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11.1 Fuzzy rules as a vehicle of knowledge representation
Rule $\equiv$ conditional statement

- If $\langle$ input variable is $A \rangle$ then $\langle$ output variable is $B \rangle$
  - $A$ and $B$: descriptors of pieces of knowledge
  - rule: expresses a relationship between inputs and outputs

- Example
  - If $\langle$ the temperature is high $\rangle$ then $\langle$ the electricity demand is high $\rangle$

- If and then parts $\langle$.......$\rangle$ formed by information granules
  - sets
  - rough sets
  - fuzzy sets

Pedrycz and Gomide, FSE 2007
Rule-based system/model (FRBS)

• FRBS is a family of rules of the form

If (input variable is \(A_i\)) then (output variable is \(B_i\))

\(i = 1, 2, ..., c\)

\(A_i\) and \(B_i\) are information granules

• More complex rules

If (input variable_1 is \(A_i\)) and (input variable_2 is \(B_i\)) and ....
then (output variable is \(Z_i\))

– multidimensional input space (Cartesian product of inputs)
– individual inputs aggregated by the \textit{and} connective
– highly parallel, modular granular model

Pedrycz and Gomide, FSE 2007
11.2 General categories of fuzzy rules and their semantics
Multi-input multi-output fuzzy rules

• If \( X_1 \) is \( A_1 \) and \( X_2 \) is \( A_2 \) and ..... and \( X_n \) is \( A_n \)
  then \( Y_1 \) is \( B_1 \) and \( Y_2 \) is \( B_2 \) and ..... and \( Y_m \) is \( B_m \)

\( X_i = \) variables whose values are fuzzy sets \( A_i \)
\( Y_j = \) variables whose values are fuzzy sets \( B_j \)

\( A_i \) on \( X_i \), \( i = 1, 2, \ldots, n \)

\( B_j \) on \( Y_j \), \( j = 1, 2, \ldots, m \)

• No loss of generality if we assume rules of the form

\textbf{If } \( X \) is \( A \) \textit{and} \( Y \) is \( B \) \textbf{then} \( Z \) is \( C \)
Certainty-qualified rules

- **If** $X$ is $A$ **and** $Y$ is $B$ **then** $Z$ is $C$ with certainty $\mu$

  $\mu \in [0,1]$

  $\mu$ : degree of certainty of the rule

  $\mu = 1$ rule is certain

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Gradual rules

- the *more* $X$ is $A$, the *more* $Y$ is $B$
  - relationships between changes in $X$ and $Y$
  - captures tendency between information granules

- Examples:

  - the *higher* the income, the *higher* the taxes
  - the *lower* the temperature, the *higher* energy consumption

Pedrycz and Gomide, FSE 2007
Functional fuzzy rules

• If $X$ is $A_i$ then $y = f(x,a_i)$

$f : X \rightarrow Y$

$x \in \mathbb{R}^n$

• Rule: confines the function to the support of granule $A_i$

$f : \text{linear or nonlinear (neural nets, etc..)}$

• Highly modular models

Pedrycz and Gomide, FSE 2007
11.3 Syntax of fuzzy rules
Backus-Naur form (BNF)

\[
\begin{align*}
\text{If_then_rule} &::= \text{if} \langle \text{antecedent} \rangle \text{ then } \langle \text{consequent} \rangle \{\langle \text{certainty} \rangle \} \\
\text{gradual_rule} &::= \langle \text{word} \rangle \langle \text{antecedent} \rangle \langle \text{word} \rangle \langle \text{consequent} \rangle \\
\langle \text{word} \rangle &::= \langle \text{more} \rangle \{\langle \text{less} \rangle \} \\
\langle \text{antecedent} \rangle &::= \langle \text{expression} \rangle \\
\langle \text{consequent} \rangle &::= \langle \text{expression} \rangle \\
\langle \text{expression} \rangle &::= \langle \text{disjunction} \rangle \{\text{and} \langle \text{disjunction} \rangle \} \\
\langle \text{disjunction} \rangle &::= \langle \text{variable} \rangle \{\text{or} \langle \text{variable} \rangle \} \\
\langle \text{variable} \rangle &::= \langle \text{attribute} \rangle \text{ is } \langle \text{value} \rangle \\
\langle \text{certainty} \rangle &::= \langle \text{none} \rangle \{\text{certainty } \mu \in [0,1] \}
\end{align*}
\]
Construction of computable representations

Main steps:

1. specification of the fuzzy variables to be used

2. association of the fuzzy variables using fuzzy sets

3. computational formalization of each rule using fuzzy relations and definition of aggregation operator to combine rules together
11.4 Basic functional modules of FRBS
General architecture of FRBS

Input Interface

Rule Base

Data Base

Fuzzy Inference

Output Interface

X → Rule Base → Fuzzy Inference → Output Interface → Y

Fuzzy if-then rules (input-output relationship)
Parameters of the FRBS
Process inputs and rules (approximate reasoning)

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Input interface

- (attribute) of (input) is (value)

  the temperature of the motor is \textit{high}

- Canonical (atomic) form

  \[ p: X \text{ is } A \]

  \( \text{temperature (motor) is } \textit{high} \)

  \( \hat{A} \)

  fuzzy set

\[ x \text{ (°C)} \]
Multiple fuzzy inputs: conjunctive canonical form

\[ p : X_1 \text{ is } A_1 \text{ and } X_2 \text{ is } A_2 \text{ and } \ldots \text{ and } X_n \text{ is } A_n \]  

\( X_i \) are fuzzy (linguistic) variables

\( A_i : \) fuzzy sets on \( X_i \)

\( i = 1, 2, \ldots, n \)

Compound proposition induces a fuzzy relation \( P \) on \( X_1 \times X_1 \times \ldots \times X_n \)

\[ P(x_1, x_2, \ldots, x_n) = A_1(x_1)tA_2(x_2)t\ldots tA_n(x_n) = \bigotimes_{i=1}^{n} A_i(x_i) \]

\( t(T) = \) t-norm

\[ p : (X_1, X_2, \ldots, X_n) \text{ is } P \]
Example

- Fuzzy relation associated with \((X,Y)\) is \(P\)
- Triangular fuzzy sets \(A_1(x,4,5,6) = A, \ A_2(y,8,10,12) = B\)
- t-norm: algebraic product

Pedrycz and Gomide, FSE 2007
Multiple fuzzy inputs: disjunctive canonical form

$q : X_1 \text{ is } A_1 \text{ or } X_2 \text{ is } A_2 \text{ or } \ldots \text{ or } X_n \text{ is } A_n$  \hspace{1cm} \text{disjunctive canonical form}

$X_i$ are fuzzy (linguistic) variables

$A_i : \text{fuzzy sets on } X_i$

$i = 1, 2, \ldots, n$

Compound proposition induces a fuzzy relation $Q$ on $X_1 \times X_1 \times \ldots \times X_n$

\[
Q(x_1, x_2, \ldots, x_n) = A_1(x_1) \land A_2(x_2) \land \ldots \land A_n(x_n) = \bigvee_{i=1}^{n} A_i(x_i) \quad s\ (S) = \text{t-conorm}
\]

$q : (X_1, X_2, \ldots, X_n) \text{ is } Q$
Example

• Fuzzy relation associated with \((X,Y)\) is \(Q\)
• Triangular fuzzy sets \(A_1(x,4,5,6) = A, \quad A_2(y,8,10,12) = B\)
• t-conorm: probabilistic sum

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Rule base

• Fuzzy rule: If $X$ is $A$ then $Y$ is $B$ ≡ relationship between $X$ and $Y$

• Semantics of the rule is given by a fuzzy relation $R$ on $X \times Y$

• $R$ determined by a relational assignment

$$R(x,y) = f (A(x),B(y)) \quad \forall (x, y) \in X \times Y$$

$$f : [0,1]^2 \rightarrow [0,1]$$

• In general $f$ can be

  – fuzzy conjunction: $f_i$
  – fuzzy disjunction: $f_s$
  – fuzzy implication: $f_i$

Pedrycz and Gomide, FSE 2007
Fuzzy conjunction

Choose a t-norm \( t \) and define:

\[
R(x,y) \equiv f_t(x,y) = A(x) \ t B(y) \quad \forall (x,y) \in X \times Y
\]

Examples:

• \( t = \text{min} \)

\[
R_c(x,y) \equiv f_c(x,y) = \min[A(x) \ t B(y)] \quad \text{(Mamdani)}
\]

• \( t = \text{algebraic product} \)

\[
R_p(x,y) \equiv f_p(x,y) = A(x)B(y) \quad \text{(Larsen)}
\]
Example: $t = \min$

$$R_c(x,y) = \min \{ A(x), B(y) \}$$

$$\forall (A(x), B(y)) \in [0,1]^2$$

References:

Pedrycz and Gomide, FSE 2007
Example: $t = \text{algebraic product}$

$$R_p(x,y) = A(x)B(y)$$

$$\forall (A(x), B(y)) \in [0,1]^2$$

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Fuzzy disjunction

Choose a t-conorm \( s \) and define:

\[
R_s(x,y) \equiv f_s(x,y) = A(x) \ s B(y) \quad \forall (x,y) \in X \times Y
\]

Examples:

- \( s = \text{max} \)
  \[
  R_m(x,y) \equiv f_m(x,y) = \max[A(x) \ t B(y)]
  \]

- \( s = \text{Lukasiewicz t-conorm} \)
  \[
  R_\ell(x,y) \equiv f_\ell(x,y) = \min[1, A(x) + B(y)]
  \]
Example: \( s = \max \)

\[
R(x, y) = \max\{A(x), B(y)\}
\]

\[
\forall (A(x), B(y)) \in [0, 1]^2
\]

\[
A(x) = A(x, 4, 5, 6)
\]

\[
B(y) = B(y, 4, 5, 6)
\]
Example: $s = \text{Lukasiewicz}$

$$R_c(x,y) = \min\{1, A(x)+B(y)\}$$

$$\forall (A(x), B(y)) \in [0,1]^2$$

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Fuzzy implication

- Choose a fuzzy implication \( f_i \) and define:
  \[
  R_i(x,y) \equiv f_i(x,y) \quad \forall (x,y) \in X \times Y
  \]

- \( f_i : [0,1]^2 \rightarrow [0,1] \) is a fuzzy implication if:
  1. \( B(y_1) \leq B(y_2) \Rightarrow f_i(A(x), B(y_1)) \leq f_i(A(x), B(y_2)) \)  \( \text{monotonicity 2}^{\text{nd}} \text{ argument} \)
  2. \( f_i(0, B(y)) = 1 \)  \( \text{dominance of falsity} \)
  3. \( f_i(1, B(y)) = B(y) \)  \( \text{neutrality of truth} \)

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Further requirements may include:

4. $A(x_1) \leq A(x_2) \Rightarrow f_i(A(x_1), B(y)) \geq f_i(A(x_2), B(y))$ \hspace{2cm} monotonicity 1st argument

5. $f_i(A(x_1), f_i(A(x_2), B(y)) = f_i(A(x_2), f_i(A(x_1), B(y))$ \hspace{2cm} exchange

6. $f_i(A(x), A(x)) = 1$ \hspace{2cm} identity

7. $f_i(A(x), B(y)) = 1 \Leftrightarrow A(x) \leq B(y)$ \hspace{2cm} boundary condition

8. $f_i$ is a continuous function \hspace{2cm} continuity

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### Examples of fuzzy implications

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lukasiewicz</td>
<td>( f_l(A(x), B(y)) = \min [1, 1 - A(x) + B(y)] )</td>
<td></td>
</tr>
<tr>
<td>Pseudo-Lukasiewicz</td>
<td>( f_{\lambda}(A(x), B(y)) = \min [1, \frac{1 - A(x) + (\lambda + 1)B(y)}{1 + \lambda A(x)}] )</td>
<td>( \lambda &gt; -1 )</td>
</tr>
<tr>
<td>Pseudo-Lukasiewicz</td>
<td>( f_w(A(x), B(y)) = \min [1, (1 - A(x)^w + B(y)^w)^{1/w}] )</td>
<td>( w &gt; 0 )</td>
</tr>
<tr>
<td>Gaines</td>
<td>( f_a(A(x), B(y)) = \begin{cases} 1 &amp; \text{if } A(x) \leq B(y) \ 0 &amp; \text{otherwise} \end{cases} )</td>
<td></td>
</tr>
<tr>
<td>Gödel</td>
<td>( f_g(A(x), B(y)) = \begin{cases} 1 &amp; \text{if } A(x) \leq B(y) \ B(y) &amp; \text{otherwise} \end{cases} )</td>
<td></td>
</tr>
<tr>
<td>Goguen</td>
<td>( f_e(A(x), B(y)) = \begin{cases} 1 &amp; \text{if } A(x) \leq B(y) \ B(y) &amp; \text{otherwise} \end{cases} )</td>
<td></td>
</tr>
<tr>
<td>Kleene</td>
<td>( f_b(A(x), B(y)) = \max [1 - A(x), B(y)] )</td>
<td></td>
</tr>
<tr>
<td>Reichenbach</td>
<td>( f_r(A(x), B(y)) = 1 - A(x) + A(x)B(y) )</td>
<td></td>
</tr>
<tr>
<td>Zadeh</td>
<td>( f_z(A(x), B(y)) = \max [1 - A(x), \min (A(x), B(y))] )</td>
<td></td>
</tr>
<tr>
<td>Klir-Yuan</td>
<td>( f_k(A(x), B(y)) = 1 - A(x) + A(x)^2 B(y) )</td>
<td></td>
</tr>
</tbody>
</table>
Example: $f_\ell = \text{Lukasiewicz}$

$$R_\ell(x,y) = \min\{1, 1 - A(x) + B(y)\}$$

$$\forall (A(x), B(y)) \in [0,1]^2$$

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Example: $f_k = \text{Klir–Yuan}$

\[
R_k (x,y) = 1 - A(x) + A(x)^2 B(y)
\]

\[
\forall (A(x), B(y)) \in [0,1]^2
\]

\[
R_k (x,y) = 1 - A(x) + A(x)^2 B(y)
\]

\[
A(x) = A(x,4,5,6)
\]

\[
B(y) = B(y,4,5,6)
\]
• Categories of fuzzy implications:

1. s-implications

\[ f_{is}(A(x), B(y)) = \overline{A}(x) sB(y) \quad \forall (x, y) \in X \times Y \]

\[ f_b(A(x), B(y)) = \max[1 - A(x), B(y)] \quad \text{Kleene} \]

\[ f_g(A(x), B(y)) = \min\{1, 1 - A(x) + B(y)\} \quad \text{Lukasiewicz} \]

2. r-implications

\[ f_{ir}(A(x), B(y)) = \sup\{c \in [0,1] \mid A(x) t c \leq B(y)\} \quad \forall (x, y) \in X \times Y \]

\[ t = \min \]

\[ f_g(A(x), B(y)) = \begin{cases} 1 & \text{if } A(x) \leq B(y) \\ B(y) & A(x) > B(y) \end{cases} \quad \text{Gödel} \]
Semantics of gradual rules

the *more* $X$ is $A$, the *more* $Y$ is $B$  $\Rightarrow$  $B(y) \geq A(x)$  $\forall x \in X$ and $\forall y \in Y$

$B_{Rd} = \{y \in Y \mid B(y) \geq A(x)\}$ for each  $x \in X$
Example: $R_d = f_a = \text{Gaines}$

$$R_d(x, y) = \begin{cases} 1 & \text{if } B(y) \geq A(x) \\ 0 & \text{otherwise} \end{cases}$$

\[ R_d(x, y) \quad \forall (A(x), B(y)) \in [0,1]^2 \]

$A(x) = A(x,3,5,7)$

$B(y) = B(y,3,5,7)$

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Main types of rule bases

• Fuzzy rule base \( \equiv \{R_1, R_2, \ldots, R_N\} \equiv \) finite family of fuzzy rules

• Fuzzy rule base can assume various formats:

  1. fuzzy graph

  \[ R_i: \text{If } X \text{ is } A_i \text{ then } Y \text{ is } B_i \text{ is a fuzzy granule in } X \times Y, \ i = 1, \ldots, N \]

  2. fuzzy implication rule base

  \[ R_i: \text{If } X \text{ is } A_i \text{ then } Y \text{ is } B_i \text{ is fuzzy implication, } i = 1, \ldots, N \]

  3. functional fuzzy rule base

  \[ R_i: \text{If } X \text{ is } A_i \text{ then } y = f_i(x) \text{ is a functional fuzzy rule, } i = 1, \ldots, N \]
Fuzzy graph

- Fuzzy rule base $R \equiv \text{collection of rules } R_1, R_2, \ldots, R_N$
- Each fuzzy rule $R_i$ is a fuzzy granule (point)
- Fuzzy graph $\equiv R$ is a collection of fuzzy granules
  - granular approximation of a function
    
    $R = \bigcup_{i=1}^{N} R_i = \bigcup_{i=1}^{N} (A_i \times B_i)$

  - $R = R_1 \text{ or } R_2 \text{ or } \ldots \text{ or } R_N$

  - general form
    
    $R(x, y) = \bigcup_{i=1}^{N} [A_i(x) t B_i(y)]$

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Point $P$ in $\mathbf{X} \times \mathbf{Y}$

$P = A \times B$

$A$ is a singleton in $\mathbf{X}$

$B$ is a singleton in $\mathbf{Y}$
Granule $G$ is an interval in $X \times Y$

$G = A \times B$

$A$ is an interval in $X$

$B$ is an interval in $Y$

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Fuzzy granules ≡ fuzzy points

fuzzy granule $R$ in $\mathbb{X} \times \mathbb{Y}$

$R = A \times B$

$A$ is a fuzzy set on $\mathbb{X}$

$B$ is a fuzzy set on $\mathbb{Y}$

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Fuzzy rule base as a set fuzzy granules

$$R_i = A_i \times B_i$$
Graph of a function $f$ and its granular approximation $R$

(a) function $y = f(x)$

(b) Granular approximation of $y = f(x)$

$x \times y$

$R_i = A_i \times B_i$

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Fuzzy rule base and fuzzy graph

Example 1

\[ R_i = A_i \times B_i \implies R_i(x,y) = \min [A_i(x), B_i(y)] \]

\[ R = \bigcup R_i \implies R(x,y) = \max [R_i(x,y), i = 1, \ldots, N] \]
Fuzzy rule base and fuzzy graph

Example 2

\[ R_i = A_i \cdot B_i \Rightarrow R_i(x,y) = A_i(x) \cdot B_i(y) \]

\[ R = \bigcup R_i \Rightarrow R(x,y) = \max \{ R_i(x,y), i = 1, \ldots, N \} \]

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**Fuzzy implication**

- Fuzzy rule base \( R \equiv \text{collection of rules } R_1, R_2, \ldots, R_N \)
- Each fuzzy rule \( R_i \) is a fuzzy implication
- Fuzzy rule base \( R \) is a collection of fuzzy relations
  - relation \( R \) is obtained using intersection
    \[
    R = \bigcap_{i=1}^{N} R_i = \bigcap_{i=1}^{N} f_i = \bigcap_{i=1}^{N} (A_i \Rightarrow B_i)
    \]
  - \( R = R_1 \text{ and } R_2 \text{ and } \ldots \text{ and } R_N \)
- general form
  \[
  R = \bigcap_{i=1}^{N} f_i(A_i(x), B_i(y))
  \]

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Fuzzy rule as an implication

- Fuzzy rule $R$ in $\mathbb{X} \times \mathbb{Y}$
  
  $R = f_\varepsilon(A, B)$

Lukasiewicz implication

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Fuzzy rule base and fuzzy implication

Example 1a

\[ R_i = f(A,B) \Rightarrow R_i(x,y) = \min [1, 1 - A_i(x) + B_i(y)] \] Lukasiewicz implication

\[ R = \cap R_i \Rightarrow R(x,y) = \min [R_i(x,y), i = 1,..., 5] \] min t-norm

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Fuzzy rule base and fuzzy implication

Example 1b

\[ R_i = f_{\cap}(A, B) \Rightarrow R_i(x,y) = \min [1, 1 - A_i(x) + B_i(y)] \]
Lukasiewicz implication

\[ R = \bigcap R_i \Rightarrow R(x,y) = R_1(x,y) t_1 R_2(x,y) t_1 \ldots t_1 R_i(x,y) \]
Lukasiewicz t-norm

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Fuzzy rule base and fuzzy implication

Example 2a

\[ R_i = f_z(A, B) \Rightarrow R_i(x, y) = \max [1 - A_i(x), \min(A_i(x), B_i(y))] \]

Zadeh implication

\[ R = \bigcap R_i \Rightarrow R(x, y) = \min [R_i(x, y), i = 1, \ldots, 5] \]

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Fuzzy rule base and fuzzy implication

Example 2b

\[ R_i = f_z(A,B) \Rightarrow R_i(x,y) = \max [1 - A_i(x), \min(A_i(x), B_i(y))] \]

\[ R = \cap R_i \Rightarrow R(x,y) = R_1(x,y) \land_1 R_2(x,y) \land_1 \ldots \land_1 R_i(x,y) \]

Zadeh implication

Lukasiewicz t-norm

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Data base

- Data base contains definitions of:
  - universes
  - scaling functions of input and output variables
  - granulation of the universes membership functions

- Granulation
  - granular constructs in the form of fuzzy points
  - granules along different regions of the universes

- Construction of membership functions
  - expert knowledge
  - learning from data
Granulation

Granular constructs in the form of fuzzy points

Granules along different regions of the universes

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Fuzzy inference

• Basic idea of inference

\[
x = a \\
y = f(x) \\
y = b
\]

\[
b = \text{Proj}_Y (a_c \cap f) \\
\downarrow \\
b = \text{Proj}_Y (I)
\]

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Inference involves operations with sets

\[ x = A \]
\[ y = f(x) \]
\[ B = f(A) = \{f(x), x \in A\} \]

\[ B = \text{Proj}_Y (A_c \cap f) \]
\[ \downarrow \]

\[ B = \text{Proj}_Y (I) \]
Inference involving sets and relations

$x$ is $A$
$(x,y)$ is $R$
$y$ is $B$

\[ B = \text{Proj}_Y (A_c \cap R) \]
\[ \Downarrow \]
\[ B = \text{Proj}_Y (I) \]
Fuzzy inference ands operations with fuzzy sets and relations

\(X\) is \(A\) (fuzzy set on \(X\))

\((X,Y)\) is \(R\) (fuzzy relation on \(X \times Y\))

\(Y\) is \(B\) (fuzzy set on \(Y\))

\[ B = \text{Proj}_Y (A_c \cap R) \]

\[ \downarrow \]

\[ B = \text{Proj}_Y (I) \]

\[ B(y) = \sup_{x \in X} \{ A(x) \cap R(x,y) \} \]
Fuzzy inference

• Compositional rule of inference

\[ X \text{ is } A \]
\[ (X,Y) \text{ is } R \]
\[ Y \text{ is } B \]

\[ B = A \circ R \]

\[ X \text{ is } A \]
\[ (X,Y) \text{ is } R \]
\[ Y \text{ is } A \circ R \]
**Fuzzy inference procedure**

```plaintext
procedure FUZZY-INERENCE (A, R) returns a fuzzy set

input: fuzzy relation: R
fuzzy set: A

local: x, y: elements of X and Y
       t: t-norm

for all x and y do
    A_c(x,y) ← A(x)
for all x and y do
    I(x,y) ← A_c(x,y) t R(x,y)
B(y) ← sup_x I(x,y)
return B
```

Pedrycz and Gomide, FSE 2007
Example: compositional rule of inference

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Example: fuzzy inference with fuzzy graph
11.5 Types of rule-based systems and architectures
Linguistic fuzzy models

\[ P: \quad X \text{ is } A \text{ and } Y \text{ is } B \text{ \hspace{1cm} input} \]

\[ R_1: \quad \text{If } X \text{ is } A_1 \text{ and } Y \text{ is } B_1 \text{ then } Z \text{ is } C_1 \]

\[ R_i: \quad \text{If } X \text{ is } A_i \text{ and } Y \text{ is } B_i \text{ then } Z \text{ is } C_i \]

\[ R_N: \quad \text{If } X \text{ is } A_N \text{ and } Y \text{ is } B_N \text{ then } Z \text{ is } C_N \]

\[ Z: \quad Z \text{ is } C \text{ \hspace{1cm} output} \]

- all fuzzy sets \( A, B, A_i, \text{s and } B_i, \text{s are given} \)
- rule and connectives (\textit{and, or}) with known semantics
- membership function of fuzzy set \( C = ?? \)
min-max fuzzy models

Assume

\( P: \ X \text{ is } A \text{ and } Y \text{ is } B \)

\[ P(x,y) = \min\{A(x), B(y)\} \]

\( R_i: \ \text{If } X \text{ is } A_i \text{ and } Y \text{ is } B_i \text{ then } Z \text{ is } C_i \)

\[ R_i(x,y,z) = \min\{A_i(x), B_i(y), C_i(z)\} \]

\( i = 1, \ldots, N \)

Using the compositional rule of inference (\( t = \min \))

\[
C = P \circ R = P \circ \bigcup_{i=1}^{N} R_i
\]

\[
C(z) = \sup\{\min\{P(x,y), \max(R_i(x,y,z), i = 1,\ldots,N)\}\}
\]

---

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\[ C = P \circ R = P \circ \bigcup_{i=1}^{N} R_i = \bigcup_{i=1}^{N} (P \circ R_i) = \bigcup_{i=1}^{N} C'_i \]

\[ C'_i = P \circ R_i \]

\[ C'_i(z) = \sup \{ \min [P(x, y), R_i(x, y, z)] \} = \sup \{ A(x) \land B(y) \land A_i(x) \land B_i(y) \land C_i(z) \} \]

\[ \sup [A(x) \land A_i(x)] = \text{Poss}(A, A_i) = m_i \]

\[ \sup [B(y) \land B_i(y)] = \text{Poss}(B, B_i) = n_i \]

\[ C'_i(z) = m_i \land n_i \land C_i(z) \]

\[ C(z) = \max \{ (m_i \land n_i) C_i, i = 1, \ldots, N \} = \max \{ \lambda_i \land C_i(z), i = 1, \ldots, N \} \]

\( \lambda_i \) is the degree of activation of \( i \) – th rule
procedure MIN-MAX-MODEL \((A,B)\) returns a fuzzy set

local: fuzzy sets: \(A_i, B_i, C_i, i = 1,.., N\)
activation degrees: \(\lambda_i\)

Initialization \(C = \emptyset\)

for \(i = 1: N\) do
\[
\begin{align*}
    m_i &= \max (\min (A, A_i)) \\
    n_i &= \max (\min (B, B_i)) \\
    \lambda_i &= \min (m_i, n_i)
\end{align*}
\]
if \(\lambda_i \neq 0\) then \(C_i' = \min (\lambda_i, C_i)\) and \(C = \max (C, C_i')\)

return \(C\)
Example: min-max fuzzy model processing

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min-max fuzzy model architecture

\[ (A, B) \]

\[ \begin{align*}
&\text{Poss} \rightarrow \lambda_i \rightarrow \text{Min} \rightarrow C_i \rightarrow \text{Max} \\
&\quad \vdots \\
&\quad \text{Poss} \rightarrow \lambda_N \rightarrow \text{Min} \rightarrow C_N
\end{align*} \]

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Special case: numeric inputs

\[ A(x) = \begin{cases} 
1 & \text{if } x = x_o \\
0 & \text{otherwise} 
\end{cases} \quad \text{and} \quad B(y) = \begin{cases} 
1 & \text{if } y = y_o \\
0 & \text{otherwise} 
\end{cases} \]

Numeric output

\[ z = \frac{\int_{Z} zC(z)dz}{\int_{Z} C(z)dz} \quad \text{centroid defuzzification} \]

\[ z = \frac{\sum_{i=1}^{N} (m_i \wedge n_i) v_i}{\sum_{i=1}^{N} (m_i \wedge n_i)} \quad \text{weighted average modal values } v_i \]
Example

\( P: \) \( X \) is \( x_o \) and \( Y \) is \( y_o \)

\( R_1: \) \textbf{If} \( X \) is \( A_1 \) and \( Y \) is \( B_1 \) \textbf{then} \( Z \) is \( C_1 \)

\( R_2: \) \textbf{If} \( X \) is \( A_2 \) and \( Y \) is \( B_2 \) \textbf{then} \( Z \) is \( C_2 \)

\( N = 2, \) centroid defuzzification

\begin{align*}
\text{inputs} & \quad (x_o, y_o), \forall x_o, y_o \in [-2, 2] \\
\text{rules} & \quad \end{align*}

(a) Input and output fuzzy sets

(b) Input-output mapping

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min-sum fuzzy models

Assume

\[ P: \text{ } X \text{ is } A \text{ and } Y \text{ is } B \quad P(x,y) = \min\{A(x), B(y)\} \]

\[ R_i: \text{ If } X \text{ is } A_i \text{ and } Y \text{ is } B_i \text{ then } Z \text{ is } C_i \quad R_i(x,y,z) = \min\{A_i(x), B_i(y), C_i(z)\} \]

\[ i = 1, \ldots, N \]

Using the compositional rule of inference (t = min)

\[ C'_i(z) = \sup_{x,y} [A(x) \land B(y) \land A_i(x) \land B_i(y) \land C_i(z)] \]

\[ C(z) = \sum_{i=1}^{N} w_i C'_i \]

Additive fuzzy models
(Kosko, 1992)
Example: min-sum fuzzy model processing

\[ \sum C_i' \]

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min-sum fuzzy model architecture

Pedrycz and Gomide, FSE 2007
Example

\[ \mathbf{P} : \quad \text{X is } x_o \text{ and Y is } y_o \]

\[ \mathbf{R}_1 : \quad \text{If X is } A_1 \text{ and Y is } B_1 \text{ then Z is } C_1 \]

\[ \mathbf{R}_2 : \quad \text{If X is } A_2 \text{ and Y is } B_2 \text{ then Z is } C_2 \]

\[ N = 2 \quad w_1 = w_2 = 1, \text{ centroid defuzzification} \]

inputs \( (x_o, y_o), \forall x_o, y_o \in [-2, 2] \)

rules

---

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product-sum fuzzy models

1- Product–probabilistic sum

\[ C_i'(z) = m_i n_i C_i(z) \]
\[ C(z) = \sum_{i=1}^{N} S_p C_i'(z) \]

2- Product–sum

\[ C_i'(z) = m_i n_i C_i(z) \]
\[ C(z) = \sum_{i=1}^{N} C_i'(z) \]
3 - Bounded product-bounded sum

\[ C_i'(z) = m_i \otimes n_i \otimes C_i(z) \]

\[ C(z) = \bigoplus_{i=1}^{N} C_i'(z) \]

\[ a \otimes b = \max\{0, a + b - 1\} \]

\[ a \oplus b = \min\{1, a + b\} \]

\[ a, b \in [0,1] \]
Functional fuzzy models

$P$: \( X \) is \( x \) and \( Y \) is \( y \)  

\[ R_1: \quad \text{If } X \text{ is } A_1 \text{ and } Y \text{ is } B_1 \text{ then } z = f_1(x,y) \]

\[ R_i: \quad \text{If } X \text{ is } A_i \text{ and } Y \text{ is } B_i \text{ then } z = f_i(x,y) \]

\[ R_N: \quad \text{If } X \text{ is } A_N \text{ and } Y \text{ is } B_N \text{ then } z = f_N(x,y) \]

\( \lambda_i(x,y) = A_i(x) \ t B_i(y) \quad t = \text{t-norm} \)

\[ z = \sum_{i=1}^{N} w_i(x,y)f_i(x,y), \quad w_i = \frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i(x,y)} \]

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Functional fuzzy model architecture

\[
\begin{align*}
(f_1(x,y)) &= \sum_{i=1}^{N} w_i A_i B_i f_i(x,y) \\
(f_N(x,y)) &= \sum_{i=1}^{N} w_i A_i B_i f_i(x,y)
\end{align*}
\]

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Example 1

\[ P: \quad X \text{ is } x \]
\[ R_1: \quad \text{If } X \text{ is } A_1 \text{ then } z = x \]
\[ R_2: \quad \text{If } X \text{ is } A_2 \text{ then } z = -x + 3 \]

inputs \hspace{1cm} x \in [0, 3]

rules

(a) Antecedent fuzzy sets

(b) Consequent functions

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\[ z = \begin{cases} 
  x & \text{if } x \in (0,1] \\
  A_1(x)x + A_2(x)(-x + 3) & \text{if } x \in [1,2] \\
  -x + 3 & \text{if } x \in [2,3) 
\end{cases} \]
Example 2

\[ P: \quad X \text{ is } x \]

\[ R_1: \quad \text{If } X \text{ is } A_1 \text{ then } y = -\sin(2x) \]

\[ R_2: \quad \text{If } X \text{ is } A_2 \text{ then } y = -0.5x \]

\[ R_3: \quad \text{If } X \text{ is } A_3 \text{ then } y = \sin(3x) \]

inputs \quad x \in [0, 3]

rules

output

---

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Example 2

\[ P: \quad X \text{ is } x \]

\[ R_1: \quad \text{If } X \text{ is } A_1 \text{ then } y = -1 \]

\[ R_2: \quad \text{If } X \text{ is } A_2 \text{ then } y = x \]

\[ R_3: \quad \text{If } X \text{ is } A_3 \text{ then } y = 1 \]

inputs \quad x \in [0, 3]

\{ \]

rules

output

(a) Antecedent fuzzy sets

(c) Output of the functional fuzzy model
Gradual fuzzy models

$R_i$: The more $X$ is $A_i$ the more $Z$ is $C_i$

$i = 1, ..., N$

$$R_i(x, y) = \begin{cases} 
1 & \text{if } C_i(z) \geq A_i(x) \\
0 & \text{otherwise}
\end{cases}$$

$$C = \bigcap_{i=1}^{N} (C_i')_{\alpha_i} = \bigcap_{i=1}^{N} C_{\alpha_i}$$
Gradual fuzzy model architecture

\[x \rightarrow \text{Poss} \rightarrow \alpha_i \rightarrow C_{\alpha l} \rightarrow C_i \rightarrow \text{Min} \rightarrow C\]

\[A_1 \rightarrow \text{Poss} \rightarrow \alpha_1 \rightarrow C_{\alpha l} \rightarrow C_1 \rightarrow \]

\[A_i \rightarrow \cdots \rightarrow A_N \rightarrow \text{Poss} \rightarrow \alpha_N \rightarrow C_{\alpha l} \rightarrow C_N \rightarrow \]

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Example: gradual fuzzy model processing

\[ A_1 \quad A_2 \]

\[ C_1 \quad C_2 \]

\[ C_1' \quad C_2' \]

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Example

\( P: \quad x \) is \( x \)

\( R_1: \quad \text{The} \ more \ x \ is \ A_1 \ \text{the} \ more \ z \ is \ C_1 \)

\( R_2: \quad \text{The} \ more \ x \ is \ A_1 \ \text{the} \ more \ z \ is \ C_1 \)

inputs \( x \in [0, 3] \)

rules

output

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11.6 Approximation properties of fuzzy rule-based models
• FRBS uniformly approximates continuous functions
  – any degree of accuracy
  – closed and bounded sets

• Universal approximation with (Wang & Mendel, 1992):
  – algebraic product t-norm in antecedent
  – rule semantics via algebraic product
  – rule aggregation via ordinary sum
  – Gaussian membership functions
  – sup-min compositional rule of inference
  – pointwise inputs
  – centroid defuzzification
• Universal approximation when (Kosko, 1992):
  – min t-norm in antecedent
  – rule aggregation via ordinary sum
  – symmetric consequent membership functions
  – sup-min compositional rule of inference
  – pointwise inputs
  – centroid defuzzification

(additive models)
• Universal approximation with (Castro, 1995):
  – arbitrary t-norm in antecedent
  – rule semantics: r-implications or conjunctions
  – triangular or trapezoidal membership functions
  – sup-min compositional rule of inference
  – pointwise inputs
  – centroid defuzzification
11.7 Development of rule-based systems
Expert-based development

- Knowledge provided by domain experts
  - basic concepts and variables
  - links between concepts and variables to form rules

- Reflects existing knowledge
  - can be readily quantified
  - short development time

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Example: fuzzy control

\[ R_i: \textbf{If} \ \text{Error is } A_i \ \textbf{and} \ \text{Change of Error is } B_i \ \textbf{then} \ \text{Control is } C_i \]

\[ R_i: \textbf{If} \ e \ \text{is } A_i \ \textbf{and} \ de \ \text{is } B_i \ \textbf{then} \ u \ \text{is } C_i \]
$R_i$: If $e$ is $A_i$ and $de$ is $B_i$ then $u$ is $C_i$
Data-driven development

• Given a finite set of input/output pairs

\[ \{(x_k, y_k), \ k = 1,..., M\} \]

\[ x_k = [x_{1k}, x_{2k},..., x_{nk}] \in \mathbb{R}^n \]

\[ z_k = [x_k, y_k] \in \mathbb{R}^{n+1}, \ k = 1,..., M \]

• Clustering \( z_k = [x_k, y_k] \in \mathbb{R}^{n+1}, \ k = 1,..., M \) (e.g. using FCM)

\[ v_1, v_2,\ldots,v_N \quad \text{prototypes/cluster centers} \]

\[ v_i \in \mathbb{R}^{n+1}, \ i = 1,..., N \]

• Idea: fuzzy clusters \( \equiv \) fuzzy rules
Example

$R_1$

$R_2$

$R_3$

$R_4$
• Projecting the prototypes in the input and output spaces

\[ v_1[y], v_2[y], \ldots, v_N[y] \] projections of prototypes in Y

\[ v_1[x], v_2[x], \ldots, v_N[x] \] projections of prototypes in X

• \( R_i: \) If \( X \) is \( A_i \) then \( Y \) is \( C_i, i = 1, \ldots, N \)
11.8 Parameter estimation for functional rule-based systems
• Functional fuzzy rules

• $R_i$: If $X_{i1}$ is $A_{i1}$ and ... and $X_{in}$ is $A_{in}$ then $z = a_{i0} + a_{i1}x_1 + ... + a_{in}x_n$

  $i = 1, ..., N$

• Given input/output data: $\{(x_1, y_1), (x_2, y_2), ..., (x_M, y_M)\}$

• Let $a_i = [a_{i0}, a_{i1}, a_{i2}, ..., a_{in}]^T$

• Output of functional models

$$\hat{y}_k = \sum_{i=1}^{N} w_{ik} f_i(x_k, a_i), \quad w_{ik} = \frac{\lambda_i(x_k)}{\sum_{i=1}^{N} \lambda_i(x_k)}$$

• Output for linear consequents

$$\hat{y}_k = \sum_{i=1}^{N} z_{ik}^T a_i, \quad z_{ik} = [1, w_{ik} x_k^T]^T$$

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Let

\[ a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \]

\[ \hat{y}_k = \begin{bmatrix} z_{1k}^T \\ z_{2k}^T \\ \vdots \\ z_{Nk}^T \end{bmatrix} \]

and

\[ b = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_M \end{bmatrix} \]

\[ B = \begin{bmatrix} z_{11}^T & z_{12}^T & \cdots & z_{1N}^T \\ z_{21}^T & z_{22}^T & \cdots & z_{2N}^T \\ \vdots & \vdots & \ddots & \vdots \\ z_{M1}^T & z_{M2}^T & \cdots & z_{MN}^T \end{bmatrix} \]

then \[ y = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \]

Then \[ y = B a \]
Global least squares approach

\[ \text{Min}_a J_G(a) = \| y - Za \|^2 \]

\[ \| y - Za \|^2 = (y - Za)^T (y - Za) \]

Solution

\[ a_{\text{opt}} = Z^# y \]

\[ Z^# = (Z^T)^{-1} Z^T \]
Local least squares approach

\[ \text{Min}_a J_L(a) = \sum_{i=1}^{N} \| y - Z_i a_i \|^2 \]

\[ Z_i = \begin{bmatrix} z_{i1}^T \\ z_{i2}^T \\ \vdots \\ z_{iM}^T \end{bmatrix} \]

Solution

\[ a_{i^{opt}} = Z_i^# y \]

\[ Z_i^# = (Z_i^T)^{-1} Z_i^T \]

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11.9 Design issues of FRBS: Consistency and completeness
Given input/output data: \{ (x_1, y_1), (x_2, y_2), \ldots, (x_M, y_M) \}

Issue: quality of the rules

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Completeness of rules

• All data points represented through some fuzzy set

\[ \max_{i=1,\ldots, M} A_i(x_k) > 0 \text{ for all } k = 1,2,\ldots, M \]

• Input space completely covered by fuzzy sets

\[ \max_{i=1,\ldots, M} A_i(x_k) > \delta \text{ for all } k = 1,2,\ldots, M \]
Consistency of rules

• Rules in conflict
  – similar or same conditions
  – completely different conclusions

<table>
<thead>
<tr>
<th>Conditions and Conclusions</th>
<th>Similar Conclusions</th>
<th>Different Conclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Similar Conditions</strong></td>
<td>rules are redundant</td>
<td>rules are in conflict</td>
</tr>
<tr>
<td><strong>Different Conditions</strong></td>
<td>different rules; could be eventually merged</td>
<td>different rules</td>
</tr>
</tbody>
</table>

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\( R_i: \) If \( X \) is \( A_i \) then \( Y \) is \( B_i \)

\( R_j: \) If \( X \) is \( A_j \) then \( Y \) is \( B_j \)

\[
\text{cons}(i, j) = \sum_{i=1}^{M} \{ |B_i(y_k) - B_j(y_k)| \Rightarrow |A_i(x_k) - A_j(x_k)| \}
\]

Alternatively

\[
\text{cons}(i, j) = \sum_{i=1}^{M} \{ \text{Poss}(A_i(x_k), A_j(x_k)) \Rightarrow \text{Poss}(B_i(y_k), B_j(y_k)) \}
\]

\( \Rightarrow \) is an implication induced by some t-norm (r-implication)

\[
\text{cons}(i) = \frac{1}{M} \sum_{j=1}^{M} \text{cons}(i, j)
\]
11.10 The curse of dimensionality in rule-based systems
• Curse of dimensionality
  – number of variables increase
  – exponential growth of the number of rules

• Example
  – $n$ variables
  – each granulated using $p$ fuzzy sets
  – number of different rules = $p^n$

• Scalability challenges
11.11 Development scheme of fuzzy rule-based models
• Spiral model of development
  – incremental design, implementation and testing
  – multidimensional space of fundamental characteristics

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