $T(a, 1) = a$.	(boundary condition)
(T2) $T(a, b) = T(b, a)$.	(commutativity)
(T3) $T(a, T(b, c)) = T(T(a, b), c)$.	(associativity)
(T4) $T(a, b) \leq T(c, d)$ if $a \leq c$ and $b \leq d$.	(monotony)

A function $L: [0, +\infty[\to [0, 1]$, such that
(i) $L(0) = 1$,
(ii) L is not increasing on $[0, +\infty[$,
is called a *reference function* of a fuzzy number.

A fuzzy number $\tilde{N} = \{(x, \mu_N(x)) \mid x \in \mathbb{R}\}$ is called of *L-R-type* if there exist reference functions L and R and scalars $\alpha, \beta > 0$ such that

$$\mu_N(x) = \begin{cases} L((n - x)/\alpha) & \text{if } x < n, \\ R((x - n)/\beta) & \text{if } x \geq n. \end{cases}$$

Symbolically $\tilde{N}$ is denoted by $(n, \alpha, \beta)_{LR}$.



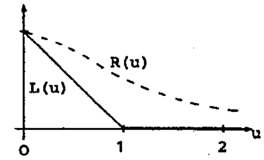


Fig. A.1. $\tilde{N} = (3; 2; 1)_{LR}$.

Fig. A.2. $L(u) = \max(0, 1-u)$; $R(u) = 1/(1+u^2)$.

A fuzzy interval $\tilde{M} = \{(x, \mu_M(x)) \mid x \in \mathbb{R}\}$ is called of *L-R-type* if there exist reference functions L and R and scalars $\alpha, \beta > 0$ such that

$$\mu_M(x) = \begin{cases} L((m_1 - x)/\alpha) & \text{if } x < m_1, \\ 1 & \text{if } m_1 \leqslant x \leqslant m_2, \\ R((x - m_2)/\beta) & \text{if } m_2 < x. \end{cases}$$

Symbolically $\tilde{M}$ is denoted by $(m_1, m_2, \alpha, \beta)_{LR}$.

The significance of fuzzy numbers or fuzzy intervals of L-R-type is that the calculations of the extended operations, based on the extension principle of Zadeh, are considerably simplified (see Dubois and Prade, 1980).

References

Bellman, R.E., and Zadeh, L.A. (1973), "Decision-making in a fuzzy environment", *Management Science* 17, 149–156.

Bortolan, G., and Degani, R. (1985), "Ranking fuzzy subsets", *Fuzzy Sets and Systems* 15, 1–19.

Buckley, J.J. (1988), "Possibilistic Linear Programming with triangular fuzzy numbers", *Fuzzy Sets and Systems* 26, 135–138.

Buckley, J.J. (1989), "Solving possibilistic Linear Programming problems", *Fuzzy Sets and Systems* 31, 329–341.

Carlsson, C., and Korhonen, P. (1986), "A parametric approach to fuzzy linear programming", *Fuzzy Sets and Systems* 20, 17–30.

Chanas, S. (1989), "Fuzzy programming in multiobjective linear programming – A parametric approach", *Fuzzy Sets and Systems* 29, 303–313.

Czyzak, P. (1990), "Application of the 'FLIP' method to farm structure optimization under uncertainty", in: R. Slowinski and J. Teghem (eds.), *Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Reidel, Dordrecht, 263–278.

Darzentas, J. (1987), "On fuzzy location models", in: J. Kacprzyk and S.A. Orlovski (eds.), *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, Dordrecht, 328–341.

Delgado, M., Verdegay, J.L. and Vila, M.A. (1989), "A general model for fuzzy linear programming", *Fuzzy Sets and Systems* 29, 21–30.

Dubois, D. and Prade, H. (1980), *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York.

Dubois, D. and Prade, H. (1989), "Ranking of fuzzy numbers in the setting of possibility theory", *Information Sciences* 30, 183–224.

Hannan, E.L. (1981), "Linear programming with multiple fuzzy goals", *Fuzzy Sets and Systems* 6, 235–248.

Hanuscheck, R. (1986), *Investitionsplanung auf der Grundlage vager Daten*, Schulz-Kirchner Verlag, Idstein.

Kacprzyk, J. (1979), "A branch and bound algorithm for the multistage control of a fuzzy system in a fuzzy environment", *Kybernetes* 8, 139–147.

Kacprzyk, J. (1983), *Multistage Decision-making under Fuzzyness*, TUV Rheinland, Köln.

Kacprzyk, J. and Orlovski, S.A. (eds.) (1987), *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, Dordrecht.

Kivijärvi, H., Korhonen, P., and Wallenius, J. (1986), "Operations Research and its practice in Finland", *Interfaces* 16, 53–59.

Lai, Y.-J., and Hwang, Ch.-L. (1992), *Fuzzy Mathematical Programming: Methods and Applications*, Springer-Verlag, Heidelberg.

Lai, Y.-J., and Hwang, Ch.-L. (1994), *Fuzzy Multiple Objective Decision Making, Methods and Applications*, Springer-Verlag, Heidelberg.

Leberling, H. (1981), "On finding compromise solutions in multicriteria problems using the fuzzy min-operator", *Fuzzy Sets and Systems* 6, 105–118.

Leberling, H. (1983), "Entscheidungsfindung bei divergierenden Faktorinteressen und relaxierten Kapazitätsrestriktionen mittels eines unscharfen Lösungsansatzes", *Zeitschrift für Betriebswirtschaftliche Forschung* 35, 398–419.

Leung, Y. (1988), "Interregional equilibrium and fuzzy linear programming", *Environment and Planning A* 20, 25–40; 219–230.

Lilien, G. (1987), "MS/OR: A mid-life crisis", *Interfaces* 17, 53–59.

Luhandjula, M.K. (1987), "Multiple Objective Programming with possibilistic coefficients", *Fuzzy Sets and Systems* 21, 135–146.

Meyer zu Selhausen, H. (1989), "Repositioning OR's products in the market", *Interfaces* 19, 79–87.

Mjelde, K.M. (1986), "Fuzzy resource allocation", *Fuzzy Sets and Systems* 19, 239–250.

Nakamura, K. (1984), "Some extensions of fuzzy linear programming", *Fuzzy Sets and Systems* 14, 211–219.

Negoita, C.V., and Ralescu, D. (1987), *Simulation, Knowledge-Based Computing, and Fuzzy Statistics*, Van Nostrand Reinhold, New York.

Negoita, C.V., and Sularia, M. (1976), "On fuzzy mathematical programming and tolerances in planning", *Economic Computation and Economic Cybernetics Studies and Research* 3, 3–15.

Negoita, C.V., Minoiu, S., and Stan, E. (1976), "On considering imprecision in dynamic linear programming", *Economic Computation and Economic Cybernetics Studies and Research* 3, 83–95.

Oder, C., and Rentz, O. (1993), "Entwicklung eines auf der Theorie unscharfer Mengen basierenden Energie-Emissions-Modells", in: *Operations Research Proceedings 1992*, Springer-Verlag, Berlin, 111–118.

Orlovski, S.A. (1985), "Mathematical Programming problems with fuzzy parameters", in: J. Kacprzyk and R.R. Yager (eds.), *Management Decision Support Systems Using Fuzzy Sets and Possibility Theory*, TUV Rheinland, Köln, 136–145.

Ostermark, R. (1988), "Profit apportionment in concerns with mutual ownership – An application of fuzzy inequalities", *Fuzzy Sets and Systems* 26, 283–297.

Ostermark, R. (1989), "Fuzzy linear constraints in the capital asset pricing model", *Fuzzy Sets and Systems* 30, 93–102.

Owsinski, J.W., Zadrozny, S., and Kacprzyk, J. (1987), "Analysis of water use and needs in agriculture through a fuzzy programming model", in: J. Kacprzyk and S.A. Orlovski (eds.), *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, Dordrecht, 377–395.

Ramik, J. (1987), "A unified approach to fuzzy optimization", in: *Reprints of the Second IFSA Congress in Tokyo, Vol. 1*, 128–130.

Ramik, J., and Rimanek, J. (1985), "Inequality between fuzzy numbers and its use in fuzzy optimization", *Fuzzy Sets and Systems* 16, 123–138.

Ramik, J., and Rimanek, J. (1987), "Fuzzy parameters in optimal allocation of resources", in: J. Kacprzyk and S.A. Orlovski (eds.), *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, Dordrecht, 359–374.

Ramik, J., and Rommelfanger, H. (1993), "A single- and a multi-valued order on fuzzy numbers and its use in linear programming with fuzzy coefficients", *Fuzzy Sets and Systems* 57, 203–208.

Rommelfanger, H. (1984), "Concave membership functions and their application in fuzzy mathematical programming", in: *Proceedings of the Workshop on the Membership Function*, EIASM, Brussels, 88–101.

Rommelfanger, H. (1986), "Rangordnungsverfahren für unscharfe Mengen", *OR Spektrum* 8, 219–228.

Rommelfanger, H. (1988), *Entscheiden bei Unschärfe – Fuzzy Decision Support-Systeme*, 2nd ed., 1994, Springer-Verlag, Berlin/Heidelberg.

Rommelfanger, H. (1989), "Inequality relations in fuzzy constraints and its use in linear fuzzy optimization", in: J.L. Verdegay and M. Delgado (eds.), *The Interface between Artificial Intelligence and Operational Research in a Fuzzy Environment*, TÜV Rheinland, Köln, 195–211.

Rommelfanger, H. (1990), "FULPAL – An interactive method for solving multiobjective fuzzy linear programming problems", in: R. Slowinski and J. Teghem (eds.), *Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Reidel, Dordrecht, 279–299.

Rommelfanger, H. (1991), "FULP – A PC-supported procedure for solving multicriteria linear programming problems with fuzzy data", in: M. Fedrizzi, J. Kacprzyk and M. Rubens (eds.), *Interactive Fuzzy Optimization and Mathematical Programming*, Springer-Verlag, Berlin/Heidelberg, 154–167.

Rommelfanger, H. (1994), "Some problems of fuzzy optimization with t-norm based addition", in: M. Delgado, J. Kacprzky, J.-L. Verdegay and M.A. Vila, *Fuzzy Optimization. Recent Advances*, Physica-Verlag, Heidelberg, 158–168.

Rommelfanger, H., and Keresztfalvi, T. (1991), "Multicriteria fuzzy optimization based on Yager's parametrized t-norm", *Foundations of Computing and Decision Sciences* 16, 99–110.

Rommelfanger, H., and Keresztfalvi, T. (1993), "Fuzzy-Optimierungsmodelle auf der Basis der Yagerschen t-Norm T_p – Theoretische und empirische Überprüfung des Einflusses des Parameters p auf die Lösungsmengen und Entwicklung geeigneter Lösungsalgorithmen", Arbeitsbericht des DFG-Projektes Ro 627/2-1, Institut für Statistik und Mathematik, J.W. Goethe-Universität Frankfurt am Main.

Rommelfanger, H., Hanuscheck, R., and Wolf, J. (1989), "Linear programming with fuzzy objectives", *Fuzzy Sets and Systems* 29, 31–48.

Roubens, M., and Teghem, J. (1988), "Comparison of methodologies for multicriteria feasibility constraint fuzzy and multiobjective stochastic linear programming", in: J. Kacprzyk and M. Fedrizzi (eds.), *Combining Fuzzy Imprecision with Probabilistic Uncertainty in Decision Making*, Springer-Verlag, Berlin/Heidelberg, 240–265.

Sakawa, M. (1983), "Interactive computer programs for fuzzy linear programming with multiple objectives", *International Journal of Man–Machine Studies* 18, 489–503.

Sakawa, M., and Yano, H. (1989), "Interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters", *Fuzzy Sets and Systems* 30, 221–238.

Sakawa, M., and Yano, H. (1990), "Interactive decision making for multiobjective programming problems with fuzzy parameters", in: R.

Slowinski, and J. Teghem (eds.), *Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Reidel, Dordrecht, 191–228.

Schwab, K.-D. (1983), *Ein auf dem Konzept der unscharfen Mengen basierendes Entscheidungsmodell bei mehrfachener Zielsetzung*, Lang-Verlag, Frankfurt.

Slowinski, R. (1986), "A multicriteria fuzzy linear programming method for water supply system development planning", *Fuzzy Sets and Systems* 19, 217–237.

Slowinski, R. (1987), "An interactive method for multiobjective linear programming with fuzzy parameters and its application to water supply planning", in: J. Kacprzyk and S.A. Orlovski (eds.), *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, Dordrecht, 396–414.

Sommer, G. (1978), "Lineare Ersatzprogramme für unscharfe Entscheidungsprobleme – Zur Optimierung bei unscharfer Problembeschreibung", *Zeitschrift für Operations Research* 22, B1–B24.

Sommer, G., and Pollatschek, M.A. (1978), "A fuzzy programming approach to an air pollution regulation problem", in: R. Trappl, G.J. Klir and F.R. Pichler (eds), *Progress in Cybernetics and Systems Research: General System Methodology, Mathematical System Theory, and Fuzzy Systems, Biocybernetics and Theoretical Neurobiology, Vol. III*, Hemisphere, Washington, DC, 303–313.

Spengler, T. (1992) "Fuzzy-Entscheidungsmodelle für die Planung der Personalbereitstellung", in: *Operations Research Proceedings 1991*, Springer-Verlag, Berlin, 501–508.

Tanaka, H., and Asai, K. (1984), "Fuzzy linear programming with fuzzy numbers", *Fuzzy Sets and Systems* 13, 1–10.

Tanaka, H., Ichihashi, H., and Asai, K. (1984), "A formulation of linear programming problems based on comparison of fuzzy numbers", *Control and Cybernetics* 13, 185–194.

Tingley, G.A. (1987), "Can MS/OR sell itself well enough?", *Interfaces* 17, 41–52.

Trappey, J.F.C., Liu, C.R., and Chang, T.C. (1988), "Fuzzy non-linear programming: Theory and application in manufacturing", *International Journal of Production Research* 26, 957–985.

Verdegay, J.L. (1984), "Application of fuzzy optimization in Operational Research", *Control and Cybernetics* 13, 229–239.

Wagenknecht, M. and Hartmann, K. (1987), "Fuzzy evaluation of Pareto points and its application to hydrocracking processes", in: J. Kacprzyk and S.A. Orlovski (eds.), *Optimization Models Using Fuzzy Sets and Possibility Theory*, Reidel, Dordrecht, 415–431.

Werners, B. (1984), *Interaktive Entscheidungsunterstützung durch ein flexibles mathematisches Programmierungssystem*, Minerwa Publikation, München.

Wolf, J. (1988), *Lineare Fuzzy-Modelle zur Unterstützung der Investitionsentscheidung*, Lang Verlag, Frankfurt.

Yazenin, A.V. (1987), "Fuzzy and stochastic programming", *Fuzzy Sets and Systems* 22, 171–180.

Zeleny, M. (1986), "Optimal system design with multiple criteria: De Novo programming approach", *Engineering Cost and Production Economics* 10, 89–94.

Zimmermann, H.-J. (1975), "Optimale Entscheidungen bei unscharfen Problembeschreibungen", *Zeitschrift für Betriebswirtschaftliche Forschung* 27, 785–795.

Zimmermann, H.-J. (1976), "Description and optimization of fuzzy systems", *International Journal of General Systems* 2, 209–216.

Zimmermann, H.-J. (1978), "Fuzzy programming and linear programming with several objective functions", *Fuzzy Sets and Systems* 1, 45–55.

Zimmermann, H.-J. (1987), *Fuzzy Sets, Decision Making and Expert Systems*, Kluwer Academic Publishers, Boston, MA.

Zimmermann, H.-J., and Zysno, P. (1979), "Latent connectives in human decision making", *Fuzzy Sets and Systems* 3, 37–51.

Zimmermann, H.-J., and Zysno, P. (1982), "Zugehörigkeitsfunktion: Modellierung, empirische Bestimmung und Verwendung in Entscheidungsmodellen", Arbeitsbericht 1. Teil des DFG-Projektes Zi 104/15-1 1982.