OPTIMAL SCHEDULING OF TRAINS ON A SINGLE LINE TRACK

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Abstract—This paper describes the development and use of a model designed to optimise train schedules on single line rail corridors. The model has been developed with two major applications in mind, namely: as a decision support tool for train dispatchers to schedule trains in real time in an optimal way, and as a planning tool to evaluate the impact of timetable changes, as well as railroad infrastructure changes. The mathematical programming model described here schedules trains over a single line track. The priority of each train in a conflict depends on an estimate of the remaining crossing and overtaking delay, as well as the current delay. This priority is used in a branch and bound procedure to allow and optimal solution to reasonable size train scheduling problems to be determined efficiently. The use of the model in an application to a “real life” problem is discussed. The impacts of changing demand by increasing the number of trains, and reducing the number of sidings for a 150 km section of single line track are discussed. It is concluded that the model is able to produce useful results in terms of optimal schedules in a reasonable time for the test applications shown here. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

1.1. Background

The model described in this paper is designed to be used in two main modes, namely: as a decision support tool for train dispatchers to schedule trains in real-time in an optimal way; and as a planning tool to evaluate the impacts of timetable changes, as well as railroad infrastructure changes on train arrival times and train delays. The model is primarily designed to optimally schedule trains on single line railroads. Under such conditions, trains operate over a single track and can only overtake and cross each other at specific locations referred to here as sidings. This type of operation is common throughout the world. Most developing countries have networks which are mainly single line track.

In Australia, the average length of haul for interstate freight movements is around 1500 km. The bulk of this interstate network consists of single line track. The extent of that network is shown in Fig. 1. The train dispatching function outside urban areas is currently performed using mainly manual methods. The dispatcher resolves train conflicts on a time–distance graph using experience and knowledge of prevailing conditions. The criteria for conflict resolution includes the priority of each train, current train delays, and expected remaining overtaking and crossing delay. This criteria is used to develop an improved lower bound. When used with the depth first search Branch and Bound procedure, optimal solutions to large problems can be found.

The main objective for developing the model described here is to provide the operator with a tool which helps him/her to perform the train dispatching function. The model is not designed to replace the dispatcher. The operator’s experience and knowledge of local conditions, will continue to be used. Train dispatching decisions, which to a certain extent involve human as well as technical factors, will require human intervention to resolve problems. However, with such an optimisation model available, the operator is able to quickly update a schedule as unplanned events occur. The new optimal schedule offered
by the model may not be fully implementable for practical reasons. However, the gap between the optimum and the practically feasible schedule, can be readily assessed. The penalty for not being able to implement the optimum schedule, in terms of operating cost and travel time reliability, can be evaluated against the practical factors which prevent implementation of the optimum schedule.

A second major use of the model relates to the planning of railroad operations. Such planning can be conveniently divided into two components, namely: short to medium term train planning; and railroad infrastructure planning associated with train operations. The model can be used to evaluate the implications of changes to a timetable in terms changed train departures, additional trains, and changes in train speeds. The optimum scheduling algorithm can be used as a simulator of proposed changes.

1.2. Train schedule reliability

When trains are scheduled on a rail corridor the objective is to achieve a given level of customer service whilst minimising overall operating costs. Customer service in this context is made up of several attributes which include overall journey time of trains and the degree to which those journey times are achieved on a daily basis. In the context of freight movements, the benefits of improved reliability need to be estimated on a train by train basis. Each train is usually loaded with freight from a range of customers and origin-destination flows. The elasticity of demand with respect to transit time reliability will differ for each customer, commodity, and origin-destination combination.

The overall timetable reliability is a measure of the likely performance of the timetable as a whole, in terms of schedule adherence. The reliability of arrivals is a critical performance measure for all rail markets. The ability of rail systems to compete effectively relies to a large extent on consistent transit time reliability (Fowkes et al., 1991; Industry Commission, 1991; Bureau of Industry Economics, 1993).
If the train operations are conducted under single line conditions, the transit time reliability is a function of a range of factors. The degree of "slackness" built into the schedule; the number and position of train conflicts; priorities for each train; terminal congestion; number and nature of scheduled stops; and train speeds are all influencing variables. Analytically based models designed to quantify the amount of delay risk associated with each track segment, train and the schedule as a whole are given by Higgins et al. (1995a).

There are several ways in which investment in track related infrastructure can reduce delays and hence improve transit time reliability. Four main investment strategies may have significant impact on the probabilities associated with train delays, namely:

(a) Investment in major track strengthening to increase maximum allowable speeds. The higher speeds have the potential to reduce conflict related delays and improve train recoverability;
(b) Investment designed to alter track alignments, both vertical and horizontal, thus increasing average train speeds;
(c) Investment in additions to the number and length of sidings where trains can cross and pass each other on single line track. Conflict related delays are directly affected by the number, length and location of sidings;
(d) Investment in advanced train control and communication systems to allow trains to proceed at shorter headways, and with less stops required for safe train operations.

The benefits of some of the above investment usually extend beyond transit reliability gains. For example, in the case of track rehabilitation and upgrading, those benefits may include reductions in overall transit times; reduction in accident risks; lower track maintenance costs; increased train productivity from higher maximum allowable axle-loads; reductions in rolling stock maintenance costs due to improved vehicle–track interaction; and improved locomotive productivity through the use of more modern equipment.

In order to obtain maximum benefits it is usually necessary to combine a number of investment strategies into a coherent and complementary package of capital expenditure projects. For example, the gains in reliability from track upgrading projects can be augmented by investment in terminal infrastructure to allow faster loading/unloading of trains, and by locomotive investment to enable higher train speeds.

2. PAST WORK

Work on an optimum solution to the train scheduling problem started in the early seventies by Szpigel (1973) who developed a linear programming model to determine the best overtaking and crossing positions given that the departure times and upper velocities of the trains are known. A Branch and Bound method is used to resolve the conflicts and lower bound to the remaining delay is generated by relaxing the remaining conflicts. Minimising the sum of the travel times was the objective and only small problems were tested. Petersen et al. (1986) considered a similar dispatch algorithm which calculates the crosses, segment transit times and determines which train takes the sidings in order to minimise the total travel times.

Kraft (1987) takes a different approach by developing a dispatching rule giving the optimal time advantage for a particular train based on train priority, track running times and the delay penalties of each train. This rule was used to resolve crossing conflicts in the Branch and Bound procedure.

Kraay et al. (1991) proposes a model which paces trains in order to conserve fuel and, at the same time, keep the lateness of trains at a minimum. The fuel consumption is a function of friction and gradient of track, speed and mass of the train, and air friction. Two heuristics were proposed which are able to find solutions to realistic size problems. Jovanovic et al. (1991) uses a similar constraint framework as part of a decision support
model called "SCAN" which is based upon combinatorial optimisation and simulation. Mills et al. (1991) formulated a discrete network type model by discretising the departure and arrival time variables. The discretisation allows the use of the shortest path algorithm to update the journey of each train. A procedure recursively updates the path of each train until a feasible schedule is found. The solution procedure is an approximation.

Mees (1991) models the single line rail as a network structure where each segment is an arc (a siding is considered as an extra arc), separated by nodes (considered as track intersections or stations). The network is time-space with a fixed schedule time span and headways are obtained by allowing only one train per arc at a time segment. A solution procedure similar to Mills et al. (1991) is used to find a feasible solution.

Due to the difficulty of finding an optimal solution to large problems the trend has been towards finding an approximate solutions. The objective in this paper is to present a lower bound that will allow the branch and bound procedure to find the optimal solution to realistic size problems in reasonable time.

3. MODEL FORMULATION

This section deals with the definition and derivation of the model. The resulting formulation is a non-linear mixed integer program for which the integer part is solved using an intelligent branch and bound procedure.

3.1. Variables and model assumptions

The set of trains is given by $I = \{1,2,\ldots,m,m+1,\ldots,N\}$ for which inbound trains are from 1 to $m$ and outbound are from $m + 1$ to $N$. The variables used in the model are listed and described in this section.

Let $P = \{P_1, P_2\}$ where

$P_1 = \text{set of single line tracks, } P_2 = \text{set of double line tracks.}$

The integer decision variables for determining which train traverses a section first (also determines the position of conflict resolution) are given by

$$A_{ijp} = \begin{cases} 1 & \text{if inbound train } i \leq m \text{ traverses track segment } p \in P_1 \text{ before } j \leq m \\ 0 & \text{otherwise} \end{cases}$$

$$B_{ijp} = \begin{cases} 1 & \text{if inbound train } i \leq m \text{ traverses track segment } p \in P_1 \text{ before } j > m \\ 0 & \text{otherwise} \end{cases}$$

$$C_{ijp} = \begin{cases} 1 & \text{if outbound train } i > m \text{ traverses track segment } p \in P_1 \text{ before } j > m \\ 0 & \text{otherwise} \end{cases}$$

The arrival and departure time decision variables are as follows:

$X_{aq}$ = arrival time of train $i \in I$ at station $q \in Q$

$X_{dq}$ = departure time of train $i \in I$ from station $q \in Q$

$X_{Oi}$ = departure time of train $i \in I$ from its origin station

$X_{Di}$ = arrival time of train $i \in I$ at its destination station.

The input parameters are defined as follows:

$h_p$ = minimum headway between two trains on segment $p \in P_1$

$d_p$ = length of segment $p \in P$
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Fig. 2. Sample of a network showing the single and double track segments.

An illustration of the ordering of a single track used for the model in this paper is given in Fig. 2 where the set of sidings is represented by $Q = \{1,2,\ldots,NS\}$ and here, track $(p-2) \in P$. The following assumptions are made with regard to the model in this section:

- The track is divided into segments which are separated by sidings.
- Crossing and overtaking can occur at any siding or double line track segments.
- Trains can follow each other on a track segment with a minimum headway. If only one train is permitted on some track segments then the headway on these track segments is increased to the track length.
- For double track sections, it is assumed one lane will be allocated for inbound trains and one lane will be allocated for outbound trains. Usually, signal points will be set up this way.
- Scheduled stops are permitted at any intermediate siding for any train.

The model will require various information to make use for the input to the model. The specific information is as follows:

- An unresolved train plan to make available the number of overtake and cross interferences for each train.
- The priorities of each train. These are determined by several factors such as the type of train, customer contract agreements and train load.
- The upper and lower velocity limits for each train (which are dependent on the physical characteristics of the track segment and the train).
- The times of any scheduled train stops. These stops may include loading/unloading, refuelling and crew changes.

3.2. Objective function and constraints

The objective function used in the model takes the following form

$$\min \sum_{i} W_{i} (\text{delay of train } i \in I \text{ at destination}) + \text{Train Operating Costs}$$

For the purposes of the solution procedure (namely Branch and Bound), the delay of train $i \in I$ is comprised of two parts. These are the current delay of train $i \in I$ at any point in time and a lower bound estimate of remaining overtake and crossing delay from this point. The model is subject to various constraints to ensure safe operation, enforce speed restrictions and permit stops. The following and overtake constraints for outbound trains $i,j \in I$ are as follows

\begin{align*}
Y_{Oi} & = \text{earliest departure time of train } i \in I \text{ from origin station} \\
Y_{Ai} & = \text{planned arrival time of train } i \in I \text{ at destination station} \\
Y_{p}^{i} & = \text{minimum allowable velocity of train } i \in I \text{ on segment } p \in P \\
V_{p}^{i} & = \text{maximum achievable average velocity of train } i \in I \text{ on segment } p \in P \\
W_{p}^{i} & = \text{priority of train } i \in I \text{ (highest for passenger trains)} \\
S_{q} & = \text{scheduled stop time for train } i \in I \text{ at station } q \in Q.
\end{align*}
\[ M \ast C_{ijp} + X_{aq+1}^i \geq X_{aq}^i + h_p \] \( \forall p \in P_1 \text{ and } i, j > m \) \( \text{(2)} \)

\[ M \ast C_{ijp} + X_{dq}^j \geq X_{d}^j + h_p \] \( \forall p \in P_1 \text{ and } i, j > m \) \( \text{(3)} \)

and for inbound trains \( i, j \in I \)

\[ M \ast A_{ijp} + X_{aq}^i \geq X_{aq}^i + h_p \] \( \forall p \in P_1 \text{ and } i, j \leq m \) \( \text{(4)} \)

\[ M \ast A_{ijp} + X_{d}^i \geq X_{d}^i + h_p \] \( \forall p \in P_1 \text{ and } i, j \leq m \). \( \text{(5)} \)

Equation (2) implies that if train \( j \in I \) goes first, then train \( i \in I \) must depart station \( q \in Q \) after train \( j \in I \) plus the minimum headway, and arrive at station \( (q + 1) \in Q \) after train \( j \in I \) plus the headway. Equation (3) is similar except train \( i \in I \) goes first. Equations (4) and (5) are the same as equations (2) and (3), but for inbound trains. The constraints for the case when two trains approach each other are

\[ h_p + X_{aq+1}^j \leq X_{aq}^i + M \ast B_{ijp} \]

\[ h_p + X_{aq+1}^i \leq X_{aq+1}^j + M \ast (1 - B_{ijp}) \] \( \forall p \in P_1, (i \leq m, j > m) \). \( \text{(6)} \)

Equation (6) implies that if outbound train \( j \in I \) goes first, inbound train \( i \in I \) must depart station \( q \in Q \) after train \( j \in I \) arrives plus a safety headway. Constant \( M \) is chosen large enough so that both equations in each crossing and overtake constraint are satisfied.

Given the upper and lower velocities for each train on each segment, the upper and lower limits for traversal time of train \( i \in I \) on segment \( p \in P_1 \) are given by

\[ \frac{d}{v_p} \leq X_{aq+1}^i + X_{d}^i \leq \frac{d}{v_p} \quad i > m, p \in P \] \( \text{(7)} \)

\[ \frac{d}{v_p^i} \leq X_{aq}^i + X_{d}^i + 1 \leq \frac{d}{v_p} \quad i \leq m, p \in P \].

To stop trains from departing before their scheduled times and trains departing intermediate stations before they arrive, the following constraints are included

\[ X_{qi}^i \geq Y_{qi}^i \]

\[ X_{aq}^i + S_q^i \leq X_{aq+1}^i \] \( \forall i, q \in Q \). \( \text{(8)} \)

The objective is to minimise equation (1) subject to constraints given by equations (2)–(8). The solution procedure to solve this model can be found in Appendix A. The algorithm for determining the lower bound is described in Appendix B.

4. MODEL APPLICATIONS

4.1. Model performance analysis

The solution procedure to solve the model of Section 3.2 along with the lower bound estimates (LBI) have been implemented in FORTRAN on a 80486 PC. The model was tested on train schedules varying from 9 trains to 30 trains and was compared to a Branch and Bound Procedure with a lower bound calculated by relaxing the remaining conflict constraints (LB2). The exact objective function used is a combination of tardiness and fuel consumption (Mills et al., 1991).
Table 1. Comparison between solution techniques: number of schedule updates

<table>
<thead>
<tr>
<th>No. of trains (No. of sidings)</th>
<th>No. of conflicts</th>
<th>Branch and Bound (LB1)</th>
<th>Branch and Bound (LB2)</th>
<th>Tabu search (TS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9(6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>24$^a$</td>
<td>97</td>
<td>620</td>
<td></td>
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<tr>
<td>12</td>
<td>24$^a$</td>
<td>211</td>
<td>411</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>24$^a$</td>
<td>282</td>
<td>543</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>28$^a$</td>
<td>452</td>
<td>436</td>
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</tr>
<tr>
<td>11(8)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>42</td>
<td>433</td>
<td>789</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>30$^a$</td>
<td>1052</td>
<td>402</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>38$^b$</td>
<td>1129</td>
<td>421</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>56</td>
<td>1026</td>
<td>1140</td>
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<tr>
<td>15(8)</td>
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<td></td>
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<tr>
<td>13</td>
<td>34$^a$</td>
<td>402</td>
<td>816</td>
<td></td>
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<tr>
<td>14</td>
<td>36$^a$</td>
<td>332</td>
<td>407</td>
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<td>20(8)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>36$^a$</td>
<td>1054</td>
<td>995</td>
<td></td>
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<tr>
<td>19</td>
<td>96</td>
<td>1190</td>
<td>860</td>
<td></td>
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<tr>
<td>25</td>
<td>56$^a$</td>
<td>1268</td>
<td>812</td>
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<td>27</td>
<td>122</td>
<td>1440</td>
<td>405</td>
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<td>25(12)</td>
<td></td>
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<tr>
<td>23</td>
<td>46$^a$</td>
<td>1338</td>
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<tr>
<td>24</td>
<td>120</td>
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<td>2820</td>
<td>1268</td>
<td></td>
</tr>
</tbody>
</table>

$^a$Optimum solution found in minimum possible time.

The results were also compared to a Tabu Search (TS) heuristic solution (Glover, 1993). The TS was allowed to run until a certain number of iterations elapses when no improvement in solution value is found. The neighbourhood was defined as the movement of a conflict to a neighbouring siding and only a sample of the neighbourhood was searched at each iteration. A full description of the use of the TS heuristic can be found in Higgins et al. (1995b).

Different examples were solved for each problem size by varying the earliest departure time of each train, upper velocities and due arrival times. As shown in Table 1, the

![Distance vs Time Graph](image-url)
examples within each problem size contain different numbers of conflicts. The latter are a representation of the complexity of each problem. Results for the synthetic problems are summarised in Table 1. The results are given in terms of the number of times the schedule had to be updated due to the addition or change of a resolved conflict. The TS results shown are the average values of 5 trials.

As shown in Table 1, the new lower bound presented in this paper (LBl) produced significant reductions in the number of calculations for all problems. The optimum solution to most of the smaller problems was found in the minimum possible time. Results compared favourably to the Tabu Search heuristic in terms of number of calculations and average travel time per train.

However, the lower bound (LBl) took longer to calculate than LB2. The significance of this towards the overall CPU time depends on the way the schedule is updated when a resolved conflict is changed or added. For the procedure used in this paper (described in Appendix B) the CPU comparison between columns LB1 and LB2 would be valid if the values for LB1 in Table 1 were multiplied by 2.5. If a network flow algorithm or a software package such as GAMS/MINOS is used to update the schedule the significance of the calculation of LBl towards the overall CPU time would be very low. The unresolved and resolved schedules for the 30 train problem is illustrated in Figs 3 and 4 respectively.

4.2. Option testing

The algorithm was tested on an actual section of railroad. The rail corridor contains 14 sidings and 31 trains are scheduled on the busiest day of the week. The four types of trains scheduled are: fast freight, heavy freight, electric passenger and locomotive hauled passenger trains. Each train class has a different maximum achievable velocity. For the purposes of this paper it is assumed all sidings are sufficiently long to accommodate any train size. The algorithm only required 46 s to determine the optimum solution to the problem. The optimum schedule is demonstrated in Fig. 5. The efficiency of the algorithm will allow the rapid updating of an optimum solution whenever an unexpected delay occurs.

The solution procedure presented here was used to demonstrate the effects of two different types of operating changes, namely: increases in demand through additional trains and reduction in the number of sidings available for conflict resolution. As shown in Fig. 5, the line is currently operating with significant spare capacity. The effects of train
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Fig. 5. Real life problem consisting of 31 trains and 14 sidings.
delays (in terms of average increase travel time per train in the optimal schedule), when
sidings are removed is demonstrated in Fig. 6. To prevent any possible bias, the siding
positions in the original problem are taken to be equally spaced. When a siding is removed
the remaining sidings are spaced equally. It is only when the number of sidings are
reduced below nine that the increase in average transit time becomes significant. If the
original number of sidings was reduced to nine, the schedule would operate with an
increase in average transit time of less than 2%.

However, this analysis does not take into account the ability of a given set of sidings to
cater for unexplained events such as track and rolling-stock related delays on the line.
Some sidings are required to provide insurance against delay risks, rather than to act as
conflict resolution locations under “ideal” conditions.

The second application consisted of scheduling additional trains at already congested
periods of the existing schedule. Three new fast freight trains were added at a time, one for
each peak period. The average effects of adding these extra trains to the original 31 train,
14 siding problem is demonstrated in Fig. 7. The average transit time increases almost
linearly with the number of trains in the schedule. The addition of 18 trains increases the
average transit time by 8.4%. The schedule with 49 trains contained nearly 100 train
conflicts and was solved within 10 min. The effects of adding extra trains and removing
sidings simultaneously is demonstrated in Table 2. The numbers in the table represent the
increase in average travel time as a proportion of the base problem. The increase in aver-
age travel time tends to be linear for the increase in number of trains and exponential for
the reduction in sidings.

![Fig. 6. Effect of decreasing number of sidings.](image1)

![Fig. 7. Effect of increasing demand.](image2)
5. CONCLUSIONS

This paper has presented an on-line model for the scheduling of trains on a single line track. The solution procedure used a lower bound estimate of the remaining overtake and crossing delay to reduce the search space in the Branch and Bound tree. The lower bound estimate was based on the calculation of the least cost path of each train, while assuming the path of a train is independent of previous accumulated conflict delay. The calculation of the lower bound is shown to be of low order polynomial time. This has allowed the optimal solution to realistic size problems to be found in a reasonable time.

As well as an online scheduling tool, the model presented here can be used for long-range planning of railroad operations. In Australia, there are two main infrastructure planning issues which are currently under investigation, namely: the upgrading of main line track to allow higher speeds and heavier axle loads; and the need to extend sidings to allow for longer trains. The scheduling optimisation model can be used to evaluate both these investment strategies. The impact on the schedule of extending some sidings and not others can be assessed by using the model to simulate the effect of the proposed changes on future schedules. The removal of sidings has a cost in terms of flexibility and feasibility of schedules. As part of the same research project, the authors have developed a model to optimise the location of sidings for a given set of train departure times and mean train speeds (Higgins et al., 1994b).

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REFERENCES


**APPENDIX A**

**Solution Procedure**

### A1. General

The solution procedure described in this section is based on Branch and Bound and uses the depth first search for the resolution of conflicts. Each node in the tree involves the solution of a non-linear program which can be solved using a specialised package program such as GAMS/MINOS. An algorithm for determining the lower bound can be found in Appendix B. The procedure is as follows:

1. Solve the objective function (1) subject to the velocity, departure time and scheduled stop constraints (7),(8) ignoring the cross and overtake constraints (2)–(6). This will give an unresolved train plan.

2. From the train graph in step 1, pick out the first conflict in time. Identify the two trains $i,j \in I$ involved and the segment $p \in P$ which this conflict occurs. There will be two alternatives to resolve this conflict, train $i \in I$ is delayed, or train $j \in I$ is delayed. For each of the alternatives, update the train journey by solving the sub problem with the conflict between these two trains, i.e. solve objective function (1) st (7,8) and the appropriate overtake or crossing constraint from eqns (2)–(6). For each of these two alternatives determine the lower bound estimate of the remaining crossing and overtake delay. Add the delay cost of the lower bound estimate of remaining delay to the overall cost.

3. Pick the alternative (resolution) with the lowest cost. If the cost is greater than the current upperbound then go to step 6(a) (the rest of the branch is pruned).

3(a). From this resolution (conflict) node, pick the next conflict in the train graph. Identify the trains involved and the segment which this conflict occurs.

4. For each of the two alternatives, update the train journey by solving the subproblem with objective function (1) subject to the constraints (7,8) and a subset of the fixed integer constraints (2)–(6) when considering the current conflict, and previous conflicts (i.e. conflicts on the same branch and higher in the branch and bound tree). Calculate the lower bound estimate of remaining conflict delay for each of these alternatives and add the delay cost of the lower bound estimate of remaining delay to the overall cost.

5. If there are no more conflicts then go to step 6 otherwise go to step 3.

6. Mark the best solution in this last alternative pair of resolutions. If the cost of this solution is less than the current upper bound then let the upper bound equal the cost of this new solution.

6(a). Trace up the Branch and Bound tree, one level at a time until a level with an unbranched node and a cost lower than the upper bound is found. Pick the node with the smallest cost in this level. If the top level in the Branch and Bound tree is reached when tracing up, then the procedure terminates and the current solution is the best solution otherwise go to step 3(a).

### A2. Solving the non-linear subproblem

The non-linear subproblems (fixed integer variables) are solved using an approximate iterative procedure which builds up a schedule given a set of resolved conflicts. The objective function [eqn (1)] in the model is assumed to have the components tardiness and fuel costs of each train. It is assumed the cost of tardiness has a higher priority than fuel costs. A train will not travel slow to arrive at the destination station late unless the trains priority is set to a minimum. Situations occur when a train is not able to make a connection on time or the due arrival time of the train is extremely late. In such a case, the train will be delayed in every conflict and the train is paced between conflicts.

When constructing the schedule using the iterative procedure, it is assumed a train will travel at the maximum achievable average velocity on any track segment, unless the train is to be delayed in a conflict at the next siding. In this case the train will be paced to arrive at the siding at the last moment. If a train has slack time, it is paced to arrive at the destination station on time from its last conflict. Full details of the algorithm can be found in Higgins et al. (1994a).

For solving the non-linear subproblem, the approximate procedure gave near identical results to GAMS/ MINOS 5.2 for all size problems tried when the objective function of minimum tardiness and fuel consumption is used. For a problem of 32 trains and 14 sidings, GAMS/MINOS 5.2 took 8 min (on a 80486DX PC) to solve one instance and required 8 megabytes of RAM. The approximate iterative procedure was able to solve nearly a thousand instances of this size problem in 1 min with near identical results.

**APPENDIX B**

**Generation of Lower Bound**

In this section the generation of a lower bound to the remaining delay costs is described. The lower bound is based on the estimation of the least cost path for each train. When the cost of the path is estimated, the path is independent of accumulation of delay that might occur from earlier conflicts. Since each path is independent of
previous delays, the sum of these least cost paths will provide a lower bound to the cost of the actual remaining delay.

The algorithm starts from the first unresolved conflict and terminates at the last. At each unresolved conflict, estimates the least cost path for each of the two trains involved is calculated. The least cost path for a train is the best path for the journey of the train up to the current conflict. The train involved in the conflict with the least cost path is chosen and the delays due to its path are recorded. Several rules are involved to prevent overwriting a delay in a conflict which belongs to the path of a different train and to prevent and double addition of delays. When the algorithm terminates after the last conflict, the delay associated with each train is added to provide the lower bound.

**Notation**

Let:

\[ CO = \text{the number of unresolved conflicts in the train schedule} \]

\[ PR_c = \text{the ordered set (in time) of previous unresolved conflicts (before and including conflict } c \text{) where one of the trains involved is the same train as the outbound train in conflict } c \]

\[ PS_c = \text{the ordered set of previous unresolved conflicts (before and including conflict } c \text{) where one of the trains involved is the same train as the inbound train in conflict } c \]

\[ NR_c = \text{the ordered set of future conflicts (including conflict } c \text{) where one of the trains involved is the same train as the outbound train in conflict } c \]

\[ NS_c = \text{the ordered set of future conflicts (including conflict } c \text{) where one of the trains involved is the same train as the inbound train in conflict } c \]

\[ T_c = \text{the difference between the arrival time of the outbound train and the departure time of the inbound train for conflict } c \text{ (see Fig. B1)} \]

\[ B_c = \text{the difference between the arrival time of the inbound train and the departure time of the outbound train for conflict } c \text{ (see Fig. B1)} \]

\[ DEL = \text{vector containing the conflict delay and train delayed for each conflict} \]

\[ DEL_c = \text{amount of delay for conflict } c, \text{ where } DEL_{c_1} = \text{delay in conflict } c, \text{ and } DEL_{c_2} = \text{delay in future conflict } c \]

\[ TR_i = \text{conflict delay to train } i \in I \]

The symbol \( \mathbf{1} \) represents the first term given that the second term has occurred.

Given a conflict under consideration \( c \), let

\[ ST_0 = \{ \} \]

\[ ST'_{p} = \begin{cases} 
\{ a_1, ST'_{p-1} \} & \text{if } \sum_{c \in PR_c} \min(T_p, B_p) \left( a_1, ST'_{p-1} \right) + a_1 < \\
\sum_{c \in PR_c} \min(T_p, B_p) \left( a_2, ST'_{p-1} \right) + a_2 & \text{otherwise}
\end{cases} \]

\[ CT_0 = 0.0 \]

\[ CT'_{p} = \begin{cases} 
a_1 + CT'_{p-1} & \text{if } ST'_{p} = \{ a_1, ST'_{p-1} \} \\
a_2 + CT'_{p-1} & \text{if } ST'_{p} = \{ a_2, ST'_{p-1} \}
\end{cases} \]

where \( a_1 = B_c \cdot ST'_{p-1} \) (the distance \( B_c \) given the sequence of conflict delays \( ST'_{p-1} \)), \( a_2 = T_c \cdot ST'_{p-1} \), \( C_p \) = conflict in the \( p \)th position of \( PR_c \), \( pos(e) \) = position of conflict \( e \) in \( PR_c \). If train \( i \in I \) is the outbound train involved in conflict \( c \), then \( ST'_{p} \) is the set of delays of the conflicts for which train \( i \in I \) is involved in from the start of the journey of train \( i \in I \) to the conflict in the \( p \)th position of \( PR_c \). Let
\[ MA^c = \arg \min_f \left( CT^*_f + \sum_{\text{pos}(v) = f} \min(T_e, B_e) \right) \]

and

\[ MB^c = \{ ST^*_{\text{MB}}, \min(T_e, B_e) \} \text{ given } e \in PR_c \text{ and } \text{pos}(e) \geq MA^c \] (9)

where \( MB^c \) is the set of delays which correspond to \( PR_c \). Let \( DT^c = \) the containing the corresponding train delayed for each conflict delay in \( MB^c \). Similarly, when considering an estimate of the best path of the inbound train from conflict \( c \), the equations become

\[ SC^c_0 = {} \]

\[ SC^c_f = \begin{cases} \{ a_1, SC^c_{f-1} \} & \text{if } \sum_{v \in \text{pos}(c)} \min(T_e, B_e) \{ a_1, SC^c_{f-1} \} + a_1 < \\ \sum_{v \in \text{pos}(c)} \min(T_e, B_e) \{ a_2, SC^c_{f-1} \} + a_2 \end{cases} \]

\[ CC^c_0 = 0.0 \]

\[ CC^c_f = \begin{cases} a_1 + CC^c_{f-1} & \text{if } SC^c_f = \{ a_1, SC^c_{f-1} \} \\ a_2 + CC^c_{f-1} & \text{if } SC^c_f = \{ a_2, SC^c_{f-1} \} \end{cases} \]

where \( a_1 = B_{C_p} \), \( a_2 = T_{C_p} \), \( C_p = \text{conflict in the } p \text{th position of } PS \text{, pos}(e) = \text{position of conflict } e \) in \( PS \). Let

\[ NA^c = \arg \min_f \left( CC^c_f + \sum_{\text{pos}(v) = f} \min(B_e, T_e) \right) \{ SC^c_f \} \]

and

\[ NB^c = \{ SC^c_{\text{NB}}, \min(T_e, B_e) \} \text{ given } e \in PS \text{ and } \text{pos}(e) \geq NA^c \] (10)

where \( NB^c \) is the set of delays corresponding to the set \( PS \). Let \( DS^c \) be the corresponding set containing the train delayed for each conflict delay in \( NB^c \). In other words, \( MB^c \) and \( NB^c \) are the set of delays of the estimated best path for the outbound and inbound train respectively involved in conflict \( c \). That is, the best path from the origin station to conflict \( c \). If conflict \( c \) is an outbound overtake then \( MB^c \) will be calculated for both trains and one of the \( MB^c \) will be treated as \( NB^c \) in the following algorithm. The same applies for an inbound overtake. A detailed description of eqns (9) and (10) can be found in Higgins et al. (1994a).

**Algorithm 1**

**Step 0.** Set \( c = 0 \). Assign zero's to \( DEL \).

**Step 1.** If \( c = CO \) GOTO Step 2. Let \( c = c + 1 \)

Determine \( MB^c, NB^c, DT^c, DS^c \).

Let \( DL^1, DL^2 = DEL \).

FOR each conflict in \( MB^c \)

Overwrite the corresponding conflict in \( DL^1 \) if the conflict in \( DL^1 \) is not previously updated from a \( NB^c, e < c \) (i.e. if the delay in \( DL^1 \) belongs to the path of a different train, it is left untouched)

END (FOR).

Let \( f \) be the neighbouring conflict of \( c \) involving the same inbound train.

IF the delay of conflict \( f \) in \( DL^1 \) (and the way the conflict is resolved) would cause a decrease in delay to conflict \( c \), subtract the decrease in delay from conflict \( c \).

IF the delay of conflict \( f \) in \( DL^1 \) (and the way the conflict is resolved) would cause extra delay to conflict \( c \) add the extra delay to conflict \( c \). FOR each conflict in \( NB^c \)

Overwrite the corresponding conflict in \( DL^2 \) if the conflict in \( DL^2 \) is not previously updated from a \( MB^c, e < c \)

END (FOR).

Let \( f \) be the neighbouring conflict of \( c \) involving the same outbound train.

IF the delay of conflict \( f \) in \( DL^2 \) (and the way the conflict is resolved) would cause a decrease in delay to conflict \( c \), subtract the decrease in delay from conflict \( c \).

IF the delay of conflict \( f \) in \( DL^2 \) (and the way the conflict is resolved) would cause extra delay to conflict \( c \) add the extra delay to conflict \( c \).

Replace \( DEL \) with either \( DL^1 \) or \( DL^2 \) depending upon which gives the least total train tardiness.

Go to step 1.

**Step 2.** Obtain the delay \( TR_i \) to each individual train \( i \in I \) from \( DEL \).

Let lower bound equal \( \sum_i \max(\{ TR_i + A_{X_{i0}} - B_{Y_{i0}} \}, 0) \cdot W_i \).

The computation complexity of the above algorithm is defined in terms of the number of IF-THEN statements and the amount of changing of arrival and departure times train. It was shown in Higgins et al. (1994a) to be \( O(CO^4) \) for the number of IF-THEN statements and \( O(CO^5) \) for the number of arrival/departure time changes.

A proof concerning the validity of the lower bound is also described in the same paper.
Explanation of $MF, NF$

Equations (9) and (10) are the delay estimates of the least cost path for the journey of the outbound and inbound train involved in conflict $c$. They are the least cost paths from the start of the train’s journey to conflict $c$. A list of the remaining conflicts for the outbound and inbound train involved is stored in $PR_c$ and $PS_c$, respectively.

Consider the outbound train involved in conflict $c$ in Fig. B2(a). Starting with the first conflict in $PR_c$ (conflict $a$ in Fig. B2(a)), the delay is $T_a$. This is because $\sum e \min (T_e, B_e)$ (outbound train delayed in conflict $a$) for the rest of the conflicts ($e = b, c$) plus $T_a$ is less than $h_0$ plus $\sum e \min (T_e, B_e)$ (inbound train delayed in conflict $a$). The term $\sum e \min (T_e, B_e)$ acts as a rough cut look ahead, since a conflict resolved at a siding which is not the nearest siding may allow the rest of the conflicts to be resolved with little cost. As an example, in Fig. B2(b), the cheapest way to resolve conflict $a$ is to delay the inbound train. However, delaying the outbound train (giving new path denoted by dotted line) will allow conflicts $b$ and $c$ to cause little delay. This means the outbound train will be delayed instead, and is why the term $\sum e \min (T_e, B_e)$ is added for the comparison part of equations $ST^*_p$ and $SC^*_p$. The equations $MA^*$ and $NA^*$ record the conflict position $f$ of a conflict for which the estimate of delay ($CT^*_f$) up to the conflict in position $f$ plus the minimum distances from the nearest siding for the remaining conflicts ($\sum e \min (T_e, B_e)$ (set of conflict delays up to $f$)) is minimum. Equations (9) and (10) record the delay to each conflict associated with this path. The term $\sum e \min (T_e, B_e)$ (set of conflict delays up to $f$) can be an underestimate or an overestimate of the actual delay of the remaining conflicts (after $f$). Since equations $MA^*$ and $NA^*$ choose the minimum cost path, it is likely that eqns (9) and (10) will be underestimates of the actual best path for the outbound and inbound trains respectively.