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DECISION-MAKING IN A FUZZY ENVIRONMENT*†

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By decision-making in a fuzzy environment is meant a decision process in which the goals and/or the constraints, but not necessarily the system under control, are fuzzy in nature. This means that the goals and/or the constraints constitute classes of alternatives whose boundaries are not sharply defined.

An example of a fuzzy constraint is: "The cost of A should not be *substantially* higher than α," where α is a specified constant. Similarly, an example of a fuzzy goal is: "x should be in the *vicinity* of x_0," where x_0 is a constant. The italicized words are the sources of fuzziness in these examples.

Fuzzy goals and fuzzy constraints can be defined precisely as fuzzy sets in the space of alternatives. A fuzzy decision, then, may be viewed as an intersection of the given goals and constraints. A maximizing decision is defined as a point in the space of alternatives at which the membership function of a fuzzy decision attains its maximum value.

The use of these concepts is illustrated by examples involving multistage decision processes in which the system under control is either deterministic or stochastic. By using dynamic programming, the determination of a maximizing decision is reduced to the solution of a system of functional equations. A reverse-flow technique is described for the solution of a functional equation arising in connection with a decision process in which the termination time is defined implicitly by the condition that the process stops when the system under control enters a specified set of states in its state space.

1. Introduction

Much of the decision-making in the real world takes place in an environment in which the goals, the constraints and the consequences of possible actions are not known precisely. To deal quantitatively with imprecision, we usually employ the concepts and techniques of probability theory and, more particularly, the tools provided by decision theory, control theory and information theory. In so doing, we are tacitly accepting the premise that imprecision—whatever its nature—can be equated with randomness. This, in our view, is a questionable assumption.

Specifically, our contention is that there is a need for differentiation between *randomness* and *fuzziness*, with the latter being a major source of imprecision in many decision processes. By fuzziness, we mean a type of imprecision which is associated with *fuzzy sets*, [20], [21] that is, classes in which there is no sharp transition from membership to nonmembership. For example, the class of *green objects* is a fuzzy set. So are the classes of objects characterized by such commonly used adjectives as large, small, substantial, significant, important, serious, simple, accurate, approximate, etc. Actually, in sharp contrast to the notion of a class or a set in mathematics, most of the classes in the real world do not have crisp boundaries which separate those objects which belong to a class from those which do not. In this connection, it is important to note that, in the discourse between humans, fuzzy statements such as "John is *several* inches taller than Jim," "x is *much larger* than y," "Corporation X has a

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bright future," "the stock market has suffered a *sharp decline*," convey information despite the imprecision of the meaning of the italicized words. In fact, it may be argued that the main distinction between human intelligence and machine intelligence lies in the ability of humans—an ability which present-day computers do not possess—to manipulate fuzzy concepts and respond to fuzzy instructions.

What is the distinction between randomness and fuzziness? Essentially, randomness has to do with uncertainty concerning membership or nonmembership of an object in a nonfuzzy set. Fuzziness, on the other hand, has to do with classes in which there may be grades of membership intermediate between full membership and nonmembership. To illustrate the point, the fuzzy assertion "Corporation X has a modern outlook" is imprecise by virtue of the fuzziness of the terms "modern outlook." On the other hand, the statement "The probability that Corporation X is operating at a loss is 0.8," is a measure of the uncertainty concerning the membership of Corporation X in the nonfuzzy class of corporations which are operating at a loss. Similarly, "The grade of membership of John in the class of tall men is 0.7," is a nonprobabilistic statement concerning the membership of John in the fuzzy class of tall men, whereas "The probability that John will get married within a year is 0.7," is a probabilistic statement concerning the uncertainty of the occurrence of a nonfuzzy event (marriage).

Reflecting this distinction, the mathematical techniques for dealing with fuzziness are quite different from those of probability theory. They are simpler in many ways because to the notion of probability measure in probability theory corresponds the simpler notion of membership function in the theory of fuzziness. Furthermore, the correspondents of $a + b$ and ab, where a and b are real numbers, are the simpler operations $\mathrm{Max}(a, b)$ and $\mathrm{Min}(a, b)$. For this reason, even in those cases in which fuzziness in a decision process can be simulated by a probabilistic model, it is generally advantageous to deal with it through the techniques provided by the theory of fuzzy sets rather than through the employment of the conceptual framework of probability theory.

Decision processes in which fuzziness enters in one way or another can be studied from many points of view. [22], [9], [14] In the present note, our main concern is with introducing three basic concepts: fuzzy goal, fuzzy constraint and fuzzy decision, and exploring the application of these concepts to multistage decision processes in which the goals or the constraints may be fuzzy, while the system under control may be either deterministic or stochastic—but not fuzzy. This, however, is not an intrinsic restriction on the applicability of the concepts and techniques described in the following sections.

Roughly speaking, by a fuzzy goal we mean an objective which can be characterized as a fuzzy set in an appropriate space. To illustrate, a simple example of a fuzzy goal involving a real-valued variable x would be: "x should be *substantially* larger than 100." Similarly, a simple example of a fuzzy constraint would be: "x should be *approximately* in the range 20–25." The sources of fuzziness in these statements are the italicized words.

A less trivial example is provided by a deterministic discrete-time system characterized by the state equations

$$x_{n+1} = x_n + u_n , \qquad n = 0, 1, 2, \cdots,$$

where x_n and u_n denote, respectively, the state and input at time n and in which for simplicity x_n and u_n are assumed to be real-valued. Here a fuzzy constraint on the input may be

$$-1 \underset{\sim}{\leq} u_n \underset{\sim}{\leq} 1$$

where the wavy bar under a symbol plays the role of a *fuzzifier*, that is, a transformation which takes a nonfuzzy set into a fuzzy set which is approximately equal to it. In this instance, $u_n \underset{\sim}{\leq} 1$, would read "$u_n$ should be *approximately* less than or equal to 1," and the effect of the fuzzifier is to transform the nonfuzzy set $-1 \leq u_n \leq 1$ into a fuzzy set $-1 \underset{\sim}{\leq} u_n \underset{\sim}{\leq} 1$. The way in which the latter set can be given a precise meaning will be discussed in §2.

Assume that the fuzzy goal is to make x_3 approximately equal to 5, starting with the initial state $x_0 = 1$. Then, the problem is to find a sequence of inputs u_0, u_1, u_2 which will realize the specified goal as nearly as possible, subject to the specified constraints on u_0, u_1, u_2.

In what follows, we shall consider in greater detail a few representative problems of this type. It should be stressed that our limited objective in the present paper is to draw attention to problems involving multistage decision processes in a fuzzy environment and suggest tentative ways of attacking them, rather than to develop a general theory of decision processes in which fuzziness and randomness may enter in a variety of ways and combinations. In particular, we shall not concern ourselves with the application to decision-making of the concept of a fuzzy algorithm [22]—a concept which may be of use in problems which are less susceptible to quantitative analysis than those considered in the sequel.

For convenience of the reader, a brief summary of the basic properties of fuzzy sets is provided in the following section.

2. A Brief Introduction to Fuzzy Sets

Informally, a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not. A more precise definition may be stated as follows.

Definition. Let $X = \{x\}$ denote a collection of objects (points) denoted generically by x. Then a *fuzzy set A in X* is a set of ordered pairs

(1) $$A = \{(x, \mu_A(x))\}, \qquad x \in X$$

where $\mu_A(x)$ is termed the *grade of membership of x in A*, and $\mu_A: X \to M$ is a function from X to a space M called the *membership space*. When M contains only two points, 0 and 1, A is nonfuzzy and its membership function becomes identical with the characteristic function of a nonfuzzy set.

In what follows, we shall assume that M is the interval [0, 1], with 0 and 1 representing, respectively, the lowest and highest grades of membership. (More generally, M can be a partially ordered set or, more particularly, a lattice [15], [6].) Thus, our basic assumption is that a fuzzy set A—despite the unsharpness of its boundaries—can be defined *precisely* by associating with each object x a number between 0 and 1 which represents its grade of membership in A.

Example. Let $X = \{0, 1, 2, \cdots\}$ be the collection of nonnegative integers. In this space, the fuzzy set A of "*several* objects" may be defined (subjectively) as the collection of ordered pairs

(2) $$A = \{(3, 0.6), (4, 0.8), (5, 1.0), (6, 1.0), (7, 0.8), (8, 0.6)\}$$

with the understanding that in (2) we list only those pairs $(x, \mu_A(x))$ in which $\mu_A(x)$ is positive.

Comment. It should be noted that in many practical situations the membership function, μ_A, has to be estimated from partial information about it, such as the values which it takes over a finite set of sample points $x_1, \cdots, x_N$. When A is defined incompletely—and hence only approximately—in this fashion, we shall say that it is partially defined by *exemplification*. The problem of estimating μ_A from the knowledge of the set of pairs $(x_1, \mu_A(x_1)), \cdots, (x_N, \mu_A(x_N))$ is the problem of *abstraction*—a problem that plays a central role in pattern recognition. [4], [18] We shall not concern ourselves with the solution of this problem in the present paper and will assume throughout—except where explicitly stated to the contrary—that $\mu_A(x)$ is given for all x in X.

For notational purposes, it is convenient to have a device for indicating that a fuzzy set A is obtained from a nonfuzzy set $\bar{A}$ by fuzzifying the boundaries of the latter set. For this purpose, we shall employ a wavy bar under a symbol (or symbols) which define $\bar{A}$. For example, if A is the set of real numbers between 2 and 5, i.e., $\bar{A} = \{x \mid 2 \leq x \leq 5\}$, then $A = \{x \mid 2 \underset{\sim}{\leq} x \underset{\sim}{\leq} 5\}$ is a fuzzy set of real numbers which are approximately between 2 and 5. Similarly, $A = \{x \mid x \underset{\sim}{=} 5\}$ or simply $\underset{\sim}{5}$ will denote the set of numbers which are approximately equal to 5. The symbol $\sim$ will be referred to as a *fuzzifier*.

We turn next to the definition of several basic concepts which we shall need in later sections.

Normality. A fuzzy set A is *normal* if and only if $\text{Sup}_x\, \mu_A(x) = 1$, that is, the supremum of $\mu_A(x)$ over X is unity. A fuzzy set is *subnormal* if it is not normal. A nonempty subnormal fuzzy set can be normalized by dividing each $\mu_A(x)$ by the factor $\text{Sup}_x\, \mu_A(x)$. (A fuzzy set A is *empty* if and only if $\mu_A(x) \equiv 0$.)

Support. The *support* of a fuzzy set A is a set $S(A)$ such that $x \in S(A) \Leftrightarrow \mu_A(x) > 0$. If $\mu_A(x) =$ constant over $S(A)$, then A is *nonfuzzy*. Note that a nonfuzzy set may be subnormal.

Equality. Two fuzzy sets are *equal*, written as $A = B$, if and only if $\mu_A = \mu_B$, that is, $\mu_A(x) = \mu_B(x)$ for all x in X. (In the sequel, to simplify the notation we shall omit the argument x when an equality or inequality holds for all values of x in X.)

Containment. A fuzzy set A is *contained in* or is a *subset of* a fuzzy set B, written as $A \subset B$, if and only if $\mu_A \leq \mu_B$. In this sense, the fuzzy set of very large numbers is a subset of the fuzzy set of large numbers.

Complementation. A' is said to be the *complement* of A if and only if $\mu_{A'} = 1 - \mu_A$. For example, the fuzzy sets: $A = \{\text{tall men}\}$ and $A' = \{\text{not tall men}\}$ are complements of one another if the negation "not" is interpreted as an operation which replaces $\mu_A(x)$ with $1 - \mu_A(x)$ for each x in X.

Intersection. The *intersection* of A and B is denoted by $A \cap B$ and is defined as the largest fuzzy set contained in both A and B. The membership function of $A \cap B$ is given by

$$\mu_{A \cap B}(x) = \text{Min}\,(\mu_A(x), \mu_B(x)), \qquad x \in X \tag{3}$$

where $\text{Min}\,(a, b) = a$ if $a \leq b$ and $\text{Min}\,(a, b) = b$ if $a > b$. In infix form, using the conjunction symbol $\wedge$ in place of Min, (3) can be written more simply as

$$\mu_{A \cap B} = \mu_A \wedge \mu_B. \tag{4}$$

The notion of intersection bears a close relation to the notion of the connective "and." Thus, if A is the class of tall men and B is the class of fat men, then $A \cap B$ is the class of men who are both tall *and* fat.

Comment. It should be noted that our identification of "and" with (4) implies that we are interpreting "and" in a "hard" sense, that is, we do not allow any tradeoff between $\mu_A(x)$ and $\mu_B(x)$ so long as $\mu_A(x) > \mu_B(x)$ or vice-versa. For example, if $\mu_A(x) = 0.8$ and $\mu_B(x) = 0.5$, then $\mu_{A \cap B}(x) = 0.5$ so long as $\mu_A(x) \geq 0.5$. In some cases, a softer interpretation of "and" which corresponds to forming the algebraic product of $\mu_A(x)$ and $\mu_B(x)$—rather than the conjunction $\mu_A(x) \wedge \mu_B(x)$—may be closer to the intended meaning of "and." From the mathematical as well as practical points of view, the identification of "and" with $\wedge$ is preferable to its identification with the product, except where $\wedge$ clearly does not express the sense in which one wants "and" to be interpreted. For this reason, in what follows "and" will be understood to be a *hard* "and" unless explicitly stated that it should be interpreted as a *soft* "and" (in the sense of corresponding to the algebraic product of membership functions).

Union. The notion of the *union* of A of B is dual to the notion of intersection. Thus, the union of A and B, denoted as $A \cup B$, is defined as the smallest fuzzy set containing both A and B. The membership function of $A \cup B$ is given by

$$\mu_{A \cup B}(x) = \text{Max}\ (\mu_A(x), \mu_B(x)), \qquad x \in X \tag{5}$$

where Max $(a, b) = a$ if $a \geq b$ and Max $(a, b) = b$ if $a < b$. In infix form, using the disjunction symbol $\vee$ in place of Max, we can write (5) more simply as

$$\mu_{A \cup B} = \mu_A \vee \mu_B \, . \tag{6}$$

As in the case of the intersection, the union of A and B bears a close relation to the connective "or." Thus, if $A = \{\text{tall men}\}$ and $B = \{\text{fat men}\}$, then $A \cup B = \{\text{tall } or \text{ fat men}\}$. Also, we can differentiate between a *hard* "or", which corresponds to (6), and a *soft* "or", corresponding to the *algebraic sum* of A and B, which is denoted by $A \oplus B$ and is defined by (9).

It is easy to verify that $\cup$ and $\cap$ are related to one another by the identity

$$A \cup B = (A' \cap B')'. \tag{7}$$

Algebraic product. The *algebraic product* of A and B is denoted by AB and is defined by

$$\mu_{AB}(x) = \mu_A(x)\mu_B(x), \qquad x \in X. \tag{8}$$

Algebraic sum. The *algebraic sum* of A and B is denoted by $A \oplus B$ and is defined by

$$\mu_{A \oplus B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \qquad x \in X. \tag{9}$$

It is easy to verify that

$$A \oplus B = (A'B')'. \tag{10}$$

Comment. It should be noted that the operations $\vee$ and $\wedge$ are associative and distributive over one another. On the other hand, $\cdot$ (product) and $\oplus$ (sum) are associative but not distributive. Note also that $\cdot$ (product) distributes over $\vee$ but not vice-versa. This property is possessed, more generally, by any operation $*$ which is monotone nondecreasing in each of its arguments. More specifically, if $b \geq b' \Rightarrow a * b \geq a * b'$ and $a \geq a' \Rightarrow a * b \geq a' * b$, then $a * (b \vee c) = (a * b) \vee (a * c)$. Many of the results described in the following sections remain valid when $\wedge$ is replaced by an operation $*$ which is associative and distributes over $\vee$.

Convexity and concavity. Let A be a fuzzy set in $X = R^n$. Then A is *convex* if and only if for every pair of points x, y in X, the membership function of A satisfies the inequality

$$(11) \qquad \mu_A(\lambda x + (1 - \lambda)y) \geq \text{Min}\,(\mu_A(x), \mu_A(y)),$$

for $0 \leq \lambda \leq 1$. Dually, A is *concave* if its complement A' is convex. It is easy to show that if A and B are convex, so is $A \cap B$. Dually, if A and B are concave, so is $A \cup B$.

Relation. A fuzzy relation, R, in the product space $X \times Y = \{(x, y)\}, x \in X, y \in Y$, is a fuzzy set in $X \times Y$ characterized by a membership function μ_R which associates with each ordered pair (x, y) a grade of membership $\mu_R(x, y)$ in R. More generally, an n-ary fuzzy relation in a product space $X = X_1 \times X_2 \times \cdots \times X_n$ is a fuzzy set in X characterized by an n-variate membership function $\mu_R(x_1, \cdots, x_n)$, $x_i \in X_i$, $i = 1, \cdots, n$.

Example. Let $X = Y = R'$, where R' is the real line $(-\infty, \infty)$. Then $x \gg y$ is a fuzzy relation in R^2. A subjective expression for μ_R in this case might be: $\mu_R(x, y) = 0$ for $x \leq y$; $\mu_R(x, y) = (1 + (x - y)^{-2})^{-1}$ for $x > y$.

Fuzzy sets induced by mappings. Let $f: X \rightarrow Y$ be a mapping from $X = \{x\}$ to $Y = \{y\}$, with the image of x under f denoted by $y = f(x)$. Let A be a fuzzy set in X. Then, the mapping f induces a fuzzy set B in Y whose membership function is given by

$$(12) \qquad \mu_B(y) = \text{Sup}_{x \in f^{-1}(y)}\, \mu_A(x),$$

where the supremum is taken over the set of points $f^{-1}(y)$ in X which are mapped by f into y.

Conditioned fuzzy sets. A fuzzy set $B(x)$ in $Y = \{y\}$ is *conditioned on* x if its membership function depends on x as a parameter. This dependence is expressed by $\mu_B(y \mid x)$.

Suppose that the parameter x ranges over a space X, so that to each x in X corresponds a fuzzy set $B(x)$ in Y. Thus, we have a mapping—characterized by $\mu_B(y \mid x)$—from X to the space of fuzzy sets in Y. Through this mapping, any given fuzzy set A in X induces a fuzzy set B in Y which is defined by

$$(13) \qquad \mu_B(y) = \text{Sup}_x \,\text{Min}\,(\mu_A(x), \mu_B(y \mid x))$$

where μ_A and μ_B denote the membership functions of A and B, respectively. In terms of $\wedge$ and $\vee$ (13) may be written more simply as

$$(14) \qquad \mu_B(y) = \vee_x (\mu_A(x) \wedge \mu_B(y \mid x)).$$

Note that this equation is analogous—but not equivalent—to the expression for the marginal probability distribution of the joint distribution of two random variables, with $\mu_B(y \mid x)$ playing a role analogous to that of a conditional distribution.

Decomposability. Let $X = \{x\}$, $Y = \{y\}$ and let C be a fuzzy set in the product space $Z = X \times Y$ defined by a membership function $\mu_C(x, y)$. Then C is *decomposable along* X and Y if and only if C admits of the representation $C = A \cap B$ or equivalently

$$(15) \qquad \mu_C(x, y) = \mu_A(x) \wedge \mu_B(y)$$

where A and B are fuzzy sets with membership functions of the form $\mu_A(x)$ and $\mu_B(y)$, respectively. (Thus, A and B are cylindrical fuzzy sets in Z.) The same holds for a fuzzy set in the product of any finite number of spaces.

Probability of fuzzy events. Let P be a probability measure on R^n. A *fuzzy event* [23]

A in R^n is defined to be a fuzzy subset A of R^n whose membership function, μ_A, is measurable. Then, the *probability of* A is defined by the Lebesgue-Stieltjes integral

$$P(A) = \int_{R^n} \mu_A(x)\, dP. \tag{16}$$

Equivalently, $P(A) = E\mu_A$ where E denotes the expectation operator. In the case of a normal nonfuzzy set, (16) reduces to the conventional definition of the probability of a nonfuzzy event.

This concludes our brief introduction to some of the basic concepts relating to fuzzy sets. In the following section, we shall use these concepts as a basis for defining the basic notions of goal, constraint and decision in a fuzzy environment.

3. Fuzzy Goals, Constraints and Decisions

In the conventional approach to decision-making, the principal ingredients of a decision process are (a) a set of alternatives; (b) a set of constraints on the choice between different alternatives; and (c) a performance function which associates with each alternative the gain (or loss) resulting from the choice of that alternative.

When we view a decision process from the broader perspective of decision-making in a fuzzy environment, a different and perhaps more natural conceptual framework suggests itself. The most important feature of this framework is its symmetry with respect to goals and constraints—a symmetry which erases the differences between them and makes it possible to relate in a relatively simple way the concept of a decision to those of the goals and constraints of a decision process.

More specifically, let $X = \{x\}$ be a given set of alternatives. Then, a *fuzzy goal* or simply a *goal*, G, in X will be identified with a given fuzzy set G in X. For example, if $X = R^1$ (the real line), then the fuzzy goal expressed in words as "x should be substantially larger than 10" might be represented by a fuzzy set in R^1 whose membership function is (subjectively) given by

$$\begin{aligned} \mu_G(x) &= 0, \qquad x < 10, \\ &= (1 + (x - 10)^{-2})^{-1}, \qquad x \geq 10. \end{aligned} \tag{17}$$

Similarly, the goal "x should be in the vicinity of 15" might be represented by a fuzzy set whose membership function is of the form

$$\mu_G(x) = (1 + (x - 15)^4)^{-1}. \tag{18}$$

Note that both of these sets are convex in the sense of (11).

In the conventional approach, the performance function associated with a decision process serves to define a linear ordering on the set of alternatives. Clearly, the membership function, $\mu_G(x)$, of a fuzzy goal serves the same purpose[1] and, in fact, may be derived from a given performance function by a normalization which leaves the linear ordering unaltered. In effect, such normalization provides a common denominator for the various goals and constraints and thereby makes it possible to treat them alike. This, as we shall see, is one of the significant advantages of regarding the concept of a goal—rather than that of a performance function—as one of the principal components of a conceptual framework for decision-making in a fuzzy environment.

In a similar manner, a *fuzzy constraint* or simply a *constraint*, C, in X is defined to

[1] Assuming, of course, that μ_G takes values in a linearly ordered set.

be a fuzzy set in X. For example, in R^1, the constraint "x should be approximately between 2 and 10," could be represented by a fuzzy set whose membership function might be of the form

$$\mu_C(x) = (1 + a(x - 6)^m)^{-1}$$

where a is a positive number and m is a positive even integer chosen in such a way as to reflect the sense in which the approximation to the interval [2, 10] is to be understood. For example, if we set $m = 4$ and $a = 5^{-4}$, then at $x = 2$ and $x = 10$ we have approximately $\mu_C(x) = 0.71$, while at $x = 1$ and $x = 11$, $\mu_C(x) = 0.5$; and at $x = 0$ and $x = 12$, $\mu_C(x)$ is approximately equal to 0.32.

An important aspect of the above definitions of the concepts of goal and constraint is that both are defined as fuzzy sets in the space of alternatives and thus, as will be elaborated upon below, can be treated identically in the formulation of a decision. By contrast, in the conventional approach to decision-making, a constraint set is taken to be a nonfuzzy set in the space of alternatives X, whereas a performance function is a function from X to some other space. Nevertheless, even in the case of the conventional approach, the use of Lagrangian multipliers and penalty functions makes it apparent that there is an intrinsic similarity between performance functions and constraints [17, Chapter 15]. This similarity—indeed identity—is made explicit in our formulation.

As an illustration, suppose that we have a fuzzy goal G and a fuzzy constraint C expressed as follows:

G: x should be substantially larger than 10, with $\mu_G(x)$ given by (17) *and*

C: x should be in the vicinity of 15, with $\mu_C(x)$ expressed by (18).

Note that G and C are connected to one another by the connective *and*. Now, as was pointed out in §2, *and* corresponds to the intersection of fuzzy sets. This implies that in the example under consideration the combined effect of the fuzzy goal G and the fuzzy constraint C on the choice of alternatives may be represented by the intersection $G \cap C$. The membership function of the intersection is given by

$$\mu_{G\cap C}(x) = \mu_G(x) \wedge \mu_C(x)$$

or more explicitly

$$\begin{aligned}\mu_{G\cap C}(x) &= \text{Min}\ ((1 + (x - 10)^{-2})^{-1}, (1 + (x - 15)^4)^{-1}) \quad \text{for} \quad x \geq 10,\\ &= 0 \quad \text{for} \quad x < 10.\end{aligned}$$

Note that $G \cap C$ is a convex fuzzy set since both G and C are convex fuzzy sets.

Turning to the concept of a *decision*, we observe that, intuitively, a *decision* is basically a choice or a set of choices drawn from the available alternatives. The preceding example suggests that a *fuzzy decision* or simply a *decision* be defined as the fuzzy set of alternatives resulting from the intersection of the goals and constraints. We formalize this idea in the following definition.

Definition. Assume that we are given a fuzzy goal G and a fuzzy constraint C in a space of alternatives X. Then, G and C combine to form a *decision*, D, which is a fuzzy set resulting from intersection of G and C. In symbols,

$$D = G \cap C \tag{19}$$

and correspondingly $\mu_D = \mu_G \wedge \mu_C$. The relation between G, C and D is depicted in Figure 1.

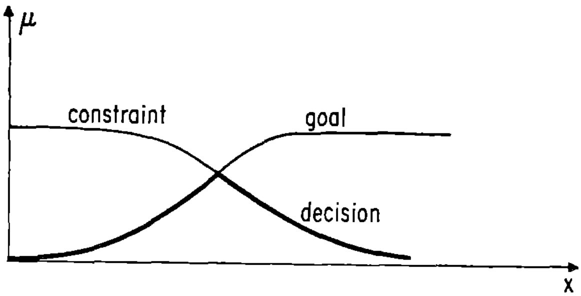

FIGURE 1

More generally, suppose that we have n goals $G_1, \cdots, G_n$ and m constraints $C_1, \cdots, C_m$. Then, the resultant decision is the intersection of the given goals $G_1, \cdots, G_n$ and the given constraints $C_1, \cdots, C_m$. That is,

$$D = G_1 \cap G_2 \cap \cdots \cap G_n \cap C_1 \cap C_2 \cap \cdots \cap C_m \tag{20}$$

and correspondingly

$$\mu_D = \mu_{G_1} \wedge \mu_{G_2} \wedge \cdots \wedge \mu_{G_n} \wedge \mu_{C_1} \wedge \mu_{C_2} \wedge \cdots \wedge \mu_{C_m}. \tag{21}$$

Note that in the above definition of a decision, the goals and the constraints enter into the expression for D in exactly the same way. This is the basis for our earlier statement concerning the identity of the roles of goals and constraints in our formulation of decision processes in a fuzzy environment.

Comment. The definition of a decision as the intersection of the goals and constraints reflects our interpretation of "and" in the "hard" sense of (4). If the interpretation of "and" is left open, we shall say that a decision—viewed as a fuzzy set—is a *confluence* of the goals and the constraints. Thus, "confluence" acquires the meaning of "intersection" when "and" is interpreted in the sense of (4); the meaning of "algebraic product" when "and" is interpreted in the sense of (8); and may be assigned other concrete meanings when a need for a special interpretation of "and" arises. (See *Comment* following (10).) In short, a broad definition of the concept of *decision* may be stated as:

Decision = Confluence of Goals and Constraints.

As an illustration of (21), we shall consider a very simple example in which $X = \{1, 2, \cdots, 10\}$ and G_1, G_2, C_1 and C_2 are defined below:

x	1	2	3	4	5	6	7	8	9	10
μ_{G_1}	0	0.1	0.4	0.8	1.0	0.7	0.4	0.2	0	0
μ_{G_2}	0.1	0.6	1.0	0.9	0.8	0.6	0.5	0.3	0	0
μ_{C_1}	0.3	0.6	0.9	1.0	0.8	0.7	0.5	0.3	0.2	0.1
μ_{C_2}	0.2	0.4	0.6	0.7	0.9	1.0	0.8	0.6	0.4	0.2

Forming the conjunction of μ_{G_1}, μ_{G_2}, μ_{C_1} and μ_{C_2}, we obtain the following table of values for $\mu_D(x)$:

x	1	2	3	4	5	6	7	8	9	10
μ_D	0	0.1	0.4	0.7	0.8	0.6	0.4	0.2	0	0

Thus the decision in this case is the fuzzy set

$$D = \{(2, 0.1), (3, 0.4), (4, 0.7), (5, 0.8), (6, 0.6), (7, 0.4), (8, 0.2)\}.$$

Note that no x in X has full (that is, unity grade) membership in D. This reflects, of course, the fact that the specified goals and constraints conflict with one another, ruling out the existence of an alternative which fully satisfies all of them.

The concept of a decision as a fuzzy set in the space of alternatives may appear at first to be somewhat artificial. In fact it is quite natural, since a fuzzy decision may be viewed as an instruction whose fuzziness is a consequence of the imprecision of the given goals and constraints. Thus, in our example, G_1, G_2, C_1 and C_2 may be respectively expressed in words as: "x should be close to 5," "x should be close to 3," "x should be close to 4" and "x should be close to 6". The decision, then, is to choose x to be close to 5. The exact meaning of "close" in each case is given by the values of the corresponding membership function.

How should a fuzzy instruction such as "x should be close to 5" be executed? Although there does not appear to be a universally valid answer to questions of this type,[2] it is reasonable in many instances to choose that x or x's which have maximal grade of membership in D. In the case of our example, this would be $x = 5$.

More generally, let D be a fuzzy decision represented by a membership function μ_D. Let K be the set of points in X on which μ_D attains its maximum, if it exists. Then, the nonfuzzy, but, in general, subnormal, subset D^M of D defined by

$$\begin{aligned} \mu_{D^m}(x) &= \operatorname{Max} \mu_D(x) \quad \text{for} \quad x \in K, \\ &= 0 \text{ elsewhere} \end{aligned}$$

will be said to be the *optimal decision* and any x in the support of D^M will be referred to as a *maximizing decision*. In other words, a maximizing decision is simply any alternative in X which maximizes $\mu_D(x)$, e.g., $x = 5$ in the foregoing example. Note that in R^n a sufficient condition for the uniqueness of a maximizing decision is that D be a strongly convex fuzzy set, i.e., that D be convex and have a unimodal membership function.

In defining a fuzzy decision D as the intersection—or, more generally, as the confluence—of the goals and constraints, we are tacitly assuming that all of the goals and constraints that enter into D are, in a sense, of equal importance. There are some situations, however, in which some of the goals and perhaps some of the constraints are of greater importance than others. In such cases, D might be expressed as a convex combination of the goals and the constraints, with the weighting coefficients reflecting the relative importance of the constituent terms. More explicitly, we may express $\mu_D(x)$ as

$$\mu_D(x) = \textstyle\sum_{i=1}^{n} \alpha_i(x)\mu_{G_i}(x) + \sum_{j=1}^{m} \beta_j(x)\mu_{C_j}(x) \tag{22}$$

where the α_i and β_j are membership functions such that

$$\textstyle\sum_{i=1}^{n} \alpha_i(x) + \sum_{j=1}^{m} \beta_j(x) \equiv 1.$$

Subject to this constraint, then, the values of $\alpha_i(x)$ and $\beta_j(x)$ can be chosen in such a way as to reflect the relative importance of $G_1, \cdots, G_n$ and $C_1, \cdots, C_m$. In par-

[2] The execution of fuzzy instructions is discussed in [22].

ticular, if $m = n = 1$, it is easy to verify that (22) can generate any fuzzy set which is contained in $G \cup C$ and contains $G \cap C$. Note that (22) resembles the familiar artifice of transforming a vector-valued criterion into a scalar-valued criterion by forming a linear combination of the components of the vector-valued objective function.

So far, we have restricted our attention to situations in which the goals and the constraints are fuzzy sets in X, the space of alternatives. A more general case which is of practical interest is one in which the goals and the constraints are fuzzy sets in different spaces. Specifically, let f be a mapping from $X = \{x\}$ to $Y = \{y\}$, with x representing an input (cause) and y, $y = f(x)$, representing the corresponding output (effect).

Suppose that the goals are defined as fuzzy sets $G_1, \cdots, G_n$ in Y while the constraints $C_1, \cdots, C_m$ are defined as fuzzy sets in X. Now, given a fuzzy set G_i in Y, one can readily find a fuzzy set $\bar{G}_i$ in X which induces G_i in Y. Specifically, the membership function of $\bar{G}_i$ is given by the equality

$$\mu_{\bar{G}_i}(x) = \mu_{G_i}(f(x)), \qquad i = 1, \cdots, n. \tag{23}$$

The decision D, then, can be expressed as the intersection of $\bar{G}_1, \cdots, \bar{G}_n$ and $C_1, \cdots, C_m$. Using (23), we can express $\mu_D(x)$ more explicitly as

$$\mu_D(x) = \mu_{G_1}(f(x)) \wedge \cdots \wedge \mu_{G_n}(f(x)) \wedge \mu_{C_1}(x) \wedge \cdots \wedge \mu_{C_m}(x), \tag{24}$$

where $f: X \to Y$. In this way, the case where the goals and the constraints are defined as fuzzy sets in different spaces can be reduced to the case where they are defined in the same space. We shall find (24) of use in the analysis of multistage decision processes in the following section.

4. Multistage Decision Processes

As an application of the concepts introduced in the preceding sections, we shall consider a few basic types of problems involving multistage decision-making in a fuzzy environment. It should be stressed that, in what follows, our main purpose is to illustrate the use of the concepts of fuzzy goal, fuzzy constraint and fuzzy decision, rather than to develop a general theory of multistage decision processes in which fuzziness enters in one way or another.

For simplicity we shall assume that the system under control, A, is a time-invariant finite-state deterministic system in which the state, x_t, at time t, $t = 0, 1, 2, \cdots$, ranges over a finite set $X = \{\sigma_1, \cdots, \sigma_n\}$, and the input, u_t, ranges over a finite set $U = \{\alpha_1, \cdots, \alpha_m\}$. The temporal evolution of A is described by the state equation

$$x_{t+1} = f(x_t, u_t), \qquad t = 0, 1, 2, \cdots \tag{25}$$

in which f is a given function from $X \times U$ to X. Thus, $f(x_t, u_t)$ represents the *successor state* of x_t for input u_t. Note that if f is a random function, then A is a *stochastic* system whose state at time $t + 1$ is a probability distribution over X, $P(x_{t+1} \mid x_t, u_t)$, which is conditioned on x_t and u_t. Analogously, if f is a fuzzy function, then A is a *fuzzy* system [21] whose state at time $t + 1$ is a fuzzy set conditioned on x_t and u_t, which means that it is characterized by a membership function of the form $\mu(x_{t+1} \mid x_t, u_t)$.[3] Since we will not be concerned with such systems in the sequel, it will be understood that f is nonfuzzy unless explicitly stated to the contrary.

[3] It should be noted that when we speak of a *fuzzy environment*, we mean that the goals and/or the constraints are fuzzy, but not necessarily the system which is under control.

We assume that at each time t the input is subjected to a fuzzy constraint C^t, which is a fuzzy set in U characterized by a membership function $\mu_t(u_t)$. Furthermore, we assume that the goal is a fuzzy set G^N in X, which is characterized by a membership function $\mu_{G^N}(x_N)$, where N is the time of termination of the process. These assumptions are common to most of the problems considered in the sequel.

Problem 1. In this case, the system is assumed to be characterized by (25), with f a given nonrandom function. The termination time N is assumed to be fixed and specified. The initial state, x_0, is assumed to be given. The problem is to find a maximizing decision.

Applying (20), the decision—viewed as a decomposable fuzzy set in $U \times U \times \cdots \times U$, may be expressed at once as

$$\mathrm{R} = C^0 \cap C^1 \cap \cdots \cap C^{N-1} \cap \bar{G}^N \tag{26}$$

where $\bar{G}^N$ is the fuzzy set in $U \times U \times \cdots \times U$ which induces G^N in X. More explicitly, in terms of membership functions, we have

$$\mu_D(u_0, \cdots, u_{N-1}) = \mu_0(u_0) \wedge \cdots \wedge \mu_{N-1}(u_{N-1}) \wedge \mu_{G^N}(x_N) \tag{27}$$

where x_N is expressible as a function of x_0 and $u_0, \cdots, u_{N-1}$ through the iteration of (25).

Our problem, then, is to find a sequence of inputs $u_0, \cdots, u_{N-1}$ which maximizes μ_D as given by (27). As is usually the case in multistage processes, it is expedient to express the solution in the form

$$u_t = \pi_t(x_t), \qquad t = 0, 1, 2, \cdots, N-1,$$

where π_t is a policy function. Then, we can employ dynamic programming to give us both the π_t and a maximizing decision $u_0^M, \cdots, u_{N-1}^M$.

More specifically, using (26) and (25), we can write

$$\begin{aligned}\mu_D(u_0^M, \cdots, u_{N-1}^M) = \mathrm{Max}_{u_0,\cdots,u_{N-2}} \mathrm{Max}_{u_{N-1}} (\mu_0(u_0) \wedge \cdots \mu_{N-2}(u_{N-2}) \\ \wedge \mu_{N-1}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1})).\end{aligned} \tag{28}$$

Now, if γ is a constant and g is any function of u_{N-1}, we have the identity

$$\mathrm{Max}_{u_{N-1}} (\gamma \wedge g(u_{N-1})) = \gamma \wedge \mathrm{Max}_{u_{N-1}}\, g(u_{N-1}).$$

Consequently, (28) may be rewritten as

$$\mu_D(u_0^M, \cdots, u_{N-1}^N) = \mathrm{Max}_{u_0,\cdots,u_{N-1}} (\mu_0(u_0) \wedge \cdots \wedge \mu_{N-2}(u_{N-2}) \wedge \mu_{G^{N-1}}(x_{N-1})) \tag{29}$$

where

$$\mu_{G^{N-1}}(x_{N-1}) = \mathrm{Max}_{u_{N-1}} (\mu_{N-1}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1}))) \tag{30}$$

may be regarded as the membership function of a fuzzy goal at time $t = N - 1$ which is induced by the given goal G^N at time $t = N$.

On repeating this backward iteration, which is a simple instance of dynamic programming, we obtain the set of recurrence equations

$$\begin{aligned}\mu_{G^{N-\nu}}(x_{N-\nu}) &= \mathrm{Max}_{u_{N-\nu}} (\mu(u_{N-\nu}) \wedge \mu_{G^{N-\nu+1}}(x_{N-\nu+1})) \\ x_{N-\nu+1} &= f(x_{N-\nu}, u_{N-\nu}), \qquad \nu = 1, \cdots, N,\end{aligned} \tag{31}$$

which yield the solution to the problem. Thus, a maximizing decision $u_0^M, \cdots, u_{N-1}^M$ is given by the successive maximizing values of $u_{N-\nu}$ in (31), with $u_{N-\nu}^M$ defined as a function of $x_{N-\nu}$, $\nu = 1, \cdots, N$.

Example. As a simple illustration, consider a system with three states σ_1, σ_2, σ_3 and two inputs α_1 and α_2. Assume $N = 2$ for simplicity. Let the fuzzy goal at time $t = 2$ be defined by a membership function μ_{G^2} whose values are given by

$$\mu_{G^2}(\sigma_1) = 0.3; \qquad \mu_{G^2}(\sigma_2) = 1; \qquad \mu_{G^2}(\sigma_3) = 0.8.$$

Furthermore, let the fuzzy constraints at $t = 0$ and $t = 1$ be defined respectively by

$$\mu_0(\alpha_1) = 0.7, \mu_0(\alpha_2) = 1; \qquad \mu_1(\alpha_1) = 1; \qquad \mu_1(\alpha_2) = 0.6.$$

The state transition table which defines the function f in (25) is assumed to be

u_t	x_t		
	σ_1	σ_2	σ_3
α_1	σ_1	σ_3	σ_1
α_2	σ_2	σ_1	σ_3

Using (30), the membership function of the fuzzy goal induced at $t = 1$ is found to be

$$\mu_{G^1}(\sigma_1) = 0.6; \qquad \mu_{G^1}(\sigma_2) = 0.8; \qquad \mu_{G^1}(\sigma_3) = 0.6$$

and the corresponding maximizing decision is given by

$$\pi_1(\sigma_1) = \alpha_2; \qquad \pi_1(\sigma_2) = \alpha_1; \qquad \pi_1(\sigma_3) = \alpha_2.$$

Similarly, for $t = 0$

$$\mu_{G^0}(\sigma_1) = 0.8; \qquad \mu_{G^0}(\sigma_2) = 0.6; \qquad \mu_{G^0}(\sigma_3) = 0.6$$

and

$$\pi_0(\sigma_1) = \alpha_2; \qquad \pi_0(\sigma_2) = \alpha_1 \text{ or } \alpha_2; \qquad \pi_0(\sigma_3) = \alpha_1 \text{ or } \alpha_2.$$

Thus, if the initial state (at $t = 0$) is σ_1, then the maximizing decision is α_2, α_1 and the corresponding value of μ_{G^2} is 0.8.

Next, we turn to a more general multistage decision process in which the system under control is stochastic, while the goal and the constraints are fuzzy.

5. Stochastic Systems in a Fuzzy Environment

As in the preceding problem, assume that the termination time X is fixed and that an initial state x_0 is specified. The system is assumed to be characterized by a conditional probability function $p(x_{t+1} \mid x_t, u_t)$. The problem is to maximize the probability of attainment of the fuzzy goal at time N, subject to the fuzzy constraints $C^0, \cdots, C^{N-1}$.

If the fuzzy goal G^N is regarded as a fuzzy event [23] in X, then the conditional probability of this event given x_{N-1} and u_{N-1} is expressed by

$$\text{(32)} \quad \text{Prob}\ (G_N \mid x_{N-1}, u_{N-1}) = E\mu_{G^N}(x_N) = \textstyle\sum_{x_N} p(x_N \mid x_{N-1}, u_{N-1})\mu_{G^N}(x_N)$$

where E denotes the conditional expectation and μ_{G^N} is the membership function of the given fuzzy goal.

We observe that (32) expresses Prob $(G_N \mid x_{N-1}, u_{N-1})$ or, equivalently, $E\mu_{G^N}(x_N)$, as a function of x_{N-1} and u_{N-1}, just as in the preceding problem $\mu_G(x_N)$ was expressed as a function of x_{N-1} and u_{N-1} via (25). This implies that $E\mu_{G^N}(x_N)$ can be treated in the same way as $\mu_{G^N}(x_N)$ was treated in the nonstochastic case, thus making it possible to reduce the solution of the problem under consideration to that of the preceding problem.

More specifically, the recurrence equations (31) are replaced by

$$\begin{aligned} \mu_{G^{N-\nu}}(x_{N-\nu}) &= \mathrm{Max}_{u_{N-\nu}}\,(\mu_{N-\nu}(u_{N-\nu}),\, E\mu_{G^{N-\nu+1}}(x_{N-\nu+1})) \\ E\mu_{G^{N-\nu+1}}(x_{N-\nu+1}) &= \sum\nolimits_{x_{N-\nu+1}} p\,(x_{N-\nu+1} \mid x_{N-\nu},\, u_{N-\nu})\mu_{G^{N-\nu+1}}(x_{N-\nu+1}) \end{aligned} \tag{33}$$

where, as before, $\mu_{G^{N-\nu}}(x_{N-\nu})$ denotes the membership function of the fuzzy goal at $t = N - \nu$ induced by the fuzzy goal at $t = N - \nu + 1$, $\nu = 1, \cdots, N$. These equations yield a solution to the problem, as is illustrated by the following example.

Example. As in the preceding example, we assume that the system has three states σ_1, σ_2, σ_3 and two inputs α_1, α_2. N is assumed to be equal to 2, and the probability function $p\,(x_{t+1} \mid x_t, u_t)$ is given by the following two tables, corresponding to $u_t = \alpha_1$, and $u_t = \alpha_2$, respectively.

I. $u_t = \alpha_1$

x_t	x_{t+1}		
	σ_1	σ_2	σ_3
σ_1	0.8	0.1	0.1
σ_2	0	0.1	0.9
σ_3	0.8	0.1	0.1

II. $u_t = \alpha_2$

x_t	x_{t+1}		
	σ_1	σ_2	σ_3
σ_1	0.1	0.9	0
σ_2	0.8	0.1	0.1
σ_3	0.1	0	0.9

The entries in these tables are the values of $p\,(x_{t+1} \mid x_t, u_t)$. Thus, the entry 0.8 in the position (σ_1, σ_2) in the first table signifies that if the system is in state σ_1 at time t and input α_1 is applied, then with probability 0.8 the state at time $t + 1$ will be σ_2.

The fuzzy goal at $t = 2$ is assumed to be the same as in the preceding example, that is

$$\mu_{G^2}(\sigma_1) = 0.3; \qquad \mu_{G^2}(\sigma_2) = 1; \qquad \mu_{G^2}(\sigma_3) = 0.8.$$

Likewise, the constraints are assumed to be the same. Thus

$$\mu_0(\alpha_1) = 0.7, \quad \mu_0(\alpha_2) = 1; \qquad \mu_1(\alpha_1) = 1, \quad \mu_1(\alpha_2) = 0.6.$$

Using (33), we compute $E\mu_{G^2}(x_2)$ as a function of x_1 and u_1. Tabulating the results, we have

u_1	x_1		
	σ_1	σ_2	σ_3
α_1	0.42	0.82	0.42
α_2	0.93	0.42	0.75

Next, using (33) with $\nu = 1$ and computing $\mu_{G^1}(x_1)$ we obtain

$$\mu_{G^1}(\sigma_1) = 0.6; \qquad \mu_{G^1}(\sigma_2) = 0.82; \qquad \mu_{G^1}(\sigma_3) = 0.6$$

which correspond to the following values of the maximal policy function

(33a) $\pi_1(\sigma_1) = \alpha_2$; $\pi_1(\sigma_2) = \alpha_1$; $\pi_1(\sigma_3) = \alpha_2$.

The final iteration with $\nu = 2$ yields

μ_0	x_0		
	σ_1	σ_2	σ_3
α_1	0.62	0.62	0.62
α_2	0.8	0.62	0.60

(33b) $\mu_{G^0}(\sigma_1) = 0.8$; $\mu_{G^0}(\sigma_2) = 0.62$; $\mu_{G^0}(\sigma_3) = 0.62$.

(33c) $\pi_0(\sigma_1) = \alpha_1$; $\pi_0(\sigma_2) = \alpha_1$ or α_2 ; $\pi_0(\sigma_3) = \alpha_1$.

The values of μ_{G^0} in (33b) represent the probabilities of attaining the given goal at $t = 2$ starting with σ_1, σ_2 and σ_3, respectively, assuming that the inputs are determined by the maximal policy function π_t, that is, $u_t = \pi_t(x_t)$ ($t = 0, 1$, $x_t = \sigma_1, \sigma_2, \sigma_3$, $u_t = \alpha_1, \alpha_2$) whose values are given in (33a) and (33c).

Comment. It should be noted that when the fuzzy goal at time N is defined in such a way that the probability of attaining it is small for all values of x_{N-1} and u_{N-1}, it may be necessary to normalize the fuzzy goal induced at time $N - 1$ before finding its intersection with C_{N-1}, for otherwise the decision would be uninfluenced by the constraints. To be consistent, such normalization may have to be carried out at each stage of the decision process. Although we shall not dwell further upon this aspect of the problem in the present paper, it should be emphasized that it is by no means a trivial one and requires a more thorough analysis.

6. Systems With Implicitly Defined Termination Time

In the preceding cases, we have assumed that the termination time, N, is fixed a priori. In the more general case which we shall consider in this section, the termination time is assumed to be determined implicitly by a subsidiary condition of the form $x_N \in T$, where T is a specified nonfuzzy subset of X termed the *termination set.*[4] Thus, the process terminates when the state of the system under control enters, for the first time, a specified subset of the state space. In this case, the goal is defined as a fuzzy set G in T, rather than in X.

More concretely, assume that the system under control, A, is a deterministic system characterized by a state equation of the form

(34) $$x_{t+1} = f(x_t, u_t), \qquad t = 0, 1, 2, \cdots$$

where x_t ranges over $X = \{\sigma_1, \cdots, \sigma_l, \sigma_{l+1}, \cdots, \sigma_n\}$, in which $T = \{\sigma_{l+1}, \cdots, \sigma_n\}$ constitutes the termination set. As before, f is assumed to be a given function from $X \times U$ to X, where $U = \{\alpha_1, \cdots, \alpha_m\}$ is the range of u_t, $t = 0, 1, 2, \cdots$. Note that if σ_i is an *absorbing* state, that is, a state in T, then we can write $f(\sigma_i, \alpha_j) = \sigma_i$ for all α_j in U.

The fuzzy goal is assumed to be a subset of T characterized by a membership func-

[4] In its conventional (nonfuzzy) formulation, this case plays an important role in the theory of optimal control and Markovian decision processes. Some of the more relevant papers on this subject are cited in the list of references.

tion $\mu_G(x_N)$, where N is the time at which $x_t \in T$, with $x_t \notin T$ for $t < N$. As for the constraints on the input, we assume for simplicity that they are independent of time but not necessarily the state. Thus, if A is in state σ_i at time t, then the fuzzy constraint on u_t is assumed to be represented by a fuzzy set $C(\sigma_i)$ (or $C(x_t)$) in U which is conditioned on σ_i. The membership function of this set will be denoted by $\mu_C(u_t \mid x_t)$.

Let x_0 be an initial state in T', where $T' = \{\sigma_1, \cdots, \sigma_l\}$ is the complement of T in X. To each such initial state will correspond a decision, $D(x_0)$, given by

$$D(x_0) = C(x_0) \cap C(x_1) \cap \cdots \cap C(x_{N-1}) \cap G \tag{35}$$

where the successive states $x_1, \cdots, x_{N-1}, x_N$ can be expressed as iterated functions of x_0 and $u_0, \cdots, u_{N-1}$ through the state equation (34). Thus

$$\begin{aligned} x_1 &= f(x_0, u_0) \\ x_2 &= f(x_1, u_1) \\ &= f(f(x_0, u_0), u_1) \\ x_3 &= f(f(f(x_0, u_0), u_1), u_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{36}$$

Note that, as in (26), the C's in (35) should be regarded as fuzzy sets in the product space $U \times U \times \cdots \times U \times T$. Another point that should be noted is that $D(x_0)$ is uniquely determined by (35) for each x_0, with the understanding that $D(x_0)$ is empty if there is no finite sequence of inputs $u_0, \cdots, u_{N-1}$ which takes the initial state x_0 into T. In this event, we shall say that T is *not reachable* from the initial state.

From (35), we can readily derive a simpler implicit equation which is satisfied by $D(x_0)$. Specifically, in virtue of the time-invariance of A and the time-independence of the goal and constraint sets, (35) implies

$$D(x_t) = C(x_t) \cap C(x_{t+1}) \cap \cdots \cap C(x_{t+N-1}) \cap G \tag{37}$$

for $t = 0, 1, 2, \cdots$. In particular,

$$D(x_{t+1}) = C(x_{t+1}) \cap \cdots \cap C(x_{t+N-1}) \cap G \tag{38}$$

and hence (37) can be written as

$$D(x_t) = C(x_t) \cap D(x_{t+1}) \tag{39}$$

or, using (34),

$$D(x_t) = C(x_t) \cap D(f(x_t, u_t)), \qquad t = 0, 1\ 2, \cdots \tag{40}$$

which is the desired implicit equation. Expressed in terms of the membership functions of the sets in question, this equation assumes the following form (for $t = 0$)

$$\mu_D(u_0, \cdots, u_{N-1} \mid x_0) = \mu_C(u_0 \mid x_0) \wedge \mu_D(u_1, \cdots, u_{N-1} \mid f(x_0, u_0)) \tag{41}$$

where the termination time N is also a function of x_0 and $u_0, u_1, u_2, \cdots$ through the state equation (34) and the termination condition $x_N \in T$, with $x_0 \notin T, \cdots, x_{N-1} \notin T$.

Now suppose that the successive inputs $u_0, u_1, \cdots, u_{N-1}$ are determined by a stationary (time-invariant) policy function π, $\pi: T' \rightarrow \mathfrak{U}$, which associates with each state x_t in T' an input u_t which should be applied to A when it is in state x_t. Thus,

$$u_t = \pi(x_t), \qquad t = 0, \cdots, N-1, \quad x_t \in T'. \tag{42}$$

Since $u_0, \cdots, u_{N-1}$ are determined by x_0 and π through (42) and the state equation (34), the membership function of $D(x_0)$ can be written as $\mu_D(x_0 \mid \pi)$. Similarly,

$\mu_C(u_0 \mid x_0)$ can be written as $\mu_C(\pi(x_0) \mid x_0)$, and $\mu_D(u_1, \cdots, u_{N-1} \mid f(x_0, u_0))$ as $\mu_D(f(x_0, \pi(x_0)) \mid \pi)$. With these substitutions, (41) assumes the more compact form

$$\mu_D(x_0 \mid \pi) = \mu_C(\pi(x_0) \mid x_0) \wedge \mu_D(f(x_0, \pi(x_0)) \mid \pi), \qquad x_0 \in T', \tag{43}$$

which in effect is a system of l equations (one for each value of x_0) in the μ_D. This system of equations determines μ_D as a function of x_0 for each π, with the understanding that $\mu_D = 0$ if under π the process does not terminate, that is, there does not exist a finite N such that $x_N \in T$. Furthermore, it is understood that $\mu_D = \mu_G$ for states in T.

It is easy to demonstrate that (43) has a unique solution. Specifically, by decomposing the set of states $T' = \{\sigma_1, \cdots, \sigma_l\}$ into disjoint subsets $T'_1, \cdots, T'_\kappa$, where T'_λ, $\lambda = 1, \cdots, \kappa$, represents the set of states from which T is reachable in λ steps, it is readily seen that the equations in (43) corresponding to the x_0 which are in T_1 yield uniquely the respective values of μ_D. In terms of these, the equations in (43) corresponding to the x_0 in T_2 yield uniquely the values of μ_D for x_0 in T_2. Continuing in this manner, all the μ_D's can be determined uniquely by successively solving subsets of the system of equations (43) for the blocks of variables in $T'_1, \cdots, T'_\kappa$.

For our purposes, it will be convenient to represent a policy π as a *policy vector*

$$\pi = (\pi(\sigma_1), \cdots, \pi(\sigma_l)) \tag{44}$$

whose ith component, $i = 1, \cdots, l$, is the input which must be applied when A is in state σ_i. Note that $\pi(\sigma_i)$ ranges over the set $U = \{\alpha_1, \cdots, \alpha_m\}$ and thus that there are m^l distinct policies in the policy space.

With reference to the system of equations (43), let

$$\mu_D(\pi) = (\mu_D(\sigma_1 \mid \pi), \cdots, \mu_D(\sigma_n \mid \pi)) \tag{45}$$

be an n-vector, termed the *goal attainment vector*, whose components are the values of the membership function of D at $\sigma_1, \cdots, \sigma_n$ (corresponding to policy π). It is natural to define a preordering in the policy space by

$$\pi' \geq \pi'' \Leftrightarrow \mu_D(\pi') \geq \mu_D(\pi'') \tag{46}$$

which means that a policy π' is better than or equal to a policy π'' if and only if $\mu_D(\sigma_i \mid \pi') \geq \mu_D(\sigma_i \mid \pi'')$ for $i = 1, \cdots, n$. Then, a policy π will be said to be *optimal* if and only if π is better than or equal to every policy in the policy space.

Does there exist an optimal policy for the problem under consideration? The answer to this question is in the affirmative. This assertion can be proved rigorously,[5] but it will suffice for our purposes to regard it as a consequence of the alternation principle [13]—a principle of broad validity which in concrete cases can be asserted as a provable theorem.

Specifically, let π' and π'' be two arbitrary policy vectors, with $\mu_D(\pi')$ and $\mu_D(\pi'')$ being the corresponding goal attainment vectors. Using π' and π'', let us construct a policy vector π in accordance with the following rules:

$$\begin{aligned} \pi_i &= \pi'_i \quad \text{if} \quad \mu_D(\sigma_i \mid \pi') \geq \mu_D(\sigma_i \mid \pi'') \\ &= \pi''_i \quad \text{if} \quad \mu_D(\sigma_i \mid \pi') < \mu_D(\sigma_i \mid \pi'') \end{aligned} \tag{47}$$

for each component π_i of π, $i = 1, \cdots, l$. Then, according to the alternation principle, $\pi \geq \pi'$ and $\pi \geq \pi''$, that is, π is better than or equal to both π' and π''. From this and the finiteness of the policy space it follows at once that there exists an optimal policy.

[5] A proof for the case of a stochastic finite-state system is given in [12].

From (43) it is a simple matter to derive a functional equation satisfied by the goal attainment vector corresponding to the optimal policy. Thus, let

$$\mu_D^M = \text{Max}_\pi \, \mu_D(\pi) \tag{48}$$

and let $P(\pi)$ be an $n \times n$ matrix of zeros and ones whose ijth element is one if and only if $\sigma_j = f(\sigma_i, \pi(\sigma_i))$, that is, the state σ_j is the immediate successor of σ_i under policy π.

Furthermore, let $\mu_C(\pi)$ denote a vector whose ith component is $\mu_C(\pi(\sigma_i) \mid \sigma_i)$. Then, on taking the maximum of both sides of (43), we obtain

$$\mu_D^M = \text{Max}_\pi \, (\mu_C(\pi) \wedge P(\pi)\mu_D^M) \tag{49}$$

which is the desired functional equation for μ_D^M. Although different in detail, equation (49) is of the same general form as the functional equations arising in the theory of Markovian decision processes [17]. Its solution, however, is considerably simpler to obtain because of the distributivity of Max and $\wedge$.

Specifically, let $\pi^1, \cdots, \pi^r$, where $r = m^l$, denote the m^l distinct policy vectors. Then, on using $\vee$ in place of Max, (49) becomes

$$\mu_D^M = (\mu_C(\pi^1) \wedge P(\pi^1)\mu_D^M) \vee \cdots \vee (\mu(\pi^r) \wedge P(\pi^r)\mu_D^M). \tag{50}$$

Taking advantage of the distributivity of $\vee$ and $\wedge$, and factoring like terms, we can put (50) into a much simpler form which, written as a system of equations in the components of μ_D^M, reads

$$\mu_D^M(\sigma_i) = \vee_j(\mu_C(\alpha_j \mid \sigma_i) \wedge \mu_D^M(f(\sigma_i, \alpha_j))), \quad i = 1, \cdots, n, \quad j = 1, \cdots, m \tag{51}$$

where $\alpha_j = \pi(\sigma_i)$ = input under policy π in state σ_i; $\mu_D^M(\sigma_i)$ = ith component of the optimal goal attainment vector; $f(\sigma_i, \alpha_j)$ = successor state[6] of σ_i for input α_j, with $f(\sigma_i, \alpha_j) = \sigma_i$ for $i = l + 1, \cdots, n$ (that is, for σ_i in the termination set T); $\mu_C(\alpha_j \mid \sigma_i)$ = value of the membership function of the constraint C in state σ_i for input α_j, with $\mu_c(\alpha_j \mid \sigma_i) = 1$ for $i = l + 1, \cdots, n$; and for $i = l + 1, \cdots, n$, $\mu_D^M(\sigma_i) = \mu_G(\sigma_i)$ = value of the membership function of the given goal G at σ_i. Thus, the $\mu_D^M(\sigma_i)$, $i = 1, \cdots, l$, are the unknowns in (51), while the $\mu_D^M(\sigma_i)$, $i = l + 1 \cdots, n$, and the $\mu_C(\alpha_j \mid \sigma_i)$, $i = 1, \cdots, n$, $j = 1, \cdots, m$, are given constants.

To make the solution of (51) more transparent, it is helpful to simplify the notation in (51) by letting the unknowns in (51) be denoted by ω_i, that is, $\omega_i = \mu_D^M(\sigma_i)$ for $i = 1, \cdots, l$. Furthermore, let the product and plus symbols denote $\wedge$ and $\vee$, respectively. Then, (51) can be written more compactly in matrix form as

$$\omega = B\omega + \gamma \tag{52}$$

where $\omega = (\omega_1, \cdots, \omega_l)$, $\gamma = (\gamma_1, \cdots, \gamma_l)$, $B = (b_{ik})$; furthermore,

$$b_{ik} = 0 \text{ if } \sigma_k \text{ is not an immediate successor of } \sigma_i;$$

$$b_{ik} = \vee_{\alpha_p} \mu_c(\alpha_p \mid \sigma_i), \text{ where the } \alpha_p \text{ are inputs which take } \sigma_i \text{ into } \sigma_k;$$

and

$$\gamma_i = \vee_j \, (\mu_c(\alpha_j \mid \sigma_i) \wedge \mu_G(f(\sigma_i, \alpha_j))) \tag{53}$$

with the understanding that $\mu_G(\sigma_i) = 0$ for states outside the termination set T.

[6] Note that the successor states in (49) are defined by $P(\pi)$.

Having put (51) into the form of a linear equation (52), it is easy to show that (53) and hence (51) can be solved by iteration. Specifically, let $\omega^0 = (0, \cdots, 0)$ and

$$\omega^{s+1} = B\omega^s + \gamma, \qquad s = 0, 1, 2, \cdots . \tag{54}$$

Then, by induction, the sequence $\omega^0, \omega^1, \omega^2, \cdots$ is monotone nondecreasing. For, assume that $\omega^{k+1} \geq \omega^k$ for some k. Using (54), we have

$$\omega^{k+2} = B\omega^{k+1} + \gamma \geq B\omega^k + \gamma = \omega^{k+1}, \tag{55}$$

and noting that $\omega^1 \geq \omega^0 = 0$, it follows that $\omega^{s+1} \geq \omega^s$ for $s = 0, 1, 2, \cdots$.

Since the sequence $\omega^0, \omega^1, \cdots$ is monotone nondecreasing and bounded from above by $\omega = (1, \cdots, 1)$, it follows that it converges to the solution of (52), that is, to the first l components[7] of the optimal goal attainment vector μ_D^M. Actually, a more detailed argument shows that (54) yields the solution of (52) in not more than l iterations. This is an immediate consequence of the following lemma.

LEMMA. *Let $B = [b_{ij}]$ be a matrix of order l with real-valued elements. Let B^s denote the sth power of B with the operations $\vee$ and $\wedge$ replacing the sum and product, respectively. Then, for all integral $s \geq l$,*

$$B + B^2 + \cdots + B^s = B + B^2 + \cdots + B^l \tag{56}$$

and

$$I + B + B^2 + \cdots + B^s = I + B + B^2 + \cdots + B^{l-1}, \qquad s \geq l - 1 \tag{57}$$

where I is the identity matrix.

PROOF. The validity of this lemma becomes rather evident when (56) is interpreted in graph-theoretic terms. Specifically, let $G(B)$ denote a graph with l nodes in which b_{ij}, $i, j = 1, \cdots$, represents the "strength" of the link between node i and node j.

Let $\gamma_{i,j,\mu}^s$ denote a chain of s links in $G(B)$,

$$\gamma_{i,j,\mu}^s = (b_{i\lambda_1}, b_{\lambda_1\lambda_2}, \cdots, b_{\lambda_{s+1},j})$$

starting at node i and ending at node j. The subscript μ serves as a label for the chain in question, with μ ranging from 1 to M, where M is the number of distinct chains of length s linking i to j.

Define the *strength* of $\gamma_{i,j,\mu}^s$, $\sigma(\gamma_{i,j,\mu}^s)$, as the strength of its weakest link, that is,

$$\sigma(\gamma_{i,j,\mu}^s) = b_{i\lambda_1} \wedge b_{\lambda_1\lambda_2} \wedge \cdots \wedge b_{\lambda_{s+1},j}. \tag{58}$$

By the definition of matrix product (with plus and product replaced by $\vee$ and $\wedge$, respectively) it is evident that b_{ij}^s, the (i, j) element of B^s, $s \geq 1$, may be expressed as

$$b_{ij}^s = \sigma(\gamma_{i,j,1}^s) \vee \sigma(\gamma_{i,j,2}^s) \vee \cdots \vee \sigma(\gamma_{i,j,M}^s) \tag{59}$$

or more compactly

$$b_{ij}^s = \vee_\mu \, \sigma(\gamma_{i,j,\mu}^s) \tag{60}$$

where $\vee_\mu$ denotes the supremum over all chains of length s linking i to j. Thus, in words,

b_{ij}^s = strength of the strongest chain among all chains of length s linking node i to node j.

[7] The remaining $n - l$ components of μ_D^M are given by the corresponding components of μ_G.

From this interpretation of the elements of B^s, it follows that the (i, j) element of $B + B^2 + \cdots + B^s$ can be expressed as

(i, j) element of $B + \cdots + B^s$ = strength of the strongest chain among all chains of length $\leq s$ linking node i to node j.

Thus, in words, the statement of the lemma implies and is implied by: If B is a matrix of order l and $s \geq l$, then:

strength of the strongest chain among all chains of length $\leq s$ linking node i to node j = strength of the strongest chain among all chains of length $\leq l$ linking node i to node j.

Stated in this form, the lemma is very easy to establish. In the first place, it is evident that, for $s \geq l$,

$$B + \cdots + B^s \geq B + \cdots + B^l. \tag{61}$$

Thus, it suffices to establish the reverse inequality $B + \cdots + B^s \leq B + \cdots + B^l$ to complete the proof.

Let $\gamma_{i,j}^s$ be a chain from i to j of length $s > l$. Clearly, in any such chain at least one node must appear more than once, implying that every chain of length $s > l$ must have one or more loops. The deletion of these loops results in a chain $\gamma_{i,j}^r$ of length $r \leq l$. Now, from the definition of the strength of a chain, (58), it follows that

$$\sigma(\gamma_{i,j}^s) \leq \sigma(\gamma_{i,j}^r) \tag{62}$$

and hence the supremum of $\sigma(\gamma_{i,j}^s)$ over chains of length s $(s > l)$ is less than or equal to the supremum of $\sigma(\gamma_{i,j}^r)$ over chains of length $\leq l$. Thus

$$B^s \leq B + B^2 + \cdots + B^l, \qquad s \geq l \tag{63}$$

and hence

$$B + B^2 + \cdots + B^s \leq B + B^2 + \cdots + B^l, \qquad s \geq l \tag{64}$$

which, in conjunction with (61), establishes (56).

As for (57), note that if $i \neq j$, then (62) is true for $s \geq l - 1$ and $r \leq l - 1$. Without this restriction $(i \neq j)$, (62) is true with $s \geq l - 1$ and $r \leq l - 1$ if $b_{ii} \geq b_{ij}$ for $i, j = 1, \cdots, l$. The latter condition is satisfied if B is replaced by $I + B$. This implies that the exponent l in (56) may be replaced by $l - 1$ if B is replaced by $I + B$. The result is (57).

Returning to the solution of (52), we note that the expression for sth iterate is given by

$$W^s = (B^{s-1} + \cdots + B + I)\gamma. \tag{65}$$

Making use of the lemma, we see that

$$W^s = W^l, \qquad s > l \tag{66}$$

which implies that (54) yields the solution to (52) in not more than l iterations.

To gain an intuitive insight into the solution of (52), it is helpful to interpret the transition from (49) to (51) with the aid of the state diagram of A. Thus, for concreteness assume that A has five states, with transitions corresponding to various inputs shown in Figure 2. In this diagram, the number associated with the branch leading from σ_i to its successor state via input α_j is the value of $\mu_c(\alpha_j \mid \sigma_i)$. States

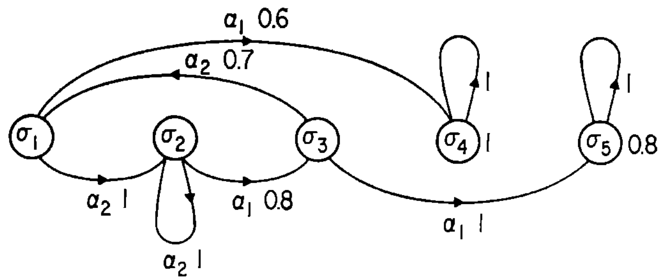

FIGURE 2

σ_4 and σ_5 are in the termination set and the corresponding values of $\mu_G(\sigma_i)$ are shown alongside. The indicated values of the $\mu_c(\alpha_j \mid \sigma_i)$ correspond to the constraint sets

$$C(\sigma_1) = \{(\alpha_1, 0.6), (\alpha_2, 1)\},$$

$$C(\sigma_2) = \{(\alpha_1, 0.8), (\alpha_2, 1)\},$$

$$C(\sigma_3) = \{(\alpha_1, 1), (\alpha_2, 0.7)\}.$$

For the system in question, the state transition function $f(\sigma_i, \alpha_j)$ is given by the following table:

α_j	σ_i: σ_1	σ_2	σ_3	σ_4	σ_5
α_1	σ_4	σ_3	σ_5	σ_4	σ_5
α_2	σ_2	σ_2	σ_1	σ_4	σ_5

From this table, it is easy to construct the matrix $P(\pi)$ for any given policy. For example, for $\pi = (\alpha_2, \alpha_1, \alpha_2)$, we have

$$P(\alpha_2, \alpha_1, \alpha_2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The system of equations (51) is obtained by reversing the direction of flow in each branch (see Figure 3) and treating the states in T, that is, σ_4 and σ_5 as sources, with the states in T', that is, σ_1, σ_2 and σ_3, playing the role of receptors (sinks). From the

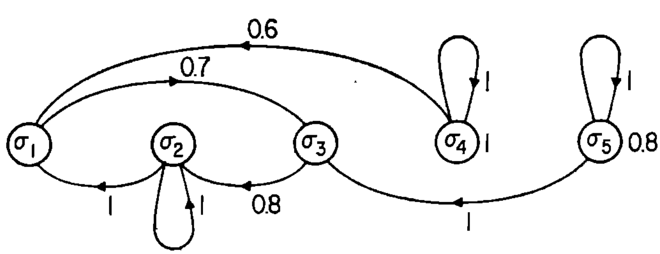

FIGURE 3

diagram shown in Figure 3, the equations in (51) can be written by inspection. Thus,

$$\begin{aligned} \mu_D^M(\sigma_1) &= (0.6 \wedge \mu_D^M(\sigma_4)) \vee (1 \wedge \mu_D^M(\sigma_2)), \\ \mu_D^M(\sigma_2) &= (0.8 \wedge \mu_D^M(\sigma_3)) \vee (1 \wedge \mu_D^M(\sigma_2)), \\ \mu_D^M(\sigma_3) &= (1 \wedge \mu_D^M(\sigma_5)) \vee (0.7 \wedge \mu_D^M(\sigma_1)), \\ \mu_D^M(\sigma_4) &= \mu_G(\sigma_4) = 1, \\ \mu_D^M(\sigma_5) &= \mu_G(\sigma_5) = 0.8. \end{aligned} \tag{67}$$

Employing the simplified notation in which $\wedge$ and $\vee$ are replaced by the product and sum, respectively, and $\omega_i = \mu_D^M(\sigma_i)$, $i = 1, 2, 3$, the system of equations (67) becomes

$$\omega = B\omega + \gamma \tag{68}$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0.8 \\ 0.7 & 0 & 0 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 0.6 \\ 0 \\ 0.8 \end{bmatrix}.$$

Letting $\omega^0 = (0, 0, 0)$, we obtain on first interation $\omega^1 = (0.6, 0, 0.8)$. Subsequent iterations yield

$$\omega^2 = (0.6, 0.8, 0.8), \qquad \omega^3 = (0.8, 0.8, 0.8), \qquad \omega^4 = (0.8, 0.8, 0.8).$$

Thus, $\omega^3 = (0.8, 0.8, 0.8)$ is the solution of (68).

To visualize the iteration process, imagine that each of the sources in Figure 3 (which are the absorbing states in Figure 2) generates balls of various diameters, with σ_i, $i = l + 1, \cdots, n$, generating balls of diameters ranging from 0 to $\mu_G(\sigma_i)$. Furthermore, imagine that a branch in Figure 2 which leaves state σ_i via input α_j, is a pipe of diameter $\mu_C(\alpha_j \mid \sigma_i)$ which can carry balls of diameter $\leq \mu_C(\alpha_j \mid \sigma_i)$ along the reverse direction, that is, along the direction shown in Figure 3. Thus, the diagram of Figure 3 may be visualized as a network of pipes whose diameters are indicated in the diagram and which can carry balls of lesser or equal diameter in the indicated directions. The states in the termination set (σ_4 and σ_5) play the role of sources of balls of diameters up to $\mu_G(\sigma_4)$ and $\mu_G(\sigma_5)$ respectively, while the remaining states (σ_1, σ_2 and σ_3) act as receptors. Because the absorbing states act as sources, we shall refer to the method of solution described above as a *reverse-flow* technique.

Now assume that it takes one unit of time for the balls to travel from a node of the network of Figure 3 to another node. If we start with no balls at σ_1, σ_2 and σ_3 at time 0, then at time $t = 1$ the maximum diameters of balls at σ_1, σ_2 and σ_3 will be, respectively, ω_1^1, ω_2^1 and ω_3^1, where $\omega^1 = (\omega_1^1, \omega_2^1, \omega_3^1)$ is the first iterate of (68). At time $t = 2$, the maximum diameters of balls will be given by ω^2 and at time $t = 3$ by ω^3. Since it takes no more than three units of time for any ball to travel from its source to any node in the network, there will be no further increase in the size of balls at each source upon further iteration. Thus, ω^3 gives the maximum diameter of balls at each receptor node and hence is the desired solution of (68).

Turning to the illustration of (43) and the alternation principle, consider the policy vector $\pi = (\alpha_1, \alpha_1, \alpha_1)$. For this π, the system of equations (43) becomes

$$\begin{aligned} \mu_D(\sigma_1 \mid \pi) &= 0.6 \wedge \mu_D(\sigma_4 \mid \pi), \\ \mu_D(\sigma_2 \mid \pi) &= 0.8 \wedge \mu_D(\sigma_3 \mid \pi), \\ \mu_D(\sigma_3 \mid \pi) &= 1 \wedge \mu_D(\sigma_5 \mid \pi). \end{aligned} \tag{69}$$

In this case, σ_1 and σ_3 are in T'_1 and σ_2 is in T'_2. Noting that $\mu_D(\sigma_4 \mid \pi) = \mu_G(\sigma_4) = 1$ and $\mu_D(\sigma_5 \mid \pi) = \mu_G(\sigma_5) = 0.8$, we find at once $\mu_D(\sigma_1 \mid \pi) = 0.6$; $\mu_D(\sigma_2 \mid \pi) = 0.8$; $\mu_D(\sigma_3 \mid \pi) = 0.8$ which is the desired solution.

Carrying out the same computation for other policy vectors, we obtain the results tabulated below

π	σ		
	σ_1	σ_2	σ_3
$(\alpha_1, \alpha_1, \alpha_1)$	0.6	0.8	0.8
$(\alpha_1, \alpha_1, \alpha_2)$	0.6	0.6	0.6
$(\alpha_1, \alpha_2, \alpha_1)$	0.6	0	0.8
$(\alpha_1, \alpha_2, \alpha_2)$	0.6	0	0.6
$(\alpha_2, \alpha_1, \alpha_1)$	0.8	0.8	0.8
$(\alpha_2, \alpha_1, \alpha_2)$	0	0	0
$(\alpha_2, \alpha_2, \alpha_1)$	0	0	0.8
$(\alpha_2, \alpha_2, \alpha_2)$	0	0	0

As a check on the alternation principle, let us take $\pi' = (\alpha_1, \alpha_1, \alpha_2)$ and $\pi'' = (\alpha_1, \alpha_2, \alpha_1)$. Using (47) leads to $\pi = (\alpha_1, \alpha_1, \alpha_1)$. Note that $\pi \geq \pi'$ and $\pi \geq \pi''$. From inspection of the table, the maximal policy is seen to be $(\alpha_2, \alpha_1, \alpha_1)$, which agrees with the result obtained by iteration.

The approach to the solution of problems involving implicitly defined termination time which we have described in this section can be extended to more complex decision processes in a fuzzy environment. In particular, the technique employed for solving the functional equation (49) can readily be extended to fuzzy systems in a fuzzy environment. Furthermore, (43) and (49) can be extended also—as in §4—to stochastic finite-state systems. Because of limitations on space, we shall not consider these cases in the present paper.

7. Concluding Remarks

The task of developing a general theory of decision-making in a fuzzy environment is one of very considerable magnitude and complexity. Thus, the results presented in this paper should be viewed as merely a first attempt at constructing a conceptual framework for such a theory.

There are many facets of the theory of decision-making in a fuzzy environment which require more thorough investigation. Among these are the question of execution of fuzzy decisions; the way in which the goals and the constraints must be combined when they are of unequal importance or are interdependent; the control of fuzzy systems and the implementation of fuzzy algorithms; the notion of fuzzy feedback and its effect on decision-making; control of systems in which the fuzzy environment is partially defined by exemplification; and decision-making in mixed environments, that is, in environments in which the imprecision stems from both randomness and fuzziness.

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