9 Interoperability aspects of fuzzy sets

Fuzzy Systems Engineering
Toward Human-Centric Computing
Contents

9.1. Fuzzy sets and its family of $\alpha$-cuts

9.2. Fuzzy sets and their interfacing with the external world

9.3. Encoding and decoding as an optimization problem of vector quantization

9.4. Decoding of a fuzzy set through a family of fuzzy sets

9.5. Taxonomy of data in structure description with shadowed sets
9.1 Fuzzy sets and its family of $\alpha$–cuts
From fuzzy set to a family of sets

- Representation theorem offers an important insight into links between a given fuzzy set and its $\alpha$-cuts.
- Any fuzzy set can be represented as an infinite family of $\alpha$-cuts.

\[
A = \bigcup_{\alpha > 0} \alpha A_{\alpha}
\]

\[
A_{\alpha} = \{ x \in X | A(x) \geq \alpha \}\]
Reconstruction
Reconstruction

Graphical representation of membership functions and the reconstruction process.
From fuzzy set to a family of sets: An optimization

- Is there an optimal level $a$ that optimizes a single $\alpha$-cut of $A$ so that $A_a$ approximates $A$ to the highest extent?

- Performance index

\[
Q = \int_{x \notin A_a} A(x)dx + \int_{x \in A_a} (1-A(x))dx
\]

\[
\min_{\alpha} Q = Q(\alpha_{opt})
\]

\[
\alpha_{opt} = \arg \min_{\alpha} Q(\alpha)
\]
Triangular fuzzy sets optimization

\[ A(x) = \max(1 - \frac{x}{b}, 0), \quad x \geq 0 \]

\[ Q = \int_{b(1-\alpha)}^{b} \left( 1 - \frac{x}{b} \right) dx + \int_{0}^{b(1-\alpha)} \left( 1 - 1 - \frac{x}{b} \right) dx \]

\[ Q = b - b(1 - \alpha) + b(1 - \alpha)^2 - \frac{b}{2} \]

\[ \frac{\partial Q}{\partial \alpha} = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2} \]
Set-based approximation of fuzzy sets

By approximating fuzzy sets by a finite family of sets we can directly exploit well-developed techniques of interval analysis and combine the partial results into a single fuzzy set (result).
9.2 Fuzzy sets and their interfacing with the external world
Fuzzy sets and interfaces

- Fuzzy sets do not exist in real-world (sets do not as well)
- To interact with the world one has to construct interfaces (encoders and decoders)
Fuzzy sets and interfaces

- Need for building interfaces exists in case of sets (interval analysis)
- Here we encounter well-known constructs of analog-to-digital (AD) and digital-to-analog (DA) converters.
Fuzzy sets and interfaces

- Two functional modules:
  - **Encoders** The objective is to translate input data into some internal format acceptable for processing at level of fuzzy sets
  - **Decoders** The objective is to convert the results of processing of fuzzy sets into some format acceptable by the external world (typically in the form of some numeric quantities)

- For encoding and decoding we engage a collection of fuzzy sets – information granules
Encoding mechanisms

- Given is a collection of fuzzy sets $A_1, A_2, \ldots, A_c$; express some numeric input $x$ in $\mathbb{R}$ in terms of these fuzzy sets

\[ x \rightarrow [ A_1(x) \ A_2(x) \ldots A_c(x) ] \]

- *Nonlinear* mapping from $\mathbb{R}$ to $c$-dimensional unit hypercube
Decoding mechanisms

- Decoding completed on a basis of a single fuzzy set
- Decoding realized on a basis of a certain finite family of fuzzy sets and levels of their activation
Decoding process: a single fuzzy set

Single fuzzy set $B \rightarrow$ develop a single numeric representative

\[ \hat{x} = \frac{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_p}{p} \]

Mean of maxima

\[ \hat{x} = \frac{\int_{-\infty}^{\infty} B(x)dx}{\tilde{x}} \]

Centre of Area

\[ \hat{x} = \frac{\int B(x)x dx}{\int B(x)dx} \]

Centre of gravity
Single fuzzy set decoding: centre of gravity

- Solution to the following optimization problem

\[ \min_{\hat{x}} V = \int_{x} B(x)[x - \hat{x}]^2 \, dx \]

\[ \frac{\partial V}{\partial \hat{x}} = 0 \quad \Rightarrow \quad 2 \int_{x} B(x)[x - \hat{x}] \, dx = 0 \]
Single fuzzy set decoding: augmented strategies

- Augmented centre of gravity

\[
\hat{x} = \frac{\int_{x \in X : B(x) \geq \beta} B(x) x \, dx}{\int_{x \in X : B(x) \geq \beta} B(x) \, dx}
\]

\[
\hat{x} = \frac{\int_{x \in X : B(x) \geq \beta} B^\gamma(x) x \, dx}{\int_{x \in X : B(x) \geq \beta} B^\gamma(x) \, dx}
\]
Single fuzzy set decoding: general requirements

- Requirements implied by:
  - monotonicity with respect to changeable membership functions
  - graphically motivated requirements (symmetry, translation, scaling…)
  - use of logic operations and logic modifiers
9.3 Encoding and decoding as an optimization problem of vector quantization
Fuzzy scalar optimization

- Decoding: a collection of fuzzy sets

```
x
ENCODER

Numeric Input (multidimensional)

Granular representation

Numeric Output (multidimensional)

DECODER

\hat{x}
```

- One-dimensional case
- Multivariable case
Decoding: one-dimensional (scalar) case

Codebook – a finite family of fuzzy sets \( \{ A_1, A_2, \ldots, A_c \} \)
Proposition

Assume:

a) \( \{A_i\} \ i = 1, \ldots, c \) forms a partition

\[
\sum_{i=1}^{c} A_i(x) = 1, \quad \forall x \in \mathbf{X}, \quad \exists i \ | \ A_i(x) > 0
\]

b) \( A_i > 0, A_{i+1} > 0 \) and \( A_k = 0 \ \forall k \neq i, i + 1 \)

c) decoding is a weighted sum of activation levels and prototypes \( v_i \)

\[
\hat{x} = \sum_{i=1}^{c} A_i(x)v_i
\]
Then

\[
A_i(x) = \begin{cases} 
\frac{x - v_{i-1}}{v_i - v_{i-1}} & \text{if } x \in [v_{i-1}, v_i] \\
\frac{x - v_{i+1}}{v_i - v_{i+1}} & \text{if } x \in [v_{i-1}, v_{i+1}] 
\end{cases}
\]

Pedrycz and Gomide, FSE 2007
Forming mechanisms of fuzzy quantization

use of sets – Vector Quantization (VQ)

use of fuzzy sets – Fuzzy Vector Quantization (FVQ)
Fuzzy vector quantization

- Codebook formed through fuzzy clustering (FCM) producing a finite collection of prototypes $v_1, v_2, \ldots, v_c$

- Given any new input $x$ we realize its encoding and decoding

- Recall
  - **encoding**: representation of $x$ in terms of the prototypes
  - **decoding**: development of external representation of the result of processing realized at the level of information granules
Coding and decoding with fuzzy codebooks

Encoding: optimization problem

\[ \sum_{i=1}^{c} u_i^m ||x - v_i||^2 \]

Minimize w.r.t. \( u_i \) subject to

\[ u_i(x) \in [0,1], \quad \sum_{i=1}^{c} u_i(x) = 1 \]

\[ u_i(x) = \frac{1}{\sum \left( \frac{||x - v_i||}{||x - v_j||} \right)^{m-1}} \]
Decoding: optimization problem

Reconstruct original multidimensional input $x$

\[
Q_2(\hat{x}) = \sum_{i=1}^{c} u_i^m \| \hat{x} - v_i \|^2
\]

minimize

\[
\hat{x} = \frac{\sum_{i=1}^{c} u_i^m v_i}{\sum_{i=1}^{c} u_i^m}
\]
Fuzzy vector quantization: decoding error

$m = 1.2$

$m = 2.0$

$m = 3.5$
9.4 Decoding of a fuzzy set through a family of fuzzy sets
Consider a family of fuzzy sets $A_1, A_2, \ldots, A_c$.

Input datum $X$ either a fuzzy set or a numeric quantity.

\[
\text{Poss}(A_i, X) = \sup_{x \in X} [X(x) \cdot A_i(x)]
\]

\[
\text{Nec}(A_i, X) = \inf_{x \in X} [X(x) \cdot (1 - A_i(x))]
\]
Possibility and necessity

Possibility

Necessity

Poss\((X, A_i)\)

Nec\((X, A_i)\)
Possibility and necessity encoding: example

\[ X = [0.0 \ 0.2 \ 0.8 \ 1.0 \ 0.9 \ 0.5 \ 0.1 \ 0.0] \]

\[ A_i = [0.6 \ 0.5 \ 0.4 \ 0.5 \ 0.6 \ 0.9 \ 1.0 \ 1.0] \]

\[ \text{Poss} (A_i, X) = \max (0.0, 0.5, 0.4, 0.5, 0.6, 0.5, 0.1, 0.0 ) = 0.6 \]

\[ \text{Nec} (A_i, X) = \min (0.4, 0.5, 0.8, 1.0, 0.9, 0.5, 0.1, 0.0 ) = 0.0 \]
Encoding and decoding: an overview

$X \lambda_1, \lambda_2, \ldots, \lambda_c$

$\mu_1, \mu_2, \ldots, \mu_c$

$\{A_1, A_2, \ldots, A_c\}$

$X^\wedge, X^\sim$

$\{A_1, A_2, \ldots, A_c\}$

Pedrycz and Gomide, FSE 2007
Design of the decoder of fuzzy data

- Given the nature of encoding (possibility and necessity measures), the decoding is regarded as a certain “inverse” problem in terms of fuzzy relational equations:
  
  - Possibility measure: sup-t composition
  
  - Necessity measure: inf-s composition
Decoding – possibility measure

Possibility measure: sup-t composition

\[ \hat{X}(x) = A(x) \varphi_\lambda = \begin{cases} 1 & \text{if } A(x) \leq \lambda \\ \lambda & \text{otherwise} \end{cases} \]

\[ \hat{X}(x) = A(x) \rightarrow \lambda = \sup\{a \in [0,1] | a \ t \ A(x) \leq \lambda\} \]

\[ \hat{X} = \bigcap_{i=1}^{c} \hat{X}_i \]
Decoding - necessity measure

Necessity measure: inf-s composition

\[ \tilde{X}(x) = (1 - A(x)) \varepsilon \mu = \begin{cases} \mu, & \text{if } 1 - A(x) < \mu \\ 0, & \text{otherwise} \end{cases} \]

\[ \tilde{X}(x) = (1 - A_i(x)) \varepsilon \mu = \inf \{ a \in [0,1] \mid \text{as } (1 - A(x)) \geq \mu \} \]

\[ \tilde{X} = \bigcup_{i=1}^{c} \tilde{X}_i \]

\[ \tilde{X} \subseteq X \subseteq \hat{X} \]
Decoding: example

Possibility measure
Decoding: example

Necessity measure
Decoding example

• Bounds of possibility and necessity measure
Taxonomy of data in structure description with shadowed sets

- Core structure
- Shadowed data structure
- Uncertain data structure
- **Core data structure**

  - patterns that belong to a core of at least one shadowed sets

  - core data structure = \{ x | \exists i x \in \text{Core}(A_i) \}
Shadowed data structure

- patterns that do not belong to a core of any shadowed set
- core fall within the shadow of one or more shadowed sets
- shadowed data structure = \{ x \mid \exists i \ x \in \text{Shadow}(A_i) \text{ and } \forall x \notin \text{Core} (A_i) \}
Uncertain data structure

- patterns that left out from all shadows

- uncertain data structure = \{ x | \exists i \ x \not\in \text{Shadow}(A_i) \text{ and } \forall x \not\in \text{Core}(A_i) \}
Three-valued characterization of data structure with shadowed sets
Three-valued characterization of data structure: Example

Pedrycz and Gomide, FSE 2007
Three-valued characterization of data structure: example