3 Characterization of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing
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3.1 Generic characterization of fuzzy sets: Some fundamental descriptors
Fuzzy sets

- Fuzzy sets are membership functions.
- In principle: any function is “eligible” to describe fuzzy sets.
- In practice it is important to consider:
  - type, shape, and properties of the function
  - nature of the underlying phenomena
  - semantic soundness

\[ A: X \rightarrow [0, 1] \]
Normality

$hgt(A) = 1$

Normal

$hgt(A) < 1$

Subnormal

$hgt(A) = \sup_{x \in X} A(x)$
Normalization

Subnormal

\[ hgt(A) < 1 \]

\[ \text{Norm}(A)(x) = \frac{A(x)}{hgt(A)} \]

Normal

\[ hgt(\text{Norm}(A)) = 1 \]
Support

\[ \text{Supp}(A) = \{ x \in X \mid A(x) > 0 \} \]

**Open set**

\[ \text{CSupp}(A) = \text{closure}\{ x \in X \mid A(x) > 0 \} \]

**Closed set**
Core

\[ \text{Core}(A) = \{ x \in X \mid A(x) = 1 \} \]
\(\alpha\)-cut

\[ A_\alpha = \{ x \in X \mid A(x) \geq \alpha \} \]

\[ A_\alpha^+ = \{ x \in X \mid A(x) > \alpha \} \]

Stronger condition
Convexity

$A[\lambda x_1 + (1-\lambda)x_2] \geq \min[A(x_1), A(x_2)]$

$x = \lambda x_1 + (1-\lambda)x_2$

$0 \leq \lambda \leq 1$

Convex fuzzy set

Nonconvex

$A_\alpha = \{x \in X | A(x) > \alpha\}$
Cardinality

\[ \text{Card}(A) = \sum_{x \in X} A(x) \quad \text{X finite or countable} \]

\[ \text{Card}(A) = \int_{X} A(x) \, dx \]

\[ \text{Card}(A) = |A| \quad \text{sigma count (\(\sigma\)-count)} \]
3.2 Equality and inclusion relationships for fuzzy sets
Equality

\[ A = B \iff A(x) = B(x) \quad \forall x \in X \]

Inclusion

\[ A \subseteq B \iff A(x) \leq B(x) \quad \forall x \in X \]
Sets

\[ A \subseteq B \]

\[ A \not\subseteq B \]
Degree of inclusion

\[ \|A(x) \subseteq B(x)\| = \frac{1}{\text{Card}(X)} \int_X (A(x) \Rightarrow B(x)) \, dx \]

\[ A(x) \Rightarrow B(x) = \begin{cases} 1 & \text{if } A(x) \leq B(x) \\ 1 - A(x) + B(x) & \text{otherwise} \end{cases} \]
Degree of equality

\[ \|A(x) = B(x)\| = \frac{1}{\text{Card}(X)} \int_X \left[ \min(A(x) \Rightarrow B(x), B(x) \Rightarrow A(x)) \right] dx \]
Example

Examples of fuzzy sets $A$ and $B$ along with their degrees of inclusion:

(a) $a = 0$, $n = 2$, $b = 3$; $m = 4$, $\sigma = 2$; $\|A = B\| = 0.637$

(b) $b = 7$; $\|A = B\| = 0.864$

(c) $a = 0$, $n = 2$, $b = 9$, $m = 4$, $\sigma = 0.5$; $\|A = B\| = 0.987$
3.3 Energy and entropy measures of fuzziness
Energy measure of fuzziness

\[ E(A) = \sum_{i=1}^{n} e[A(x_i)] \]

\[ E(A) = \int_{\mathbf{X}} e[A(x)]dx \]

\[ \text{Card (X)} = n \]

\[ e : [0, 1] \rightarrow [0, 1] \text{ such that} \]

\[ e(0) = 0 \]

\[ e(1) = 1 \]

\[ e: \text{monotonically increasing} \]

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Pedrycz and Gomide, FSE 2007
Example

\[ e(u) = u \quad \forall u \in [0, 1] \]

\[ E(A) = \sum_{i=1}^{n} A(x_i) = \text{Card}(A) \]

\[ E(A) = \sum_{i=1}^{n} A(x_i) = \sum_{i=1}^{n} |A(x_i) - \phi(x_i)| = d(A, \phi) \]

\[ d = \text{Hamming distance} \]
$e(u)$ non-linear

Emphasis on high membership values

Emphasis on low membership values
Inclusion of probabilistic information

\[ E(A) = \sum_{i=1}^{n} p_i e[A(x_i)] \]

\[ E(A) = \int_X p(x) e[A(x)] dx \]

\( p_i \): probability of \( x_i \)

\( p(x) \): probability density function

Pedrycz and Gomide, FSE 2007
Entropy measure of fuzziness

\[ H(A) = \sum_{i=1}^{n} h[A(x_i)] \]

\[ H(A) = \int_{X} h(A(x)) \, dx \]

\[ h : [0,1] \rightarrow [0,1] \]

1-monotonically increasing \([0, \frac{1}{2}]\]

2-monotonically decreasing \((\frac{1}{2}, 1]\]

3-boundary conditions:

\[ h(0) = h(1) = 0 \]
\[ h(\frac{1}{2}) = 1 \]
Specificity of fuzzy sets

Specific fuzzy set

Lack of specificity

Pedrycz and Gomide, FSE 2007
Specificity

1-\(\text{Spec}(A) = 1\) if and only if \(\exists x_0 \in A(x_0) = 1, A(x) = 0 \ \forall x \neq x_0\)

2-\(\text{Spec}(A) = 0\) if and only if \(A(x) = 0 \ \forall x \in X\)

3-\(\text{Spec}(A_1) \leq \text{Spec}(A_2)\) if \(A_1 \supset A_2\)
Examples

\[ Spec(A) = \int_{0}^{\alpha_{\text{max}}} \frac{1}{\text{Card}(A_{\alpha})} d\alpha \]

\[ Spec(A) = \sum_{i=1}^{m} \frac{1}{\text{Card}(A_{\alpha_i})} \Delta\alpha_i \]

*Yager (1993)*
Geometric interpretation of sets and fuzzy sets

\[ X = \{ x_1, x_2 \} \quad P(X) = \{ \emptyset, \{ x_1 \}, \{ x_2 \}, \{ x_1, x_2 \} \} \]
3.4 Granulation of information
Motivation

- **Need of granulation:**
  - abstract information
  - summarize information

- **Purpose:**
  - comprehension
  - decision making
  - description
Discretization, quantization, granulation

Discretization

Quantization

Granulation

Discretization, quantization, granulation

Pedrycz and Gomide, FSE 2007
Formal mechanisms of granulation

\[ \langle X, G, S, C \rangle \]

- \( X \): universe
- \( G \): formal framework of granulation
- \( S \): collection of information granules
- \( C \): transformation

Pedrycz and Gomide, FSE 2007
3.5 Characterization of families of fuzzy sets
Frame of cognition

- Codebook of conceptual entities
  - family of linguistic landmarks
  \[ \Phi = \{ A_1, A_2, \ldots, A_m \} \]
  \[ A_i \text{ is a fuzzy set on } X, \ i = 1, \ldots, m \]

- Granulation that satisfies semantic constraints
  - coverage
  - semantic soundness
Coverage

$\Phi = \{A_1, A_2, \ldots, A_m\}$ covers $X$ if, for any $x \in X$

$\exists i \in I \mid A_i(x) > 0$

$\exists i \in I \mid A_i(x) > \delta$  ($\delta$-level coverage) $\delta \in [0, 1]$

$A_i$'s are fuzzy set on $X$, $i \in I = \{1, \ldots, m\}$
Semantic soundness

- Each $A_i$, $i \in I = \{1, \ldots, m\}$ is unimodal and normal
- Fuzzy sets $A_i$ are disjoint enough ($\lambda$-overlapping)
- Number of elements of $\Phi$ is low
Characteristics of frames of cognition

- Specificity: $\Phi_1$ more specific than $\Phi_2$ if $\text{Spec}(A_{1i}) > \text{Spec}(A_{2j})$
- Granularity: $\Phi_1$ finer than $\Phi_2$ if $|\Phi_1| > |\Phi_2|$
Focus of attention

Regions of focus of attention implied by the corresponding fuzzy sets
• Information hiding

\[ x \in [a_2, a_3] \text{ indistinguishable for } A, \text{ but not for } B \]
3.6 Fuzzy sets, sets and the representation theorem
Any fuzzy set can be viewed as a family of sets:

\[ A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha \]

\[ A(x) = \sup_{\alpha \in [0,1]} \alpha A_\alpha (x) \]
Example

\( X = \{1, 2, 3, 4\} \)

\( A = \{0/1, 0.1/2, 0.3/3, 1/4, 0.3/5\} = [0, 0.1, 0.3, 1, 0.3] \)

\( A_{0.1} = \{0/1, 1/2, 1/3, 1/4, 1/5\} = [0, 1, 1, 1, 1] \rightarrow 0.1A_{0.1} = [0, 0.1, 0.1, 0.1, 0.1] \)

\( A_{0.3} = \{0/1, 0/2, 1/3, 1/4, 1/5\} = [0, 0, 1, 1, 1] \rightarrow 0.3A_{0.3} = [0, 0, 0.3, 0.3, 0.3] \)

\( A_1 = \{0/1, 0/2, 0/3, 1/4, 0/5\} = [0, 0, 0, 1, 0] \rightarrow 1.0A_1 = [0, 0, 0, 1, 0] \)

\( A = \text{max} \left( 0.1A_{0.1}, 0.3A_{0.3}, 1A_1 \right) \)

\( A = [\text{max} \left( 0, 0, 0 \right), \text{max} \left( 0.1, 0, 0 \right), \text{max} \left( 0.1, 0.3, 0 \right), \text{max} \left( 0.1, 0.3, 1 \right), \text{max} \left( 0.1, 0.3, 0 \right)] \)

\( A = [0, 0.1, 0.3, 1, 0.3] \)