Fuzzy Sets*

L. A. Zadeh

Department of Electrical Engineering and Electronics Research Laboratory,
University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.
History of the Theory of Fuzzy Sets

• **Prehistory of the Theory of Fuzzy Sets**
  **1920s-1960s**

• **Genesis of the Theory of Fuzzy Sets**
  **1960s**

• **Applications of the Theory of Fuzzy Sets**
  **1970s**

• **Enforcement of the Theory of Fuzzy Sets as a scientific paradigm**
  **1980s - 1990s**
History of the Theory of Fuzzy Sets
History of the Theory of Fuzzy Sets

- Thinking Machines and Communication Systems: A New Field of Electrical Engineering
- System Theory: A New Scientific Discipline
- A New View on System Theory: Fuzzy Sets and Systems
- A short Outlook: The First Real World Application Fuzzy System
Lotfi Aliasker Zadeh

• born 1921 in Baku, Azerbaijan
• since 1942: Electrical Engineering, University Tehran
• then: Technical Associate of the US Army Forces in Iran
• 1944: Emigration into the USA, International Electronic Laboratories, New York
  Studies of Electrical Engineering at the MIT
• 1946: Master of Science, Supervisor: Robert Fano, Then: Columbia University, New York
• 1949: Ph. D. Thesis: *Frequency Analysis of Variable Networks*
  Supervisor: John Ralph Ragazzini
• 1950: *An Extension of Wiener’s Theory of Prediction* (with Ragazzini)
• since 1952: Scientific Work: *Information Theory and System Theory*
• since 1964: Fuzzy Sets
Thinking Machines


**Example 1**

**Characterization of ENIAC as a brain**

17 box 80.

Feb. 15, 1946:

"Army's New Wonder Brain and its Inventors."—*Philadelphia Inquirer.*

"Mathematical Brain Enlarges Man's Horizon."—*Philadelphia Inquirer.*

"Mechanical Mathematician 'Brain Child' of Hopkins Man."—*The Baltimore Sun.*

"Magic Brain Spurs Science and Technology."—*New York World-Telegram.*

"Electronic Brain Computes 100-Year Problem in 2 Hours."—*New York Herald Tribune.*


"Fastest Mechanical Brain Disclosed; Weighs 30 Tons: Giant Calculating Machine Said to Work 1,000 Times Faster Than Any Previously Built."—*Chicago Sun.*

"Computing Super-Brain Aids Army."—*Newark Star Ledger.*
I’ll be damned. It says ‘Cogito, ergo sum.’
UNIVAC (Universal Automatic Computer)

Plate 20. UNIVAC digital computer in the U.S.A., showing a bank of magnetic-tape storage units on the right.
The BINAC was a bit-serial binary computer with two independent CPUs, each with its own 512-word acoustic mercury delay line memory. The CPUs continuously compared results to check for errors caused by hardware failures. The 512-word acoustic mercury delay line memories were divided into 16 channels each holding 32ords (31-bit with an additional 11-bit space between words to allow for circuit delays in switching. The clock rate was 4.25 MHz which yielded a word time of about 10 microseconds. New programs or data had to be entered manually in octal using an eight-key keypad.

BINAC was an early electronic computer designed for *Northrop Aircraft Company* by Eckert and Mauchly in 1949.
Lotfi A. Zadeh, 1950: Example of a „Thinking Machine“

The two units of Robert Haufes Tit-Tat-Toe machine.

**Figure 1**—A schematic diagram illustrating how the basic elements of a thinking machine are arranged.

<table>
<thead>
<tr>
<th>Relay Circuit Element</th>
<th>Symbolic Logic Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circuit A</td>
<td>Statement A</td>
</tr>
<tr>
<td>Closed circuit</td>
<td>A is false</td>
</tr>
<tr>
<td>Open circuit</td>
<td>A is true</td>
</tr>
<tr>
<td>Series connection of A and B</td>
<td>A and/or B ((A \lor B))</td>
</tr>
<tr>
<td>Parallel connection of A and B</td>
<td>A and B ((A \land B))</td>
</tr>
</tbody>
</table>
Lotfi Zadeh, 1952: Some Basic Problems in Communication of Information

System Theory

L. A. Zadeh
Associate Professor
Electrical Engineering
System:
„an aggregation or assemblage of objects united by some form of interaction or interdependence“

(Webster's dictionary)

A: $z = f(x, y, w)$
B: $u = g(z, v)$
C: $v = h(u)$
$w = k(u)$

Lotfi Zadeh, 1952: Some Basic Problems in Communication of Information
Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*
Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

**Input-output-relationship:**

\[ y = f(u) \]
Problem:

Let $X = \{x(t)\}$ be a set of signals. An arbitrarily selected member of this set, say $x(t)$, is transmitted through a noisy channel $\Gamma$ and is received as $y(t)$.

As a result of the noise and distortion introduced by $\Gamma$, the received signal $y(t)$ is, in general, quite different from $x(t)$.

Nevertheless, under certain conditions it is possible to recover $x(t)$ – or rather a time-delayed replica of it – from the received signal $y(t)$.

$$ y = \Gamma x \quad \text{resp.} \quad x = \Gamma^{-1} y $$
Special case: reception process:

Let $X = \{x(t)\}$ consist of a finite number of discrete signals $x_1(t)$, $x_2(t)$, ..., $x_n(t)$, which play the roles of symbols or sequences of symbols.

The replicas of all these signals are assumed to be available at the receiving end of the system. Suppose that a transmitted signal $x_k$ is received as $y$.

To recover the transmitted signal from $y$, the receiver evaluates the ‘distance’ between $y$ and all possible transmitted signals $x_1$, $x_2$, ..., $x_n$, by the use of a suitable distance function $d(x, y)$, and then selects that signal which is ‘nearest’ to $y$ in terms of this distance function.
Distance functions:

1. \( d(x, y) = \text{l.u.b.} \ |x(t) - y(t)| \)

2. \( d(x, y) = \left\{ \frac{1}{T} \int_{0}^{T} [x(t) - y(t)]^2 \, dt \right\}^{1/2} \)

3. \( d(x, y) = \text{l.u.b.} \left\{ \frac{1}{T} \int_{0}^{t+T} [x(t) - y(t)]^2 \, dt \right\}^{1/2} \)

4. \( d(x, y) = \frac{1}{T} \int_{0}^{T} |x(t) - y(t)| \, dt \)
In many practical situations it is inconvenient, or even impossible, to define a quantitative measure, such as a distance function, of the disparity between two signals.

In such cases we may use instead the concept of neighborhood, which is basic to the theory of topological spaces.
Problem: multiplex transmission of two or more signals; the system has two channels.

\[ X = \{x(t)\} \quad \text{and} \quad Y = \{y(t)\}: \text{ sets of signals assigned to their respective channels.} \]

At the receiving end: sum signal: \[ u(t) = x(t) + y(t). \]

To do: Extract \( x(t) \) and \( y(t) \) from \( u(t) \)!

That means: Find two filters \( N_1 \) and \( N_2 \) such, that, for any \( x \) in \( X \) and any \( y \) in \( Y \),

\[ N_1(x + y) = x \quad \text{and} \quad N_2(x + y) = y \]
Lotfi Zadeh, 1952: *Some Basic Problems in Communication of Information*

The New York Academy of Sciences (1952)

Geometrical representation of

nonlinear filtering

and

linear filtering

in terms of two-dimensional signal spaces.
The Bandwagon

What is Information Theory?

CLAUDE E. SHANNON

NORBERT WIENER
Indeed, the hard core of information theory is, essentially, a branch of mathematics, a strictly deductive system.

Research rather than exposition is the keynote, and our critical thresholds should be raised.
I am pleading in this editorial that Information Theory go back of its slogans and return to the point of view from which it originated: that of the general statistical concept of communication.

I hope that these Transactions may encourage this integrated view of communication theory by extending its hospitality to papers which, why they bear on communication theory, cross its boundaries, and have a scope covering the related statistical theories. In my opinion we are in a dangerous age of overspecialization.
Richard Bellman, Robert Kalaba, 1957:
On the Role of Dynamic Programming in Statistical Communication Theory

In mathematical terms, let

\[ x = \text{the pure signal emanating from } S. \]

\[ r = \text{the noise associated with the signal.} \]

\[ x' = F(x, r), \text{ the input to the communication system.} \]

\[ y = \text{the signal transmitted to the observer by the communication channel.} \quad (1) \]

Let us further write

\[ y = T(x') = T(F(x, r)), \quad (2) \]
What Is Optimal?

Criterion A:
- Design $D_1$ might be better than $D_2$, and
- Design $D_2$ might be better than $D_3$.

Criterion B:
- Design $D_2$ might be better than $D_3$, and
- Design $D_3$ might be better than $D_1$.

Criterion C:
- Design $D_3$ might be better than $D_1$, and
- Design $D_1$ might be better than $D_2$. 
Electrical Filters, Sieves
Electrical Filters, Sieves

System with two variables $v_1$ and $v_2$;

\[ \frac{dv_2}{dt^2} = \frac{d^2 v_1}{dt^2} + v_1 \]

This system can be realized in different forms.
Lotfi A. Zadeh: 1963, Linear System Theory

**Physical Realization 1:**

*elektrical network.*

$v_1$: voltage

$v_2$: current.

**Physical Realization 2:**

*mechanical system.*

$v_2$: force at particle

$v_1$: velocity of the particle

---

**Fig. 1.4.1** A network realization of the object of Example 1.4.14.

**Fig. 1.4.2** A mechanical realization of the object of Example 1.4.14.

$C = L = 1$

$M = 1$, spring constant = 1
A System is a big black box
Of which we can‘t unlock the locks,
And all we can find out about
Is what goes in and what goes out.
Perceiving input-output pairs,
Related by parameters,
Permits us, sometimes, to relate
An input, output, and a state.
If this relation‘s good and stable
Then to predict we may be able,
But if this fails us – heaven forbid!
We‘ll be compelled to force the lid!

Kenneth E. Boulding

In fact, there is a fairly wide gap between what might be regarded as „animate“ system theorists and „inanimate“ system theorists at the present time, and it is not at all certain that this gap will be narrowed, much less closed, in the near future.

There are some who feel that this gap reflects the fundamental inadequacy of the conventional mathematics – the mathematics of precisely-defined points, functions, sets, probability measures, etc. - for coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. Indeed, the need for such mathematics is becoming increasingly apparent even in the realm of inanimate systems, for in most practical cases the a priori data as well as the criteria by which the performance of a man-made system is judged are far from being precisely specified or having accurately-known probability distributions.
R. Bellman, R. Kalaba, L. A. Zadeh, 1964: *Abstraction And Pattern Classification*

MEMORANDUM
RM-4307-PR
OCTOBER 1964

ABSTRACTION
AND PATTERN CLASSIFICATION
R. Bellman, R. Kalaba and L. A. Zadeh

This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 39(60)0-200 monitored by the Directorate of Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

DDC AVAILABILITY NOTICE
Qualified requesters may obtain copies of this report from the Defense Documentation Center (DDC).
9 September 1964

Professor Lotfi Zadeh
Department of Electrical Engineering
University of California
Berkeley 4, California

Dear Lotfi:

I think that the paper is extremely interesting and I would like to publish it in JMAA, if agreeable to you. When I return, or while in Paris, I will write a companion paper on optimal decomposition of a set into subsets along the lines of our discussion.

Cordially,

Richard Bellman

RB: jb
S is a fuzzy system if $u(t)$ or $y(t)$ or $s(t)$ or any combination are fuzzy sets.
Fuzzy Sets*

L. A. ZADEH

Department of Electrical Engineering and Electronics Research Laboratory, University of California, Berkeley, California

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

I. INTRODUCTION

More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1.

Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.

The purpose of this note is to explore in a preliminary way some of the basic properties and implications of a concept which may be of use in

* This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Grant No. AF-AFOSR-129-64 and by the National Science Foundation under Grant GP-2413.
Georg Cantor, 1895/97: Set Theory

Georg Cantor (1845-1918):
Definition:
'A set is a collection into a whole $M$ of definite and separate objects $m$ of our intuition or thought.'

„Unter einer Menge verstehen wir jede Zusammenfassung $M$ von bestimmten, wohlunterschiedenen Objekten $m$ unserer Anschauung oder unseres Denkens (welche die Elemente von $M$ genannt werden) zu einem Ganzen.“

Georg Cantor, 1895/97: Set Theory
Lotfi A. Zadeh, 1965: Fuzzy Sets
Definition:
„A fuzzy set (class) $A$ in $X$ is characterized by a membership function (characteristic function) $\mu_A(x)$ which associates with each point in $X$ a real number in the interval $[0,1]$, with the value of $\mu_A(x)$ at $x$ representing the ‘grade of membership‘ of $x$ in $A$.“
Set Theory

(a) $A \cap B$

(b) $A \cup B$

(c) $A \setminus B$

(d) $\bar{A}$
A fuzzy set is empty iff: \( \mu_A(x) = 0, \quad x \in X. \)

Equal fuzzy sets, \( A = B, \) iff: \( \mu_A(x) = \mu_B(x), \quad x \in X. \)

The complement \( A' \) of a fuzzy set \( A \) is defined by:

\[
\mu_{A'}(x) = 1 - \mu_A(x), \quad x \in X.
\]

Containment: \( A \subseteq B \) iff:

\[
\mu_A(x) \leq \mu_B(x), \quad x \in X.
\]
Union $A \cup B$ of two fuzzy sets

$A$ and $B$ with resp. membership functions

$$\mu_{A \cup B}(x) = \max \{\mu_A(x), \mu_B(x)\}, x \in X$$

Intersection $A \cap B$ of fuzzy sets

$A$ and $B$ with resp. membership functions

$$\mu_{A \cap B}(x) = \min \{\mu_A(x), \mu_B(x)\}, x \in X$$
“Specifically, let $f_i(x)$ $i = 1, \ldots, n$, denote the value of the membership function of $A_i$ at $x$. Associate with $f_i(x)$ a sieve $S_i(x)$ whose meshes are of size $f_i(x)$. Then, $f_i(x) \cup f_j(x)$ and $f_i(x) \cap f_j(x)$ correspond, respectively, to parallel and series combinations of $S_i(x)$ and $S_j(x)$. ...”

Fig. 2. Parallel and series connection of sieves simulating $\cup$ and $\cap$. 

Lotfi A. Zadeh, 1965: Fuzzy Sets
“More generally, a well formed expression involving $A_1, \ldots, A_n$, $\cup$ and $\cap$ corresponds to a network of sieves $S_1(x), \ldots, S_n(x)$ which can be found by the conventional synthesis techniques for switching circuits.”

Fig. 2. A network of sieves simulating $\{[f_1(x) \lor f_2(x)] \land f_3(x) \} \lor f_4(x)$
First Ph. D Thesis on Fuzzy Sets

Fuzzy Sets and Pattern Recognition
By
Chin-Liang Cheng
Grad. (Taiwan Provincial Taipie Institute of Technology) 1958
M.S. (Lehigh University) 1964
Dissertation
Submitted in partial satisfaction of the requirements for the degree of
DOCTOR OF PHILOSOPHY
in
Engineering
in the
GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

Approved:

Committee in Charge

Degree conferred: Dec 1st 1987

Categories of Fuzzy Sets:
Applications of Non-Cantor Set Theory
By
Joseph Emile Coquen, Jr.
A.B. (Harvard University) 1963
M.A. (University of California) 1966
Dissertation
Submitted in partial satisfaction of the requirements for the degree of
DOCTOR OF PHILOSOPHY
in
Mathematics
in the
GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

Approved:

Committee in Charge

Degree conferred: Jun 16th 1988
First Papers on Fuzzy Sets (Part 1)


<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Title and Publication Details</th>
</tr>
</thead>
</table>
### Table 3

*Distribution of year of publication of papers classified as fuzzy*

<table>
<thead>
<tr>
<th>Year</th>
<th>Number</th>
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<tbody>
<tr>
<td>1965</td>
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<td>1966</td>
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<td>1967</td>
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<tr>
<td>1974</td>
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<tr>
<td>1975</td>
<td>227</td>
</tr>
<tr>
<td>1976</td>
<td>143 (incomplete)</td>
</tr>
<tr>
<td>Total</td>
<td>763</td>
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</table>
Statistics on the impact of fuzzy logic

A measure of the wide ranging impact of Lotfi Zadeh's work on fuzzy logic is the number of papers in the literature which contain the word 'fuzzy' in title. The data drawn from the INSPEC and Mathematical Reviews databases are summarized below. The data for 2000 are not complete.

**STATISTICS**

**INSPEC/fuzzy**
- 1980-1990: 2,361
- 1990-2000: 23,733
- Total: 26,680

**MathSciNet/fuzzy**
- 1970-1980: 403
- 1980-1990: 2,416
- 1990-2000: 6,428
- Total: 11,337

**INSPEC/software computing**
- 1990-2000: 1,994

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Number of citations in the Citation Index over 11,000.

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For optimal display use resolution-mode of 1600x1200 dpi.

JavaScript and style sheets should be enabled in your browser.
Man has two principal objectives in the scientific study of his environment:

He wants to understand and to control.

The two goals reinforce each other, since deeper understanding permits firmer control, and, on the other hand, systematic applications of scientific theories inevitably generates new problems which require further investigations, and so on.

Richard Bellman,
Selected Papers on Mathematical Trends in Control Theory,
Fuzzy-Regelung (fuzzy control)
Fuzzy-Regelung (fuzzy control)

Fig. 2.2 The Plant
Fuzzy-Regelung (fuzzy control)

- Zadeh, 1973: *Outline of a New Approach to the Analysis of Complex Systems and Decision Processes*


*Fig. 2.1 The System*

*Fig. 2.9 Process Variables*
Fuzzy-Regelung (fuzzy control)

Assilian, Mamdani, 1974: *An Experiment in Linguistic Synthesis with a Fuzzy Logic Controller*

<table>
<thead>
<tr>
<th>Fuzzy Control Variables</th>
<th>Fuzzy Subsets</th>
</tr>
</thead>
<tbody>
<tr>
<td>PE Pressure Error</td>
<td>PB Positive Big</td>
</tr>
<tr>
<td>CPE Change in pressure error</td>
<td>PM Positive Medium</td>
</tr>
<tr>
<td></td>
<td>PS Positive Small</td>
</tr>
<tr>
<td>HC Heat Change</td>
<td>NO Nil</td>
</tr>
<tr>
<td></td>
<td>NS Negative Small</td>
</tr>
<tr>
<td></td>
<td>NM Negative Medium</td>
</tr>
<tr>
<td></td>
<td>NB Negative Big</td>
</tr>
</tbody>
</table>
Fuzzy-Regelung (fuzzy control)

Input variables:

• **Pressure Error**
  (Difference between tatsächlichem und vorgegebenem Druck.)

• **Change in Pressure Error**
  (Velocity of the movement oft the presureeschwindigkeit mit der sich
der tatsächliche Druck
  vom Sollwert entfernt bzw. nähert.)

Output variable:

• **Heat Change**
Fuzzy-Regelung (fuzzy control)
Fuzzy-Regelung (fuzzy control)

Regel 1:
Wenn die Druckabweichung klein und positiv ist und sich die Druckabweichung nicht viel ändert, dann vermindere die Wärmezufuhr ein wenig.

Wenn PK und Null, dann NK.

Regel 2:
Wenn die Druckabweichung etwa Null ist und sich die Druckabweichung nicht viel ändert, dann verändere die Wärmezufuhr nicht.

Wenn Null und Null, dann Null.

Regel 3:
Wenn die Druckabweichung klein und positiv ist und sich die Druckabweichung langsam vergrößert, dann vermindere die Wärmezufuhr ein wenig.

Wenn PK und NK, dann NK.
<table>
<thead>
<tr>
<th>Änderung in der Druckabweichung</th>
<th>Negativ, groß</th>
<th>Negativ, mittel</th>
<th>Negativ, klein</th>
<th>Null</th>
<th>Positiv, klein</th>
<th>Positiv, mittel</th>
<th>Positiv, groß</th>
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<tbody>
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<td>PG</td>
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<td>NM</td>
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<td>NK</td>
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<td>Null Regel 1</td>
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<td>Null Regel 2</td>
<td>Null Regel 3</td>
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<tr>
<td>PK</td>
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</table>
Fuzzy-Regelung (fuzzy control)
Fuzzy-Regelung (fuzzy control)

Fixed controller (DDC algorithm), $\times$, $\square$; Fuzzy controller, $\circ$. 