Max-Plus: a Network Algebra

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Abstract. Networks can support Quality of Service (QoS) architectures for a wide variety of applications. The Network Calculus (NC) appeared as a set of rules and results for computing bounds for QoS parameters. The goal of this paper is to improve the systematic use of the Max-Plus algebra in the formulation and derivation of NC results. To illustrate the presented methodology, we analyze the problem of optimizing the playback delay and the buffer size of an optimal video decoder with some look-ahead.

1 Introduction

Networks that support Integrated Services (IntServ) or Differentiated Services (DiffServ) can carry traffic from a wide variety of applications such as video-on-demand (VoD), videoconference and voice over IP (VoIP). Each application demands different Quality of Service (QoS) requirements, such as maximum end-to-end delay and jitter and minimum transmission rate. In this sense, traffic shaping mechanisms can be proposed to limit the traffic flows appropriately, in order to guarantee that none of them monopolize the network resources, causing the QoS seen by the other traffics to degrade.

On this context, the Network Calculus (NC) appeared as a set of rules and results that can be used to compute bounds for QoS parameters of communication networks [1]. The theory is mainly presented in [2, 3, 4] and a short introduction is found in [5, 6]. NC is based on the idea that a detailed analysis of traffic flows is not required in order to specify a network performance in terms of service requirements. Chang and colleagues [7, 4] showed that the calculus is simple and elegant if the mathematical constraints of the systems are represented using the Max-Plus algebra\(^1\). However, the potential of this algebra for solving NC problems is still unexplored.

The goal of this article is to improve the systematic use of the Max-Plus algebra in the formulation and derivation of NC results. From this Max-Plus algebraic methodology, the task of modeling a typical network system is

\(^1\) In this work, we use the term “Max-Plus algebra” as a reference to the generic algebraic structure also known as “doid” or “idempotent semiring”.
simplified and systemized. The analysis is based on two new concepts: basic constraints\(^2\) and basic elements.

The article is organized as follows: Section 2 introduces the proposed methodology; Section 3 presents definitions and results from the Max-Plus algebra; Section 4 defines basic constraints and basic elements and Section 5 illustrates our methodology by the analysis of an optimal video smoother.

2 Methodology

As previously mentioned, the objective of this article is to consider a new Max-Plus algebraic methodology to manipulate problems related to network performance. The elements manipulated are positive functions of the type \(x : (\mathbb{Z}^+)^2 \to \mathbb{R}^+ \cup \{+\infty\}\), where \(x(s, t)\) denotes the amount of data that arrive at or leave a given network element during the interval \((s, t]\). Function \(x(s, t)\), therefore, must be a non-decreasing function in \(t\). As Cruz [8], we consider a discrete time model. We are interested in the determination of lower and upper bounds for functions \(x\). The adopted strategy is to present an algebraic structure over which functions \(x\) are manipulated. Based on this structure, the constraints of a network are described as a composition of basic elements (defined in Section 4). Therefore, the proposed methodology is modular.

In this approach, the problem of obtaining bounds for network performance is treated as follows: (1.) the particular problem is described as a composition of basic elements; (2.) the algebraic constraints of each basic element of 1. are written; (3.) the system of equations obtained in 2. is algebraically solved; (4.) the results obtained are instantiated on the context of the particular network problem. For the sake of clarity, we illustrate this methodology by analyzing the problem of optimizing the playback delay and the buffer size of an optimal video decoder with some look-ahead.

3 Network Calculus and the Max-Plus algebra

In this section, we present definitions and results of the Max-Plus algebra that are necessary to understand the further analysis. Due to the lack of space, proofs of theorems and propositions are omitted. Theorem 3 and Propositions 1, 2, 4 and 5 are new. Systems are considered empty at \(s = t = 0\).

Definition 1 (Dioid and complete dioid). ([9]) A dioid \(\mathcal{D}\) is a set \(\mathcal{D}\) endowed with two closed operations called “addition” (\(\oplus\)) and “multiplication” (\(\otimes\)). The usual notation is \((\mathcal{D}, \oplus, \otimes)\). The operations of a dioid must satisfy the following axioms: associativity of addition and multiplication, commutativity

\(^2\) The idea of “mathematical constraints” in network modeling is not new (see Chang et al. [4]). However, in this work it is presented under a systematic Max-Plus framework, allowing the definition of basic constraints.
of addition, distributivity of multiplication over finite sums and idempotency (i.e. \( \forall a \in \mathcal{D}, a \oplus a = a \)). Besides, there must exist neutral elements for both operations, where \( e \) is the unity element and \( \epsilon \) is the null element, that must be absorbing (i.e. \( \forall a \in \mathcal{D}, \epsilon \otimes a = a = a \otimes \epsilon = \epsilon \)). If the dioid is closed for infinite sums, and distributivity applies for infinite sums, the dioid is called complete.

**Remark 1 (Order in a complete \( \mathcal{D} \)).** ([9]) Let \( a, b \in \mathcal{D} \), with \( \mathcal{D} \) complete. A partial order in \( \mathcal{D} \) can be defined by the following way: \( a \geq b \iff a = a \oplus b \). In particular, define \( 1 \) as the sum of all elements in \( \mathcal{D} \). \( 1 \) is, thus, the maximum element of \( \mathcal{D} \).

**Theorem 1.** (Analogous to Theorem 4.75 in [9]) Given \( a, b \in \mathcal{D} \), with \( \mathcal{D} \) complete. The minimum solution of equation \( x = xa \oplus b \) is given by \( x = b \otimes a^* \), where \( a^* = \bigoplus_{i \in \mathbb{N}} a^i \), \( a^i = e \), and \( a^i = a \otimes \cdots \otimes a \) (\( i \) times).

**Theorem 2.** (Analogous to Theorem 4.73 in [9]) Let \( a, b \in \mathcal{D} \) and define \( (a \land b) \in \mathcal{D} \) as the greatest element of \( \mathcal{D} \) that is less than or equal to \( a \) and \( b \). The maximum solution of equation \( x = x \wedge a \land b \) is given by \( x = b \wedge a^* \).

**Definition 2 (Residuated function and residual).** ([9]) Let \( f : \mathcal{A} \to \mathcal{B} \) be an isotone mapping from the complete dioid \( \mathcal{A} \) into the complete dioid \( \mathcal{B} \). If, \( \forall b \in \mathcal{B} \), the set of solutions of \( f(x) \leq b \) is non-empty and has a maximum element, denoted \( f^\#(b) \) (unique), then \( f \) is called a residuated function and \( f^\#(b) \) is called the residual of \( f \) at \( b \).

Define functions \( l_a(x) = a \otimes x \) and \( r_a(x) = x \otimes a \). It can be proved that, for any dioid, \( l_a(x) \) and \( r_a(x) \) are residuated. The residual functions are denoted respectively \( l^\#_a(y) = a \wedge y \) and \( r^\#_a(y) = y \wedge a \).

Consider, now, the set \( \mathcal{C} \) of functions of the type \( x : (\mathbb{Z}^+)^2 \to \mathbb{R}^+ \cup \{+\infty\} \),

\[
\mathcal{C} = \{ x : x(s,t) \geq 0, \text{ if } s \leq t; \text{ and } x(s,t) = 0, \text{ otherwise} \}.
\]

Given \( a, b \in \mathcal{C} \), the following operations are defined in \( \mathcal{C} \):

\[
(a \oplus b)(s,t) = \min \{a(s,t), b(s,t)\},
\]

\[
(a \otimes b)(s,t) = \begin{cases} \min \{a(s,\tau) + b(\tau, t)\}, & \text{if } s \leq t, \\ 0, & \text{if } s > t. \end{cases}
\]

**Theorem 3.** \( \mathcal{C} = (\mathcal{C}, \oplus, \otimes) \) is a complete dioid, where \( \mathcal{C} \), \( \oplus \) and \( \otimes \) are defined by (1)-(3), and the zero and unit elements are defined as follows:

\[
e(s,t) = \begin{cases} 0, & \text{if } t < s, \\ +\infty & \text{if } t \geq s, \end{cases}
\]

\[
e(s,t) = \begin{cases} 0, & \text{if } t \leq s, \\ +\infty & \text{if } t > s. \end{cases}
\]

For dioid \( \mathcal{C} \), \( (a \land b)(s,t) = \max \{a(s,t), b(s,t)\} \) and \( \top(s,t) = 0, \forall(s,t) \).

**Proposition 1.** For \( \mathcal{C} = (\mathcal{C}, \oplus, \otimes) \): \( (a \land y)(s,t) = \max_{\tau < s < t} \{y(\tau, t) - a(\tau, s)\} \) and \( (y \land a)(s,t) = \max_{s < t < \tau} \{y(s, \tau) - a(t, \tau)\} \), if \( s \leq t \); and \( (a \land y)(s,t) = (y \land a)(s,t) = 0, \text{ if } s > t. \)
At the beginning of this section, we mentioned that a function $x(s, t)$ that represents the number of packets that arrive at or leave a given network element during the interval $[s, t]$ must be a non-decreasing function in $t$. Thus, it is natural to seek non-decreasing bounds for $x$. By the other side, the non-decreasing property is not required for general elements in $\mathcal{C}$ and, for this reason, the result of a multiplication in $\mathcal{C}$ can be a function with decreasing portions, even when the terms are non-decreasing functions. Let $C_t$ be the set of non-decreasing functions of $\mathcal{C}$: $C_t = \{ x \in \mathcal{C} : x(s, t_1) \geq x(s, t_2), \text{ if } t_1 > t_2 \}$. Propositions 2 and 4, presented next, show that, under certain conditions, if $a, b \in C_t$, then $(a \otimes b) \in C_t$.

**Proposition 2.** Let $a, b \in C_t$ and $b(s, s) = 0$, $\forall s$. Then $(a \otimes b) \in C_t$.

**Definition 3 (Vertical displacement function).** ([4]) For a given function $B : \mathbb{Z}^+ \to \mathbb{R}^+$. Define the vertical displacement function, $\lambda_B \in \mathcal{C}$, as follows:

$$
\lambda_B(s, t) = \begin{cases} 
0, & s > t \\
B(t), & s = t \\
+\infty, & s < t
\end{cases}
$$

**Proposition 3.** ([4, Example 6.7]) For all $x \in \mathcal{C}$: $(x \otimes \lambda_B)(s, t) = x(s, t) + B(t)$, if $s \leq t$; $(x \otimes \lambda_B)(s, t) = 0$, if $s > t$.

**Proposition 4.** $\forall x \in C_t$ and $B(t) \in \mathbb{R}^+$ non-decreasing in $t$, then $(x \otimes \lambda_B) \in C_t$.

**Definition 4 (Delay function).** For a given number $D \in \mathbb{R}^+$, define the delay function $\delta_D \in \mathcal{C}$ by the following way:

$$
\delta_D(s, t) = \begin{cases} 
0, & t - s \leq D \\
+\infty, & \text{otherwise}
\end{cases}
$$

**Proposition 5.** For all $x \in C_t$ and $x(s, s) = 0$, $\forall s$: $(x \otimes \delta_D)(s, t) = x(s, t - D)$.

**Definition 5 (Function $\widehat{x} \in \mathcal{C}$).** In cases where the evaluation of a function $x$ is known just in the interval $[0, t]$, it is convenient to associate $x(0, t)$ to function $\widehat{x} \in \mathcal{C}$ defined as follows:

$$
\widehat{x}(s, t) = \begin{cases} 
x(0, t), & \text{if } s = 0 \\
\varepsilon(s, t), & \text{if } s \neq 0
\end{cases}
$$

4 Basic network constraints

This paper concerns the determination of QoS parameters such as end-to-end delay, maximum backlog and maximum throughput. Mathematical constraints can be associated to these parameters, expressing performance specifications as well as physical limitations of the system. Further, we describe four basic types of constraints: flux constraints, regulation constraints, service
constraints and backlog constraints. We do not claim the completeness of this set of constraints to solve all network problems. However, it covers a large set of fundamental problems related to the QoS parameters described above and for this reason we call it the basic set of constraints.

The flux constraint guarantees that the quantity of data that leaves a system is never greater than the one that enters in it. Mathematically, \( y(0, t) \leq x(0, t) \), \( \forall t \). Now, according to (2) and Definition 5, this equation can be written in \( \mathcal{C} \) by the following way: \( \hat{y} = \hat{y} \oplus \hat{x} \) (or \( \hat{y} \triangleright \hat{x} \), by Remark 1).

The regulation constraint shapes the output flow by limiting it. Mathematically: \( y(0, t) \leq (y \otimes \sigma)(0, t) \), \( \forall t \), where \( \sigma \in \mathcal{C} \) is called the regulation curve of the system. In \( \mathcal{C} \), this equation can be written simply: \( \hat{y} = \hat{y} \oplus \hat{y}\sigma \).

The service constraint is related to the minimum quantity of data served by a network element. Mathematically, it is expressed by the following way: \( y(0, t) \geq (x \otimes \beta)(0, t) \) or, in \( \mathcal{C} \), \( \hat{x}\beta = \hat{x}\beta \oplus \hat{y} \hat{\beta} \), where \( \beta \in \mathcal{C} \) is called the service curve or the minimum service function of the system.

Finally, the backlog constraint represents limitations on buffer capacity. Define the backlog of a system at instant \( t \) as the vertical deviation between the input and output curves at instant \( t \), i.e. \( x(0, t) - y(0, t) \). We say that a given queue has maximum backlog capacity \( B(t) \), if the following inequality is satisfied, \( \forall t \): \( x(0, t) - y(0, t) \leq B(t) \). From Proposition 3, this inequality can be rewritten by the following way: \( x(0, t) \leq (y \otimes \lambda_B)(0, t) \) or \( \hat{x} = \hat{x} \oplus \hat{y}\lambda_B \).

For practical purposes, we define pictorial elements based on the mathematical constraints described before (the flux constraint is always present in causal systems): the \( \sigma \)-regulator, the \( \beta \)-network and the \( B \)-queue. These definitions allow the modular description of more complex network systems, as will be exemplified in Section 5. Table 1 presents the systems of equations in \( \mathcal{C} \) that characterize each basic element.

| Table 1. Basic elements and their constraints in \( \mathcal{C} \) |
|-----------------|-----------------|-----------------|
| \( \sigma \)-regulator | \( \beta \)-network | \( B \)-queue |
| \( x \) | \( y \) | \( \hat{y} = \hat{y} \oplus \hat{x} \) |
| \( y \) | \( \hat{y} = \hat{y} \oplus \hat{\sigma} \) |
| \( \hat{x} \beta = \hat{x} \beta \oplus \hat{\beta} \hat{\sigma} \) | \( \hat{x} \beta = \hat{x} \beta \oplus \hat{y} \hat{\beta} \) |
| \( x \) | \( y \) | \( \hat{y} = \hat{y} \oplus \hat{x} \) |
| \( \hat{x} \beta \) | \( \hat{x} \beta \oplus \hat{\beta} \hat{\sigma} \) |

5 The optimal video smoother

In this section, we illustrate the methodology presented in Section 2, by analyzing an optimal video smoother (VS). Consider that a prerecorded video stream \( a \) is transmitted over a network that guarantees a service curve \( \beta \). The guaranteed service requires that the input flow of the network conforms to a regulation curve \( \sigma \). In order to satisfy \( \sigma \), the video is led to a \( \sigma \)-regulator, before being transmitted to the \( \beta \)-network. Fig. 1 presents each part of the system. We assume that the \( \sigma \)-regulator can perform some look-ahead (or prefetching) such that the video signal is disposed to the \( \beta \)-network as fast as the regulator curve allows. This is represented by the input \( e \) (impulse) in
Fig. 2. At the destination, to minimize the effect of the jitter, the video is kept in a buffer with size $B$ during an interval of time $D$, the playback delay. We want to determine the minimum possible values of $B$ and $D$ that guarantee that the output flow, $\hat{y}$ (see Definition 5), is equal to the input function, $\hat{a}$, delayed by the fixed amount of time $D$, i.e. $y(0,t) = a(0,t-D)$ or, by Proposition 5, $\hat{y} = \hat{a}\delta_D$. This constraint is represented by a constant delay line, as shown in Fig. 2. Assume that $a, x_1, x_2, y, \sigma, \beta \in \mathcal{C}$, and that Propositions 2 or 4 apply for all the multiplications that appear next.

Fig. 1. VS: physical description.  
Fig. 2. VS: mathematical constraints.

The analysis of Fig. 2 can lead us to the following conclusions: 1. if $B$ is given, then the worst playback delay, $D$, is achieved when $\hat{x}_2$ is minimum in $\mathcal{C}$; 2. if $a$ and $D$ are given, then the worst buffer utilization, $B$, is achieved when $\hat{x}_2$ is maximum in $\mathcal{C}$. Therefore, we need to find the maximum and minimum $\hat{x}_2$ that satisfy the following equations (see Table 1):

$$\begin{align*}
\hat{x}_1 &= \hat{x}_i \oplus \hat{e}, \\
\hat{x}_1 &= \hat{x}_i \oplus \hat{x}_i \sigma, \\
\hat{x}_2 &= \hat{x}_2 \oplus \hat{x}_i.
\end{align*}$$

$$\begin{align*}
\hat{x}_1 &= \hat{x}_1 \oplus \hat{e}, \\
\hat{a}\delta_D &= \hat{a}\delta_D \oplus \hat{x}_2, \\
\hat{x}_2 &= \hat{x}_2 \oplus \hat{e}.
\end{align*}$$

To find the minimum $\hat{x}_2$, define $u = \hat{x}_1 \beta$, such that $u \not\geq \hat{x}_1 \beta$ (or $u = u \oplus \hat{x}_1 \beta$) and $u \not\lessdot \hat{x}_1 \beta$. Thus, (4)-(9) can be rewritten by the following way:

$$\begin{align*}
[X \ u] &= [X \ u] \oplus K \oplus L, \\
\hat{x}_2 &\lessdot \hat{a}\delta_D, \\
u &\not\geq \hat{x}_1 \beta,
\end{align*}$$

with $X^T = \left[ \begin{array}{c} \hat{x}_1 \\ \hat{x}_2 \end{array} \right]$; $K = \left[ \begin{array}{ccc} (e \oplus \sigma) & e & \beta \\ e & e & e \\ e & e & e \end{array} \right]$; $L^T = \left[ \begin{array}{c} \hat{e} \\ \hat{a}\delta_D \lambda_B \end{array} \right]$.

From Theorem 1, the optimal response of (10) is $X^b = LK^*$, given by the following way

$$\begin{align*}
\left[ \begin{array}{c} \hat{x}_1 \\ \hat{x}_2 \\ \hat{a} \end{array} \right] &= \left[ \begin{array}{ccc} \hat{e} \hat{a}\delta_D \lambda_B \ & e \\ (e \oplus \sigma)^* & (e \oplus \sigma)^* & (e \oplus \sigma)^* \beta \\ \hat{e} & e & e \\ \hat{e} & e & e \end{array} \right] \\
&= \left[ \begin{array}{c} \hat{e}(e \oplus \sigma)^* \oplus \hat{e}(e \oplus \sigma)^* \oplus \hat{e}(e \oplus \sigma)^* \beta \oplus \hat{a}\delta_D \lambda_B \end{array} \right].
\end{align*}$$

\[A constant delay line is a special case of a $\delta_D$-network (in fact, $\hat{y} = \hat{a}\delta_D = \hat{a}\delta_D \oplus \hat{y}$).

10 emphasize this fact, a different symbol was chosen to represent this system.
Besides, from (11) and (12), we must have \( \hat{x}'_1 \leq \hat{\alpha} \delta_D \) and \( u' \leq \hat{x}'_1 \beta \), such that:

\[
\begin{align*}
\hat{\alpha} \delta_D \triangleright \hat{\alpha}(e \oplus \sigma)^* \oplus \hat{\alpha} \delta_B \lambda_B \\
\hat{\alpha}(e \oplus \sigma)^* \beta \triangleright \hat{\alpha}(e \oplus \sigma)^* \beta \oplus \hat{\alpha} \delta_D \lambda_B \\
\Rightarrow \hat{\alpha}(e \oplus \sigma)^* \beta \triangleright \hat{\alpha} \delta_B \lambda_B.
\end{align*}
\]

Therefore, the optimal solution, \( D \), is the minimum value of \( D \) that guarantees that \( \hat{\alpha} \delta_D \triangleright \hat{\alpha}(e \oplus \sigma)^* \), i.e. \( D = \min_{D > 0} \{ \hat{\alpha} \delta_D \triangleright \hat{\alpha}(e \oplus \sigma)^* \} \).

To show that \( D \) is optimal, assume that there exists a \( D < D \) such that (4)-(9) are satisfied. If \( D < D \), there must exist at least one \( t \) such that \( (\hat{\alpha} \delta_D)(0,t) > (e \oplus \sigma)^*(0,t) \). But \( \hat{x}'_1 = \hat{\alpha}(e \oplus \sigma)^* \) and then we must have \( (\hat{\alpha} \delta_D)(0,t) > \hat{x}'_1(0,t) \). Besides, \( \hat{x}'_1 \) is minimum in \( C \). Therefore, \( \hat{x}_1 \nleq \hat{x}'_1 \) or equivalently \( \hat{x}_1(0,t) \leq \hat{x}'_1(0,t) \), \( \forall t \). Thus, \( \hat{x}_1(0,t) < (\hat{\alpha} \delta_D)(0,t) \) for at least one \( t \). Moreover, from (6), \( \hat{x}_2(0,t) \leq \hat{x}_1(0,t) \), \( \forall t \). Consequently, there must exist at least one \( t \) such that \( \hat{x}_2(0,t) < (\hat{\alpha} \delta_D)(0,t) \). This contradicts (8).

Now, from (5), \( \hat{x}_1 \nleq \hat{x}_1 \sigma \) such that \( \hat{x}_1 \leq \hat{x}_1 \phi \sigma \). Moreover, from (7), \( \hat{x}_2 \leq \hat{x}_1 \beta \). Thus \( \hat{x}_2 \leq \hat{x}_1 \beta \leq (\hat{x}_1 \phi \sigma) \beta \leq \hat{x}_1 \phi (\beta \psi \sigma) \) (see [9, Table 4.1]). Therefore, (4)-(9) can be rewritten as follows (see [9, Section 4.6]):

\[
\begin{align*}
X &= X \phi M \wedge N, \\
\hat{x}_1 &= \hat{\alpha}, \\
\hat{x}_2 &= \hat{\alpha} \delta_D \lambda_B,
\end{align*}
\]

with \( X^T = [\hat{x}_1 \quad \hat{x}_2] \); \( M = \begin{bmatrix} e & e & c \\ b \psi \sigma & c & \end{bmatrix} \) and \( N^T = \begin{bmatrix} T \hat{\alpha} \delta_D \end{bmatrix} \).

From Theorem 2, the optimal response of (13) is \( X^* = N \phi M^* \), that is:

\[
[\hat{x}'_1 \quad \hat{x}'_2] = [T \hat{\alpha} \delta_D] \phi \begin{bmatrix} (\beta \psi \sigma)^* \\ (\beta \psi \sigma)^* \end{bmatrix}.
\]

Then, \( \hat{x}'_1 = \hat{x}'_2 = (\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^* \). Besides, from (14) and (15), we must have \( \hat{x}'_1 \triangleright \hat{\alpha} \) and \( \hat{x}'_2 \triangleright \hat{\alpha} \delta_D \lambda_B \), such that:

\[
\begin{align*}
(\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^* \triangleright \hat{\alpha} \\
(\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^* \triangleright \hat{\alpha} \delta_D \lambda_B \\
\Rightarrow \lambda_B \leq (\hat{\alpha} \delta_D) \psi (\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^*.
\end{align*}
\]

The optimal solution \( B \), thus, is the minimum value of \( B \) that guarantees that \( \lambda_B \leq (\hat{\alpha} \delta_D) \psi (\hat{\alpha} \delta_D) \psi (\beta \psi \sigma)^* \), i.e. \( B = \max_{\lambda_B > 0} \{ [(\hat{\alpha} \delta_D) \psi (\hat{\alpha} \delta_D) \psi (\beta \psi \sigma)^*] (s,s) \} \).

Now, assume that there exists a \( B < B \) such that (4)-(9) are satisfied. If \( B < B \), there must exist at least one \( t \) such that \( (\hat{\alpha} \delta_D \lambda_B)(0,t) < [(\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^*](0,t) = \hat{x}'_2(0,t) \). But \( \hat{x}'_2 \) is maximum in \( C \). Thus, \( \hat{x}_2 \leq \hat{x}'_2 = (\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^* \) or, equivalently, \( \hat{x}_2(0,t) \leq [(\hat{\alpha} \delta_D) \phi (\beta \psi \sigma)^*](0,t) \), \( \forall t \). Then, \( \hat{x}_2(0,t) > (\hat{\alpha} \delta_D \lambda_B)(0,t) \) for at least one \( t \). This contradicts (9).

For the particular case where \( a, \sigma \) and \( \beta \) are time-invariant functions, the optimal solutions of \( B \) and \( D \) can be calculated by the following equations:
Le Boudec and Thiran [6] solved this video smoother problem for the particular case where $\hat{x}_1 \beta \preceq \hat{\delta}_D$ and $\hat{x}_1 \succ \hat{\delta}_D \lambda_B$ and that $\sigma = (\epsilon \oplus \sigma)^*$ and $\beta = (\epsilon \oplus \beta)$. The results obtained where $D_{\text{LeBT}} = \min_{D \geq 0} \{ aD \succ (\beta \sigma) \}$ and $B_{\text{LeBT}} = [(a \epsilon \sigma) \delta (\beta \sigma)](0)$. As expected, $B_{\text{LeBT}} \geq B$ and $D_{\text{LeBT}} \geq D$.

6 Conclusion

In this article, we proposed a Max-Plus algebraic methodology, for the determination of QoS parameters bounds. To exemplify the introduced methodology, we analyzed the problem of minimizing the buffer and the playback delay of a video smoother with prefetching. The results obtained show that the defined methodology simplify the solution of complex systems.

References