4 Design of Fuzzy Sets

Fuzzy Systems Engineering
Toward Human-Centric Computing
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4.1 Semantics of fuzzy sets: General observations
Semantics of fuzzy sets

• Generic constructs/building conceptual blocks to describe systems in a meaningful way

• Each fuzzy set comes with a well-delineated semantics (meaning)
  – Example: small – medium – large error

• Limited number of fuzzy sets
  – “magic” number of 7 +/- 2 (Miller, 1956) (short –term memory)
• Fuzzy sets require calibration
  – determination of their membership functions

• Two main approaches to the problem:
  – Expert –driven (designer, user, decision-maker…)
  – Data driven (from data to fuzzy sets)
4.2 Fuzzy sets as a descriptor of feasible solutions
Consider some function $f(x)$ defined in $\Omega$,

$$f: \Omega \rightarrow \mathbb{R}, \text{ where } \Omega \subseteq \mathbb{R}$$

Determine its maximum

$$x^{\text{opt}} = \arg \max_x f(x).$$

Fuzzy set $A$ of optimal solutions $\equiv$ a collection of feasible solutions that could be labeled as optimal with some degrees of membership.

$$A(x) = \frac{f(x) - f_{\text{min}}}{f_{\text{max}} - f_{\text{min}}}$$

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Consider some function $f(x)$ defined in $\Omega$,

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Determine its minimum

$$x^{\text{opt}} = \arg \max_x f(x).$$

Fuzzy set $A$ of optimal solutions $\equiv$ a collection of feasible solutions that could be labeled as optimal with some degrees of membership.

$$A(x) = 1 - \frac{f(x) - f_{\text{min}}}{f_{\text{max}} - f_{\text{min}}}$$
Fuzzy sets as descriptors of feasible solutions

Example

Linearization error

Linearize function $y = g(x) = \exp(-x)$ around $x_0=1$ and assess the quality of this linearization in the range $[-1, 7]$.

Linearization formula: $y - y_0 = g'(x_0)(x - x_0)$

$y_0 = g(x_0)$ and $g'(x_0)$ is the derivative of $g(x)$ at $x_0$.

Linearized version of the function $\exp(-1)(2 - x)$.

Quality of linearization $f(x) = |g(x) - \exp(-1)(2 - x)|$.

$$A(x) = 1 - \frac{f(x) - f_{\text{min}}}{f_{\text{max}} - f_{\text{min}}}$$

$f_{\text{max}} = f(7) = 1.84$ and $f_{\text{min}} = 0.0$
Fuzzy sets as descriptors of feasible solutions

Example

\[ A(x) = 1 - \frac{f(x) - f_{\text{min}}}{f_{\text{max}} - f_{\text{min}}} \]
4.3 Fuzzy sets as a descriptor of the notion of typicality
Fuzzy sets as notions of typicality

- Fuzzy set as collection of elements of varying degrees of typicality

- Geometric figures: squares, circles....

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4.4 Membership functions in the visualization of preferences solutions
Fuzzy sets in visualization of preferences of solutions

\[ P = i^2 R = \left( \frac{E}{R+r} \right)^2 R \]
4.5 Nonlinear transformations of fuzzy sets
4.6 Vertical and horizontal schemes of membership estimation
Horizontal scheme of membership estimation

Finite elements of the universe of discourse X
Question of the form

-does x belong to concept A?

Accepted are binary answers (yes-no)

“n” experts – count of positive (yes) answers: p/n

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Horizontal scheme of membership estimation

Binary replies follow binomial distribution; we can determine confidence interval

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

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Vertical scheme of membership estimation

Estimation of membership function by determining $\alpha$-cuts and aggregating them (see representation theorem)

- What are the elements of $X$ which belong to fuzzy set $A$ at degree not lower than $\alpha$?
Horizontal and vertical schemes of membership estimation

Simple and intuitively appealing

Reflective of domain knowledge

Lack of continuity – elements of $X$ considered independently

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Saaty’s priority method of pairwise comparison

Collection of elements $x_1, x_2, \ldots, x_n$

Membership degrees are given $A(x_1), A(x_2) \ldots A(x_n)$

Reciprocal matrix $R$

$$R = [r_{ij}] = \begin{bmatrix} \frac{A(x_1)}{A(x_1)} & \frac{A(x_1)}{A(x_2)} & \cdots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_1)}{A(x_2)} & \frac{A(x_2)}{A(x_2)} & \cdots & \frac{A(x_2)}{A(x_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A(x_1)}{A(x_n)} & \frac{A(x_2)}{A(x_n)} & \cdots & \frac{A(x_n)}{A(x_n)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{A(x_1)}{A(x_2)} & \cdots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & 1 & \cdots & \frac{A(x_2)}{A(x_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_n)}{A(x_2)} & \cdots & 1 \end{bmatrix}$$

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Saaty’s priority method of pairwise comparison

Reciprocal matrix $R = [r_{ij}] = \begin{bmatrix} \frac{A(x_1)}{A(x_i)} & \frac{A(x_1)}{A(x_2)} & \cdots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & \frac{A(x_2)}{A(x_2)} & \cdots & \frac{A(x_2)}{A(x_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_n)}{A(x_2)} & \cdots & \frac{A(x_n)}{A(x_n)} \end{bmatrix} = 1 \begin{bmatrix} 1 & \frac{A(x_1)}{A(x_2)} & \cdots & \frac{A(x_1)}{A(x_n)} \\ \frac{A(x_2)}{A(x_1)} & 1 & \cdots & \frac{A(x_2)}{A(x_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A(x_n)}{A(x_1)} & \frac{A(x_2)}{A(x_1)} & \cdots & 1 \end{bmatrix}$

Reciprocal matrix $R$ – main properties:
(a) reflexivity
(b) reciprocality
(c) transitivity

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Saaty’s priority method of pairwise comparison: computing

\[
\begin{bmatrix}
A(x_i) & A(x_i) & \cdots & A(x_i) \\
A(x_1) & A(x_2) & & A(x_n) \\
& & & \\
& & & \\
A(x_1) & A(x_2) & & A(x_n) \\
& & & \\
& & & \\
& & & \\
A(x_1) & A(x_2) & & A(x_n)
\end{bmatrix}
\begin{bmatrix}
A(x_1) \\
A(x_2) \\
\vdots \\
A(x_n)
\end{bmatrix}

i-th row of R

\[ RA = nA \]

n-the largest eigenvalue of R

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Saaty’s priority method of pairwise comparison

Estimation of reciprocal matrix:

Scale (typically 1-7 range, could be larger, 1-9)
- strong preference: high values on the scale (7-9)
- preference: 4-7
- weak preference or no preference 1-3

Solving the eigenvalue problem for R, max eigenvalue, $\lambda_{\text{max}}$
Saaty’s priority method: consistency of results

\[ \nu = \frac{(\lambda_{\text{max}} - n)}{(n-1)} \]

lack of consistency \( \nu > 0.1 \)
Saaty's priority method: Example

*high* temperature

Universe of discourse: 10, 20, 30, 40, 45

Scale 1-5

\[
R = \begin{bmatrix}
1 & 1/2 & 1/4 & 1/5 \\
2 & 1 & 1/3 & 1/4 \\
4 & 3 & 1 & 1/3 \\
5 & 4 & 3 & 1 \\
\end{bmatrix}
\]

max eigenvalue = 4.114 eigenvector [0.122 0.195 0.438 0.869] after normalization [0.14 0.22 0.50 1.00].

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Fuzzy sets as granular representation of numeric data

The principle of justifiable granularity

experiment-driven and intuitively appealing rationale:

(a) we expect that A reflects (or matches) the available experimental data to the highest extent, and

(b) second, the fuzzy set is kept specific enough so that it comes with a well-defined semantics.
The principle of justifiable granularity

(a) we expect that A reflects (or matches) the available experimental data to the highest extent, and

Maximize “coverage” of data

(b) second, the fuzzy set is kept specific enough so that it comes with a well-defined semantics.

Minimize spread of fuzzy set

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The principle of justifiable granularity: unimodal fuzzy set

Numeric data $x_1, x_2, \ldots, x_n$

Determine its “modified” median

Consider separately data to the left and right from the median

\[ \max_{a \neq m} \left( \sum_k A(x_k) \right) \]

\[ \frac{\sum_k A(x_k)}{|m - a|} \]
The principle of justifiable granularity: examples

Distance of point from geometric figure

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The principle of justifiable granularity: examples

Distance between two geometric figures A and B

\[ d_H(A, B) = \max \{ \sup_{x \in A} [\min_{y \in B} d(x, y)], \sup_{y \in B} [\min_{x \in A} d(x, y)] \} \]

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Clustering: Fuzzy C-Means (FCM)

Given a collection of n-dimensional data set \( \{x_k\}, k=1,2,\ldots,N, \) determine its structure – a collection of “c” clusters.

Minimize the following objective function (performance index) \( Q \)

\[
Q = \sum_{i=1}^{c} \sum_{k=1}^{N} u_{ik}^m \| x_k - v_i \|^2
\]

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Fuzzy clustering: structure representation

Partition matrix \( U \)

Prototypes \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_c \)

\[
\sum_{i=1}^{c} u_{ik} = 1, \quad k = 1, 2, \ldots, N
\]

\[
0 < \sum_{k=1}^{N} u_{ik} < N, \quad i = 1, 2, \ldots, c
\]

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FCM – optimization procedure

Optimization with respect to

- Partition matrix $U$, and
- Prototypes $v_1, v_2, \ldots, v_c$
Optimization: partition matrix

use of Lagrange multipliers

\[ V = \sum_{i=1}^{c} u_{ik} m^2 + \lambda(\sum_{i=1}^{c} u_{ik} - 1) \]

\[ \frac{\partial V}{\partial u_{st}} = 0 \quad \frac{\partial V}{\partial \lambda} = 0 \]
Optimization: partition matrix

\[ V = \sum_{i=1}^{c} u_{ik} d_{ik}^2 + \lambda \left( \sum_{i=1}^{c} u_{ik} - 1 \right) \]

\[ \frac{\partial V}{\partial u_{st}} = 0 \quad \frac{\partial V}{\partial \lambda} = 0 \]

\[ \frac{\partial V}{\partial u_{st}} = m u_{st}^{m-1} d_{st}^2 + \lambda \]

\[ u_{st} = -\left( \frac{\lambda}{m} \right)^{1 \over m-1} d_{st}^{2 \over m-1} \]

\[ -\left( \frac{\lambda}{m} \right)^{1 \over m-1} = \frac{1}{\sum_{j=1}^{c} d_{jt}^{m-1}} \]

\[ u_{st} = \frac{1}{\sum_{j=1}^{c} \left( \frac{d_{st}^2}{d_{jt}^2} \right)^{m-1}} \]

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Optimization: prototypes

\[ Q = \sum_{i=1}^{c} \sum_{k=1}^{N} u_{ik}^m \sum_{j=1}^{n} (x_{kj} - v_{ij})^2 \]

Gradient of \( Q \) w.r.t. prototype \( v_s \)

\[ \sum_{k=1}^{N} u_{ik}^m (x_{kt} - v_{st}) = 0 \]

\[ v_{st} = \frac{\sum_{k=1}^{N} u_{ik}^m x_{kt}}{\sum_{k=1}^{N} u_{ik}^m} \]

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FCM: an overview of the algorithm

**procedure** FCM-CLUSTERING (x)  **returns** prototypes and partition matrix

**input**: data $x = \{x_1, x_2, ..., x_k\}$

**local**: fuzzification parameter: $m$

threshold: $\varepsilon$

norm: $\|\|$ 

**INITIALIZE-PARTITION-MATRIX**

t ← 0

**repeat**

for $i = 1:c$  do

\[
\sum_{k=1}^{N} u_{ik}^{m}(t)x_k
\]

compute prototypes

\[
v_i(t) \leftarrow \frac{\sum_{k=1}^{N} u_{ik}^{m}(t)x_k}{\sum_{k=1}^{N} u_{ik}^{m}(t)}
\]

for $i = 1:c$  do

for $k = 1:N$  do

update partition matrix

\[
u_{ik}(t+1) = \frac{1}{\sum_{j=1}^{c} \left( \frac{\|x_k - v_j(t)\|}{\|x_k - v_j(t)\|} \right)^{2(m-1)}}
\]

update partition matrix

\[
\text{t} \leftarrow \text{t} + 1
\]

until $\|U(t+1)-U(t)\| \leq \varepsilon$

**return** $U, V$

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FCM and its parameters

Number of clusters \((c)\)

Objective function \(Q\)

Distance function \(\| . \|\)

Fuzzification coefficient \((m)\)

Termination criterion

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Geometry of clusters and fuzzification coefficient (m)

\[ m = 1.2 \quad m = 2.0 \quad m = 3.5 \]
Cluster sharing: a separation measure

\[ \phi(u_1, u_2, ..., u_c) = 1 - c^c \prod_{i=1}^{c} u_i \]

Data fully belongs to a single cluster (1- 0)

Data belongs to all clusters at the same level (1/c)

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Hierarchical format of FCM: Successive refinements of clusters

\[ V_i = \sum_{k=1}^{N} u_{ik}^m \| x_k - v_i \|_2^2 \]

\[ X(i_0) = \{ x_k \in X | u_{i_0k} = \max_i u_{ik} \} \]

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Fuzzy equalization

Construct triangular fuzzy sets $A_1$, $A_2$, ..., $A_c$ defined in $\mathbb{R}$ such that they come with the same level of experimental evidence (support)

\[\sum_{k=l_k}^{N} A_1(x_k) = \frac{N}{2(c-1)}\]

\[\sum_{k=l_k}^{N} A_2(x_k) = \frac{N}{(c-1)}\]

\[\sum_{k=l_k}^{N} A_{c-1}(x_k) = \frac{N}{(c-1)}\]

\[\sum_{k=l_k}^{N} A_c(x_k) = \frac{N}{2(c-1)}\]

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Linguistic approximation

Given is a family of reference fuzzy sets \( \{A_i\} \) defined in some space \( X \)

We have at disposal is a family of linguistic modifiers \( \tau_j \), say

- more or less (dilution),
- very (concentration)

Represent (approximate) \( B \) in \( X \) with the use of reference fuzzy sets and linguistic modifiers == linguistic approximation

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Linguistic approximation: optimization

\[ B \approx \tau_i(A_j) \]

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Construction of fuzzy sets: Design guidelines (1)

Strive for highly visible and well-defined semantics of information granules.

Keep the number of information granules quite low (7 ± 2 fuzzy sets).

There are several fundamental views at fuzzy sets and depending upon them, consider the use of various estimation techniques.

Fuzzy sets are context-sensitive constructs and require careful calibration. This
The calibration mechanisms being used in the design of the membership function are reflective of human-centricity of fuzzy sets.

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Construction of fuzzy sets: Design guidelines (2)

two major categories of approaches supporting the design of membership functions:

data-driven and expert (user)-driven.

The user-driven membership estimation uses the statistics of data yet in an *implicit* manner.

The granular term – fuzzy sets come into existence once there is some experimental evidence behind them

the development of fuzzy sets can be carried out in an stepwise manner.