# Modeling Fabrics as a Special Case of Elastic Cosserat Surface 

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#### Abstract

Aiming at realistic visual effects and an intuitive interface, we present in this paper a formulation of a Cosserat surface that is not only convenient for numerical calculations, but also preserves direct relations between the fabrics' material properties and their easily interpretable geometrical quantities. It is not a novelty, at least to textile scientists, that the theory of a Cosserat surface is promising for cloth modeling. The main barrier for its practical applications, at least to computer graphics practitioners, is its numerical complexities. The theory involves several geomerical quantities that are not readily accessible for computing. These computation issues are the main concern of this paper. The calculations of the normal vectors and the geometric properties associated with them are detailed in this paper. With our formulation, we successfully reproduce fabrics' responses to a variety of forces, including the axial forces, by only altering a set of comparatively few parameters. Promising visual results comprising of fabric folding, draping, buckling, and wrinkling, are also given.


Categories and Subject Descriptors: I.3.5 [Computer Graphics]: Object Modeling—physically based modeling; I.6.8 [Simulation and Modeling]: Types of Simulation—animation

General Terms: Theory, Algorithms
Additional Key Words and Phrases: Fabric bending behavior, cloth modeling and animation, continuum mechanics, computer animation.

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Fig. 1. Buckling of fabric under axial forces at $\Delta t=1.25 \mathrm{~s}$ : (a) silk; (b) cotton; (c) jeans; and (d) sailcloth.

## 1. INTRODUCTION

Cloth modeling is of particular interest in several applications, ranging from the entertainment and advertisement purposes to the highly lucrative fashion business. Although a variety of strategies for the computer support of cloth modeling and animation has been rapidly evolved since the mid seventies, realistic cloth modeling is still a challenging problem both for the graphics and for the textile engineering communities. While the textile scientists look for a mechanical model that relates the dynamics cloth behavior to the material parameters, such as the ones produced by the cantilever beam test [ASTM 1987] or the Kawabata's Evaluation System [Kawabata 1980], the graphics researchers pursue an efficient numerically robust and stable model for animating clothes on virtual actors in motion. To our knowledge, the first work that attempts to create an enginnering model taking into account the graphics tools is owed to Breen et al. [Breen et al. 1994a; 1994b].

Fabrics are complex structures consisting of interwoven threads, which are themselves made of twisted fibers. The frictional (internal) forces between these fibers give the fabric a very peculiar physical behavior under applied (external) forces. It strongly resists to the length/area variations while is being very permissive to bending deformations, which may lead to a numerical stiff problem. Figure 1 exemplifies the shape that four distinct fabrics assume under axial forces.

Physically based models are recognized to be the most promising ones for producing natural appearance to the clothes in motion. They consider that the cloth dynamics are ruled by the partial differential equilibrium equation at each point $\mathbf{r}$ [Terzopoulos et al. 1987]

$$
\begin{equation*}
\mu \frac{\partial^{2} \mathbf{r}}{\partial t^{2}}+\varrho \frac{\partial \mathbf{r}}{\partial t}+\frac{\delta \mathcal{A}(\mathbf{r}, t)}{\delta \mathbf{r}}=\mu \frac{\partial^{2} \mathbf{r}}{\partial t^{2}}+\varrho \frac{\partial \mathbf{r}}{\partial t}+\mathbf{K}(\mathbf{r}, t) \mathbf{r}(t)=\mu \mathbf{F}(\mathbf{r}, t) \tag{1}
\end{equation*}
$$

[^1]where $\mu$ is the mass density (mass per unit area), $\varrho$ is the damping density (damping coefficient or time variation of mass per unit area) that reduces the kinetic energy, $\mathbf{F}$ denotes the total contribution of external forces per unit mass on $\mathbf{r}$ at time $t$, and the term $\frac{\delta \mathcal{A}(\mathbf{r}, t)}{\delta \mathbf{r}}$ corresponds to the internal energy per unit area governing the cloth's flexible appearance. The parameter $\mathbf{K}(\mathbf{r}, t)$ is usually called the cloth's stiffness. Eq. 1 may further be rewritten as the basic Newton's law if we consider the damping force as the external or the internal force [Provot 1995]
\[

$$
\begin{align*}
\mu \frac{\partial^{2} \mathbf{r}}{\partial t^{2}} & =-\mathbf{K}(\mathbf{r}, t) \mathbf{r}(t)+\left(\mu \mathbf{F}(\mathbf{r}, t)-\varrho \frac{\partial \mathbf{r}}{\partial t}\right) \\
& =-\left(\mathbf{K}(\mathbf{r}, t) \mathbf{r}(t)+\varrho \frac{\partial \mathbf{r}}{\partial t}\right)+\mu \mathbf{F}(\mathbf{r}, t) \tag{2}
\end{align*}
$$
\]

The stability and efficiency of a cloth simulation system may rely on the solution scheme for Eq. 1 or Eq. 2. The explicit integration methods reigned in '90s [Michel Carignan and Thalmann 1992; Provot 1995; Volino et al. 1995; Eberhardt et al. 1996]. Only at the end of 90s', Baraff and Witkin demonstrate in [Baraff and Witkin 1998] the superiority of implicit (backward) numerical integration scheme in comparison to the explicit (forward) ones. This is because that in an implicit scheme the new velocities $\dot{\mathbf{r}}(t+\Delta t)$ are computed in terms of the force conditions at $t+\Delta t$ instead $t$. For alleviating the time-consuming computation of a large non-symmetric sparse linear system, Baraff and Witkin also develop a modified conjugate gradient method. The results are so promising that, since then, a series of works have been devoted to improve the implicit framework [Desbrun et al. 1999; Kang et al. 2001; Choi and Ko 2002; Bridson et al. 2003].

Due to its typical geometry (high ratio of area to thickness), realistic cloth animation requires proper handling of its interactions with its environment (collision) as well as with itself (self-collision). Collision handling involves three extensively investigated issues: efficient collision detection [Benoit Lafleur and Thalmann 1991; van den Berger 1998; Bigliani and Eischen 2000; Teschner et al. 2005], cloth's response [Terzopoulos et al. 1987; Volino et al. 1995; Eischen et al. 1996; Volino and Magnenat-Thalmann 2000; Bridson et al. 2002; Choi and Ko 2002], and consistent post-collision behavior [Volino et al. 1995; Provot 1997; Baraff et al. 2003; Bridson et al. 2003; Ngoc 2004]. Nevertheless, simply employing implicit integration method or applying robust collision handling cannot overcome all instability problems [Choi and Ko 2002]. Because of its particular behavior (comparatively resistant to stretching and shearing - in-plane inextensibility, and permissive to bending -out-of-plane flexibility) the cloth's model itself deserves special attention.

There are essentially two approaches for modeling the microstructure of fabrics: the particle or mass-spring paradigm, in which a fabric is considered as a collection of material points held together by linear (or possibly nonlinear) structural, shear and flexion springs for simulating its material mechanics properties [Breen et al. 1992; Breen et al. 1994b; Provot 1995; Baraff and Witkin 1998; Desbrun et al. 1999; Choi and Ko 2002; Grispun et al. 2003; Bridson et al. 2003; Ngoc 2004; Ji et al. 2006]; and the continuum mechanics based technique, in which a fabric is regarded as a continuous media to which the nonlinear shell theory is applied for analyzing its stretching, shearing, and bending/flexural behavior [Feynman 1986; Terzopoulos et al. 1987; Chen and Govindaraj 1995; Eischen et al. 1996; Wang et al. 1998; Grispun et al. 2003; Hu 2004; Thomaszewiski et al. 2005]. It is worth remarking that, after spatial and time finite differentiations, the particle and the con-
tinuum approaches have similar ordinary differential formulations [Desbrun et al. 1999]. They differ essentially in the constitutive equations ${ }^{1}$, which are responsible for the internal force $\mathbf{K}(\mathbf{r}, t) \mathbf{r}(t)$ due to the cloth deformation.

In the particle or mass-spring approach, the internal force is modeled as the resultant of the tensions of the springs linking a point $\mathcal{P}_{i}$ to all its neighboring points $\mathcal{P}_{j}$. Whereas in the continuum mechanics one, the internal force is expressed as a function of the variation of $\mathcal{A}(\mathbf{r}, t)$ to the strain measures $\varepsilon$ and to the bending measures $\kappa$ :

$$
\begin{equation*}
\mathbf{K}(\mathbf{r}, t) \mathbf{r}(t)=\mu \frac{\delta \mathcal{A}(\mathbf{r}, t)}{\delta \mathbf{r}(t)}=\frac{\partial \mathcal{A}(\mathbf{r}, t)}{\partial \varepsilon(t)}+\frac{\partial \mathcal{A}(\mathbf{r}, t)}{\partial \kappa(t)} \tag{3}
\end{equation*}
$$

and the internal energy $\mathcal{A}(\mathbf{r}, t)$ assumes the following aspect [Terzopoulos et al. 1987]

$$
\begin{equation*}
\mu \mathcal{A}(\mathbf{r}, t)=\varepsilon(t)^{T} \mathbf{C}_{11} \varepsilon(t)+\kappa(t)^{T} \mathbf{C}_{22} \kappa(t) . \tag{4}
\end{equation*}
$$

The changes of the coefficients of the first fundamental form (Eq. 53) are universally accepted as strain measures, but it is still polemic the quantities to be used as bending measures [Mollmann 1981]. Moreover, there is no commonsense about the contribution of the coupling effects of strain and bending measures to the fabrics internal energy $\mathcal{A}$ [Amirbayat and Hearle 1989; Chen and Govindaraj 1995; Choi and Ko 2002; Bridson et al. 2003]. Section 2 provides an overview of the state-of-art.

The main concern of this paper is the cloth's bending modeling. Our motivation is twofold. First, we would like to get a cloth's model that can reproduce high levels of visual details without resorting to fictitious damping forces to subdue any oscillations. To achieve impressive results recent works have to resort to some cumbersome artifacts for correctly dealing with deformations under pure axial compression, such as damping forces to attenuate the oscillations caused by the stretching, shearing, and bending motions [Bridson et al. 2002; Baraff et al. 2003; Grispun et al. 2003], and distinct stretching and compression models [Choi and Ko 2002]. Instead, with help of the geometrically exact shell model - the theory of a Cosserat surface [Green et al. 1965], we show in Section 3 how to get around this difficulty and present a readily implementable solution in Section 5.

Applying the general Cosserat's shell theory to model fabrics is not a novelty. Simo and Fox have already published a series of three papers [Simo and Fox 1989a; 1989b; 1989c] to demonstrate that, despite its awful formulation, a classical shell theory is conducive to an efficient numerical implementation. The key point for their finding is a new parametrization that avoids the terms such as the Christoffel symbols and the coefficients of the second fundamental form. The price that they pay is to adopt relations that do not explicitly associate shape quantities with the textile mechanics ones. The main contribution of our work is to demonstrate that it is feasible to implement the classical Cosserat surface without disregarding the explicit relations between the geometrical and mechanical properties. A better balance between the model's usability and its implementational suitability may, thus, be achieved.

Second, we would like to get a more accurate expression for calculating the surface normal at the sample points. It is a continuation of our previous efforts in improving the implementation of a physically based deformation model [Horta and Wu 1995; Ramos and Wu 1997; Wu and de Melo 2003]. A variety of approximations to normal vectors have

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already been proposed, such as [Meyer et al. 2003; Wu and de Melo 2003; Agam and Tang 2005]. Because of availability, we depart from the formulation proposed by [Wu and de Melo 2003], which is based on the Gauss-Weingarten relations between the first and the second fundamental forms (Eq. 62) and which requires that the points are not planar. Instead of setting to zero the normal of locally planar points, we further propose to compute those vectors from Eq. 58. We also average the normals determined in the three directions from Eq. 62 to cope with surface variations along distinct directions in the vicinity of a point. In Section 4 we will detail our proposal. Results presented in Section 6 demonstrate that, with this approximation, the surface regularity tends to be preserved and the deformation dynamics becomes much more stable.

An explicit integration scheme was employed to implement our proposed model. Spatial and time discretization details are provided in Section 5. In Section 6 we show results obtained with our proposal. Qualitative visual comparisons of the simulation results with real behavior of some cloth are provided. We furnish the parameters that we used for each simulation to show that the proposed model is, at least, appropriate for direct control on the fabrics behavior along distinct directions. Running times are also provided to hint our simulator's performance. Finally, we conclude this paper by providing in Section 7 some further work directions.

To be self-contained, we also provide in Appendices A and B, respectively, the notations of Tensor Calculus and some fundamental concepts of Differential Geometry. Throughout the paper, with the exception of Section 5, Latin indices will have the range $1,2,3$ whereas Greek indices with the range 1,2 are used for components of space tensor or components of surface tensor.

## 2. RELATED WORK

It is a fact that the bending measures are crucial for enhancing the realism in the appearance of simulated clothes. The questions that researchers have been worked aroud are which quantities can be used as bending measures and how to appropriately formulate them for finite difference or finite element implementations. To not be extensive, we will refer in this section some pioneering works that allow us to illustrate the efforts in the last two decades.

Several authors have used the coefficients of the second fundamental form (Eq. 57) as bending measures. Expressing these coefficients in terms of the unknown partial derivatives of the position vector $\mathbf{r}$ leads to a form that is not readily solvable by the existing numerical techniques. Physically accurate visual results provided by them have, nevertheless, motivated several works focusing on a suitable formulation for them.

Assuming that the Christoffel symbols ( $\Gamma_{\alpha \beta}^{\lambda}$ ) are negligible and $b_{\alpha \beta}$ (the coefficients of the second fundamental form) are equal to 1 in Eq. 62, Terzopoulos et al. [Terzopoulos et al. 1987] approximate the normal vectors at time $t$ that appear in Eq 57 to the second derivatives of the position vector $\mathbf{r}$ with respect to the coordinate curves $x^{\alpha}$ and $x^{\beta}$

$$
\mathbf{n}(t) \approx \frac{\partial^{2} \mathbf{r}(t)}{\partial x^{\alpha} \partial x^{\beta}}
$$

The coefficients are, then, reduced to the second partial derivatives. This approximation may distort not only the magnitude but also the direction and the sense of bending deformations, since the directions of the second derivatives do not necessary coincide with the direction of the normal vector. Therefore, unexpected stretching and bending effects might be yielded [Wu and de Melo 2003]. For diminishing such effects, fictitious damping forces
have been introduced [Michel Carignan and Thalmann 1992]. Chen and Govindaraj use an equivalent expression to Eq. 57 for representing bending measures [Chen and Govindaraj 1995]

$$
b_{\alpha \beta}(t)=\frac{-1}{2}\left(\frac{\partial \mathbf{n}(t)}{\partial x^{\alpha}} \frac{\partial \mathbf{r}(t)}{\partial x^{\beta}}+\frac{\partial \mathbf{n}(t)}{\partial x^{\beta}} \frac{\partial \mathbf{r}(t)}{\partial x^{\alpha}}\right)
$$

It is not explicit in their paper how they get the normal vectors. From their statement that $\mathbf{n}$ is not exactly normal to the surface in the current configuration, we may only infer that they use the normals of $t-\Delta t$ to compose the stiffness matrix $\mathbf{K}(\mathbf{r}, t)$. Wang et al. enhance in [Wang et al. 1998] the bending quantities provided by Terzopoulos, but they do not explain how they determine the normal vectors. Again, from their presentation, we believe that they adopt the same solution as Chen and Govindaraj.

The computation of the normal vectors from the position vectors of the sample points at $t-\Delta t$ is still not straightforward when only a polygonal approximation of the surface is given. The most known way for determining the surface normal at each point of a polygonal mesh is to average the normal vectors of all of its adjacent faces. This coarse estimation may lead to a physically unrealizable surface and, thus, to unstable numerical behaviors. Wu and Melo [Wu and de Melo 2003] propose to use Eq. 62 for expressing the normal vectors in the component term $\frac{\delta \mathcal{A}(\mathbf{r}, t)}{\delta \mathbf{r}}$ of Eq. 1. More stable and visually plausible results have been achieved when they apply a semi-implicit integration scheme in their implementation. In this work, we show that even for an explicit integration scheme the approximation paradigm proposed by Wu and Melo delivers better results. This is, however, still not enough to get convincing visual effects when a cloth is subjected to compressing forces, such as bucklings and creases. We conjecture that the root of the problem might be the bending model itself.

Based on the work by Simo and Foxet al. [Simo and Fox 1989a; 1989b; 1989c], Eischen et al. [Eischen et al. 1996] present a cloth model founded on a Cosserat surface, originally proposed by the Cosserats in 1909, rediscovered during the 50 s for oriented bodies modeling [Ericksen and Truesdell 1958] and, later, for shell modeling [Green et al. 1965]. A Cosserat surface is a surface embedded in $\mathbf{R}^{3}$ to which an out-of-plane vector $\mathbf{d}$, called a director, is assigned to every point. A brief overview of the general theory of an elastic Cosserat surface is given in Section 3.1. In this work we will show that our proposed cloth's model is a special case of Cosserat surfaces, which is considered a "geometrically exact" shell model.

The objective of Simo and Fox is to provide a computational feasible framework for a Cosserat surface. For bypassing the coefficients of the second fundamental form in a finite element implementation, they consider the changes of the position vectors $\mathbf{r}$ and the directors $\mathbf{d}$ as two independent unkowns and propose to measure the bending behaviors of an inextensible one-director Cosserat surface with the following quantities:

Bending strains $\kappa_{\alpha \beta}(t)$.

$$
\begin{equation*}
\kappa_{\alpha \beta}(t)=\left(\frac{\partial \mathbf{r}(t)}{\partial x^{\alpha}} \cdot \frac{\partial \mathbf{d}(t)}{\partial x^{\beta}}-\frac{\partial \mathbf{R}}{\partial x^{\alpha}} \cdot \frac{\partial \mathbf{D}}{\partial x^{\beta}}\right) \tag{5}
\end{equation*}
$$

Transverse shear strains $\gamma_{\alpha}(t)$.

$$
\begin{equation*}
\gamma_{\alpha}(t)=\left(\frac{\partial \mathbf{r}(t)}{\partial x^{\alpha}} \cdot \mathbf{d}(t)-\frac{\partial \mathbf{R}}{\partial x^{\alpha}} \cdot \mathbf{D}\right) \tag{6}
\end{equation*}
$$

where $\mathbf{R}$ and $\mathbf{D}$ are, respectively, the position vectors and directors of the reference configuration. Figure 2 illustrates the relation of all of these quantities. Nevertheless, they convey


Fig. 2. Relations of $\mathbf{r}$ and $\mathbf{d}$.
the problem to their model's user, who must suitably assign the directors for obtaining plausible visual results. Instead, we model the cloth as a special case of Cosserat surfaces, for which $\mathbf{d}$ is along the normal to the surface at each time step. The reference system associated to $r$ varies, therefore, with the changes of their derivatives at time $t$. Although this leads to more complex formulation envolving Chistoffel symbols (responsible for connecting the local and the global reference systems) and the coefficients of the second fundamental form, the model is much more tractable from anS application's standpoint. As already stated, this is because that it keeps explicitly the relation of the physical (textile) material properties and the easily interpretable geometrical quantities, as detailed in Section 3.

Breen et al. [Breen et al. 1992] depart completely from the differential geometry paradigm and propose an angular expression for the bending measures. They observe that a single thread can bend "out-of-plane" around crossing threads and describe this phenomenon by modeling the angle formed between each set of three adjacent crossing nodes (or particles) in a retangular mesh. More accurate control on the bending shape on the basis of the angle and the radius of curvature is later proposed by Volino et al. [Volino et al. 1995]. Describing the behavior of particles displacements and particles rotations independently of each other without recognizing that these quantities must be compatible is the main flaw of this paradigm. To improve the realism, Provot introduces in [Provot 1995] the flexion springs to implicitly control the angular variations and formulates the constitutive equations solely in terms of $\mathbf{r}$ and its derivatives. Differently from the differential geometry approach, it is not ensured that the angles can actually reflect the shape of the surface in the vicinity of $\mathbf{r}$ and, therefore, the accuracy of the internal force response. To attenuate unrealistic residues due to the deviated force directions, Provot also proposes to define damping forces for dissipating them. The results are so convincing that Baraff and Witkin reformulate the angular expression in terms of the dihedral angles [Baraff and Witkin 1998]. Detailed derivation of their expression may be found in [Bridson et al. 2003]. As the previous angular based algorithms, they still need the fictitious damping forces to attenuate the unrealistic residual stretching, shearing, and bending forces for avoiding undesired oscillations.

The problem regarding with the residual forces has been carefully analyzed by Choi and

Ko [Choi and Ko 2002] who conclude that it stems from the fact that the existing bending model cannot appropriately deal with the cloth's behavior under compression. As a solution, they propose to separately treat the "distension" (type 1 interaction) and the "compression" (type 2 interaction) cases, and provide a way to compute the bending energy in terms of the initial arc length, the distances between the particles (which may be smaller than the initial arc length), and the bending stiffness. In essence, they turn to the point that Chen and Govindaraj have already emphasized in [Chen and Govindaraj 1995]: the link between the strain and bending measures are crucial in cloth's modeling. From the theory of thin shells, this link may be represented by the product of the (current) strain and (current) bending measures [Mollmann 1981]. For avoiding complex expression, the initial bending measures are instead used in [Chen and Govindaraj 1995].

From Eq. 60 one may observe that the mean curvature may be expressed in terms of the products of stretching and bending quantities. Hence, the mean curvature may also be used as the coupling quantities. It is, therefore, not surprising that Grinspun et al. achieve very impressive effects of crease and crumple by simply using the mean curvature as the bending measure [Grispun et al. 2003]. The main flaw of their algorithm is that, due to their angular approach, it inherits the residual force problems. In addition, it is not clear in their proposal how they distinguish for example the bending effects in a sail fabric from the ones in a silk textile material, without resorting to the Gaussian curvature (Eq. 61). Both fabrics have the same "creasing" or "buckling" behavior. Nevertheless, in the former, the shape in the vicinity of each particle tends to be preserved - high resistance to the Gaussian curvature (Figure 1.(d)), and in the latter, wrinkles are easily formed - low resistance to the Gaussian curvature (Figure 1.(a)). The theory of a Cosserat surface, on which our work is based, provides a uniform framework for modeling the both classes of bendings. In this work, we also present in Section 5 a computational framework for implementing our proposed model.

## 3. A BENDING MODEL

We present in this section a bending model with emphasis on an intuitive interface and realistic cloth's appearance. Two issues should, then, be addressed:
(1) "what you control is what you get" paradigm: the applied forces and the changes in the surface's shape should be directly related, and
(2) bending representativeness: a variety of cloth's bending behaviors should be distinguishable and reproduceable.
The first issue suggests us to express the strain and bending measures in terms of the coefficients of the first and the second forms, as proposed in [Terzopoulos et al. 1987]

$$
\begin{aligned}
& \varepsilon_{\alpha \beta} \propto\left(a_{\alpha \beta}-A_{\alpha \beta}\right) \\
& \kappa_{\alpha \beta} \propto\left(b_{\alpha \beta}-B_{\alpha \beta}\right),
\end{aligned}
$$

where $a_{\alpha \beta}$ are coefficients of the first fundamental forms and $b_{\alpha \beta}$, the coefficients of the second form at the current time step. Their respective values of the initial configuration are $A_{\alpha \beta}$ and $B_{\alpha \beta}$. Our option is justified by the fact that the first fundamental form (Eq. 53) provides us direct access to the metric measures of a surface, such as curve length, angles of tangent vectors and areas, without further reference to the ambient space, whereas the second fundamental form (Eq. 59) gives us elements to quantitate the shape of surface in the neighborhood of a point, or how far the surface is from a plane.

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Fig. 3. Geometrical quantities in a vicinity of a point $\mathcal{P}$.
For the second issue we also meet the solution in Differential Geometry. There are two important concepts related to the shape of a surface in the vicinity of a point: the intrinsic and the extrinsic geometries. An instrinsic geometric property is the one that may be measured without leaving the surface; while an extrinsic one can only be perceived by an observer located in the ambient space. Examples for intrinsic properties of a surface are the coefficients of the first fundamental form, the surface's area, the length of a curve on the surface, and the Gaussain curvature. For exemplifying extrinsic characteristics we may mention the coefficients of the second fundamental form and the mean curvature (Appendix B).

The intrinsic properties of most of inextensible fabrics, such as linen, cotton and jeans, are almost invariant while they deform. To distinguish the shape states, that are indistinguishable by the intrinsic properties such as buckling, we should use an extrinsic quantity. The mean curvature is an extrinsic measure and Eq. 60 says that it involves the products of the coefficients of the first and the second fundamental form. It is, consequently, opportune to add to Eq. 4 one more term to explicitly represent the part of internal energy that corresponds to the change of the area enclosed by a curve $\mathcal{C}$ in a vicinity of a point $\mathcal{P}$ (Figure 3). Eq. 4, then, becomes
$\mu \mathcal{A}(\mathbf{r}, t)=\sum_{\alpha, \beta=1}^{2}\left[\Phi^{\alpha \beta}(t) \varepsilon_{\alpha \beta}(t) \varepsilon_{\alpha \beta}(t)+\Psi^{\alpha \beta}(t) \kappa_{\alpha \beta}(t) \kappa_{\alpha \beta}(t)+\Theta^{\alpha \beta}(t) \varepsilon_{\alpha \beta}(t) \kappa_{\alpha \beta}(t)\right]$.
The other two terms $\Phi^{\alpha \beta}(t) \varepsilon_{\alpha \beta}(t) \varepsilon_{\alpha \beta}(t)$ and $\Psi^{\alpha \beta}(t) \kappa_{\alpha \beta}(t) \kappa_{\alpha \beta}(t)$ in Eq. 7 correspond, respectively, to the energy that causes the variation of the curve length and the angle $\theta$ between the curves that cover the deforming surface and to the energy that leads to the variations of the bending radius $R$ along any curve through $\mathcal{P}$ on the surface, which might be roughly related to the Gaussian curvature given in Eq. 61. Figure 4 illustrates visually their contributions to the shape of a deforming silk satin fabric under five pairs of opposite axial forces


Fig. 4. The appearence of the silk satin fabric under axial forces (a) without and (b) with coupling term.
with magnitude equal to 10 N . Despite lower resistance to curving, observe that, without the coupling term, the surface tends to preserve its local planarity (Figure 4.(a)). When we superposed the coupling term, bucklings have been naturally formed (Figure 4.(b)).

Now, two questions must be solved to make our proposal usable. First, how close our proposed model is to a physically accurate shell model, whose geometrical and physical quantities can be assumed with sufficient accuracy to be functions of only two independent variables. In other words, how representative is our model. If we assume that

$$
\begin{align*}
& \varepsilon_{\alpha \beta}=\frac{1}{2}\left(a_{\alpha \beta}-A_{\alpha \beta}\right)  \tag{8}\\
& \kappa_{\alpha \beta}=-\left(b_{\alpha \beta}-B_{\alpha \beta}\right) \tag{9}
\end{align*}
$$

it is interesting to observe the similarities between Eq. 8 and Eq. 12; Eq. 9 and Eq. 13; Eq. 2 and Eq. 18, which is derived on the basis of the Cosserat's theory. Further, Eq. 7 and Eq. 28, that represents the energy stored in a "geometrically exact" shell possessing holohedral isotropy, are almost algebraically identical. Only the component term $\mathcal{A}^{*}\left(\kappa_{3 \alpha}, \mathcal{T}\right)$ of the latter is neglected. Therefore, it is expected that our model is appropriate to reproduce with realism not only a large range of fabrics but also elastic (Figure 5.(a)) or rigid (very) thin material (Figure 5.(b)). In the both figures, the thin flat square sheet are hanging at a (top) corner.


Fig. 5. Visual effects of (a) melting cheese and (b) metallic plate under the gravity force.

Second, how the proposed model can be implemented in a computational framework. The simplicity of Eq 18, Eq. 8 and Eq. 9 does not reveal the complexity of its component equations. While Eq. 8 requires the first derivatives, Eq. 9 may involve the products of normal vectors and second derivatives. To make it worser, the internal force is a covariant differentiation of the contravariant vector $\mathbf{N}^{\alpha}$ with respect to the coordinate curve $x^{\alpha}$. From our standpoint, the main task that we undertook is to expand each equation to make it readily processable by a computer, as we detail in the subsequent sections. A brief introduction to the basic idea of the theory of a Cosserat surface precedes in Section 3.1.

For the sake of conciseness, we adopt in the remaining of exposition the following notation

$$
\mathbf{a}_{, \alpha}=\frac{\partial \mathbf{a}}{\partial x^{\alpha}}
$$

and the summation convention which consists in omitting the sign $\sum$. If in a product a Greek letter figures twice, once as superscript and once as subscript, summation must be performed from 1 to 2 with respect to this letter, and if a Latin letter appears, summation must be carried out from 1 to 3 .

### 3.1 An inextensible normal-director elastic Cosserat Surface

In this section only formulas that we use in our work are transcribed in order to make clear our real contribution. We refer to the detailed explanation of Green et al. in [Green et al. 1965] as further reading.


Fig. 6. A moving trihedron.

Let $\mathcal{S}(t)=\mathbf{r}\left(x^{1}, x^{2}, t\right)$ be the elastically deformable surface at time $t$, whose metric and curvature tensors are denoted by $a_{\alpha \beta}(t)$ and $b_{\alpha \beta}(t)$, respectively. Particularly, at $t=t_{0}$ we refer the undeformed surface by $\mathcal{S}\left(t_{0}\right)=\mathbf{R}\left(x^{1}, x^{2}\right)=\mathbf{r}\left(x^{1}, x^{2}, t_{0}\right)$ and its metric and curvature tensors by $A_{\alpha \beta}=a_{\alpha \beta}\left(t_{0}\right)$ and $B_{\alpha \beta}=b_{\alpha \beta}\left(t_{0}\right)$. Also, let $x^{1}$ - and $x^{2}$-curves be the coordinate curves lie on $\mathcal{S}$ and $x^{3}$ be along the normal to $\mathcal{S}(t)$. The $x^{i}$ are identified as convected coordinates because any point on $\mathcal{S}$ has the same curvilinear coordinates in the reference state and in the deformed state ${ }^{62}$. Additionally, the first derivatives along the $x^{\alpha}$-curves

$$
\begin{equation*}
\mathbf{a}_{\alpha}(t)=\frac{\partial \mathbf{r}}{\partial x^{\alpha}}(t) \tag{10}
\end{equation*}
$$

[^3]

Fig. 7. A moving Cosserat surface.
and the unit normal to $\mathcal{S}(t)$

$$
\begin{equation*}
\mathbf{n}(t)=\mathbf{a}_{3}(t)=\frac{\frac{\partial \mathbf{r}}{\partial x^{1}}(t) \times \frac{\partial \mathbf{r}}{\partial x^{2}}(t)}{\left|\frac{\partial \mathbf{r}}{\partial x^{1}}(t) \times \frac{\partial \mathbf{r}}{\partial x^{2}}(t)\right|} \tag{11}
\end{equation*}
$$

are linearly independent and build the base vectors of a moving trihedron, of which $\frac{\partial \mathbf{r}}{\partial x^{1}}$ and $\frac{\partial \mathbf{r}}{\partial x^{2}}$ lie in the tangent plane normal to $\mathbf{n}$. Further, let $\mathbf{D}$ be a vector, pointing outwards the surface and not necessarily along its normal, assigned to every point of $\mathcal{S}\left(t_{0}\right)$ and $\mathbf{d}(t)$ be its dual at time $t$ (Figure 6).

Let $\mathbf{a}^{i}(t)$ be the reciprocal vector of $\mathbf{a}_{i}(t)$, whose relations are given by Eq. 49. The motion of a Cosserat surface is defined by

$$
\mathbf{r}(t) \equiv \mathbf{r}\left(x^{1}, x^{2}, t\right), \quad \mathbf{d}(t) \equiv \mathbf{d}\left(x^{1}, x^{2}, t\right)
$$

under the restriction that the vector $\mathbf{d}(t)$ must satisfy

$$
\mathbf{d}(t)=d_{i} \mathbf{a}^{i}(t)=d_{\alpha} \mathbf{a}^{\alpha}+d_{3} \mathbf{a}^{3}
$$

and its components, relative to $\mathbf{a}^{i}(t)$, remain invariant when the motion is only due to the superposed rigid transformations. Figure 7 illustrates the change of the shape $\mathcal{S}\left(t^{*}\right)$ to the shape $\mathcal{S}\left(t^{* *}\right)$ by moving each point $\mathbf{r}$.
The directors of a Cosserat surface must not necessarily be along the normal vectors, neither to be unit. When the directors coincide with the (unit) normal vectors at any $t$, that is $\mathbf{d}(t)=\mathbf{a}_{3}(t)$, we call it an inextensible normal-director Cosserat surface. We apply this surface to model clothes.

For analyzing the continuous deformation that $\mathcal{S}$ suffers, the metric tensor (Eq. 54), the director displacement and its derivatives may be used for defining the kinematic variables (strains) that are invariant under the rigid transformations. In the case of an inextensible normal-director Cosserat surface, these derivatives may be expressed in terms of the coefficients of the first (Eq. 53) and the second fundamental form (Eq. 59):

Membrane strains $\varepsilon_{\alpha \beta}$. that measure the amount that $\mathcal{S}$ stretches or compresses

$$
\begin{equation*}
\varepsilon_{\alpha \beta}=\frac{1}{2}\left(a_{\alpha \beta}-A_{\alpha \beta}\right)=\frac{1}{2}\left(\mathbf{r}_{, \alpha} \cdot \mathbf{r}_{, \beta}-\mathbf{R}_{, \alpha} \cdot \mathbf{R}_{, \beta}\right) \tag{12}
\end{equation*}
$$

Bending strains $\kappa_{\beta i}$. that measure the amount that $\mathcal{S}$ bends or twists

$$
\begin{align*}
\kappa_{\beta \alpha} & \left.=\left(\mathbf{r}_{, \beta} \cdot \mathbf{d}_{, \alpha}-\mathbf{R}_{\beta} \cdot \mathbf{D}_{, \alpha}\right)\right)=-\left(b_{\beta \alpha}-B_{\beta \alpha}\right) \\
\kappa_{\beta 3} & =0 \tag{13}
\end{align*}
$$



Fig. 8. Internal forces.

Director or transverse shear strains $\gamma_{i}$. that measure the rotations and distensions of the directors. In the case of an inextensible normal-director Cosserat surface, $\mathbf{d}=\mathbf{D}$. Thus,

$$
\begin{equation*}
\gamma_{i}=d_{i}-D_{i}=0 \tag{14}
\end{equation*}
$$

The strains are results of the internal forces due to the action of the proper surface or the external loading forces. We proceed to characterize the internal forces, or stresses, of $\mathcal{S}$.

Let $\sigma$, the area of an inextensible normal-director Cosserat surface $\mathcal{S}$ at time $t$, be bounded by a closed curve $\mathcal{C}$. If we cut $\mathcal{S}$ along the curve $\mathcal{C}$ and designate one side of $\mathcal{S}$ as positive and the other side as negative, the portion on the positive side exerts a force on the negative part of $\mathcal{S}$. This force per unit length is transmitted through an incremental length $d s$ of $\mathcal{C}$ by direct contact of the two portions of $\mathcal{S}$. Let the vector $\nu=\nu_{\alpha} \mathbf{a}^{\alpha}$, lying in the surface and perpendicular to $\mathcal{C}$, be the exterior normal vector. If for all arbitrary velocity fields $\mathbf{v}$, there is a three-dimensional force field $\mathbf{N}=\mathbf{N}^{i} \mathbf{a}_{i}$, such that the scalar $\mathbf{N} \cdot \mathbf{v}$ represents a rate of work per unit length $d s$ of $\mathcal{C}$, then $\mathbf{N}$ is a curve force vector measured per unit length. Similarly, if $\mathbf{M}=\mathbf{M}^{i} \mathbf{a}_{i}$ is a three-dimensional vector field and if, for all arbitrary director velocity fields $\mathbf{w}$, the scalar $\mathbf{M} \cdot \mathbf{w}$ represents a rate of work per unit length of $\mathcal{C}$, then $\mathbf{M}$ is a director force vector (momentum) measured per unit length (Figure 8). They may be expressed in terms of the base vectors $\mathbf{a}_{i}$

$$
\begin{gather*}
\mathbf{N}=N^{i} \mathbf{a}_{i}=\nu_{\alpha} \mathbf{N}^{\alpha}=\left(\nu_{\alpha} N^{i \alpha}\right) \mathbf{a}_{i}  \tag{15}\\
\mathbf{M}=M^{i} \mathbf{a}_{i}=\nu_{\alpha} \mathbf{M}^{\alpha}=\left(\nu_{\alpha} M^{i \alpha}\right) \mathbf{a}_{i} \tag{16}
\end{gather*}
$$

where $N^{\beta \alpha}, N^{3 \alpha}, M^{\beta \alpha}$ and $M^{3 \alpha}$ are the surface tensors under transformation of surface coordinates, and $\mathbf{N}^{\alpha}$ and $\mathbf{M}^{\alpha}$ are curve force vector and director force vector over the $x^{\alpha}$-curve, respectively.

Also, let $\mathbf{F}=F^{i} \mathbf{a}_{i}$ and $\mathbf{L}=L^{i} \mathbf{a}_{i}$ be the force fields per unit mass of $\mathcal{S}$ at time $t$, such that $\mathbf{F} \cdot \mathbf{v}$ and $\mathbf{L} \cdot \mathbf{w}$ represent rate of work per unit area of $\mathcal{S}$ for all arbitrary $\mathbf{v}$ and $\mathbf{w}$, respectively.

If $\mu$ is the mass density at time $t$ per unit area of $\mathcal{S}$ and $\mathcal{U}$ is the internal energy per unit mass, Green et al. show in [Green et al. 1965] that the equation of balance of energy may
be expressed as

$$
\begin{align*}
& \int_{\sigma}[\mu \mathbf{v} \cdot \dot{\mathbf{v}}+\mu \dot{\mathcal{U}}] d \sigma+\int_{\sigma}\left[\frac{1}{2} \mathbf{v} \cdot \mathbf{v}+\mathcal{U}\right]\left[\dot{\mu}+\mu \sum_{\alpha}\left(v_{\mid \alpha}^{\alpha}-b_{\alpha}^{\alpha} v^{3}\right)\right] d \sigma \\
& =\int_{\sigma} \mu\left[h_{\text {supply }}+\mathbf{F} \cdot \mathbf{v}+\overline{\mathbf{L}} \cdot \mathbf{w}\right] d \sigma+\int_{c}\left[\mathbf{N} \cdot \mathbf{v}+\mathbf{M} \cdot \mathbf{w}-h_{f l u x}\right] d c \tag{17}
\end{align*}
$$

where $h_{\text {supply }}$ is the heat supply function per unit mass per unit time, $h_{\text {flux }}$ is the flux of heat across $c$ per unit length per unit time and $\overline{\mathbf{L}}$ is the difference of the assigned director force per unit mass $\mathbf{L}$ and the inertia terms due to the director displacement $\mathbf{d}$. From Eq. 17, with the assumption that the state of $\mathcal{S}$ remains unchanged under superposed uniform rigid body translational velocities, the following conservation equation for the linear momentum is derived

$$
\begin{equation*}
\mu \dot{\mathbf{v}}-\mathbf{N}^{\alpha}{ }_{\mid \alpha}=\mu \mathbf{F}(\mathbf{r}, t) \tag{18}
\end{equation*}
$$

and for the angular momentum,

$$
\begin{equation*}
\mathbf{m}=\mathbf{M}^{\alpha}{ }_{\mid \alpha}+\mu \overline{\mathbf{L}}(\mathbf{r}, t), \tag{19}
\end{equation*}
$$

where $\dot{\mathbf{v}}=\frac{\partial^{2} \mathbf{r}}{\partial t^{2}}$, with $\mathbf{v}=\frac{\partial \mathbf{r}}{\partial t}$, and $\mathbf{m}$ is the director force vector.
For implementation purpose, it is convenient to write Eq. 18 and Eq. 19 in the tensor components form. This may be achieved by performing the scalar product of them with $\mathbf{a}^{\beta}$ and again with $\mathbf{a}_{3}$

$$
\begin{align*}
N^{\beta \alpha}{ }_{\mid \alpha}-b_{\alpha}^{\beta} N^{3 \alpha}+\mu F^{\beta} & =\mu \dot{v}^{\beta}  \tag{20}\\
N^{3 \alpha}{ }_{\mid \alpha}+b_{\alpha \beta} N^{\beta \alpha}+\mu F^{3} & =\mu \dot{v}^{3}  \tag{21}\\
M^{\beta \alpha}{ }_{\mid \alpha}-b_{\alpha}^{\beta} M^{3 \alpha}+\mu \bar{L}^{\beta} & =m^{\beta}  \tag{22}\\
M^{3 \alpha}{ }_{\mid \alpha}+b_{\alpha \beta} M^{3 \alpha}+\mu \bar{L}^{3} & =m^{3} \tag{23}
\end{align*}
$$

We use the following expression deduced by Green et al.

$$
\begin{equation*}
N^{3 \alpha}=M_{\mid \beta}^{\alpha \beta}+\mu \bar{L}^{\alpha} \tag{24}
\end{equation*}
$$

Considering $\mathcal{A}$ the Helmholtz free energy function per unit mass, Green et al. also show the constitutive equations that an inextensible normal-director elastic Cosserat surface, at constant temperature and entropy, must hold for all time $t$

$$
\begin{align*}
N^{* \alpha \beta} & =\mu \frac{\partial \mathcal{A}}{\partial \varepsilon_{\alpha \beta}} \\
M^{i \alpha} & =\mu \frac{\partial \mathcal{A}}{\partial \kappa_{i \alpha}} \tag{25}
\end{align*}
$$

with

$$
\begin{equation*}
N^{* \alpha \beta}=N^{\alpha \beta}+b_{\lambda}^{\alpha} M^{\beta \lambda} \tag{26}
\end{equation*}
$$

that is

$$
\begin{equation*}
N^{\alpha \beta}=N^{* \alpha \beta}-b_{\lambda}^{\alpha} M^{\beta \lambda} \tag{27}
\end{equation*}
$$

If the surface $\mathcal{S}$ is initially homogeneous, free from curve and director forces, and the deformations undergo at constant temperature and entropy, then an approximation to the
internal energy $\mathcal{A}$ may be expressed in terms of the geometric properties of the Cosserat surface, $\varepsilon_{\alpha \beta}$ and $\kappa_{\alpha \beta}$, and the parameters that characterize its material properties, $H^{\alpha \beta \lambda \rho}$, $B^{\alpha \beta \lambda \rho}$ and $C^{\alpha \beta \lambda \rho}$

$$
\begin{equation*}
\mu_{0} \mathcal{A}=\left[H^{\alpha \beta \lambda \rho} \varepsilon_{\alpha \beta} \varepsilon_{\lambda \rho}+B^{\alpha \beta \lambda \rho} \kappa_{\alpha \beta} \kappa_{\lambda \rho}+C^{\alpha \beta \lambda \rho} \varepsilon_{\alpha \beta} \kappa_{\lambda \rho}\right]+\mathcal{A}^{*}\left(\kappa_{3 \alpha}, \mathcal{T}\right) \tag{28}
\end{equation*}
$$

where $\mathcal{A}^{*}\left(\kappa_{3 \alpha}, \mathcal{T}\right)$ is the energy that involves the terms $\kappa_{3 \alpha}, \gamma_{i}$ and $\mathcal{T}$, which may be neglected in most of cases. The first and second terms on the right-hand side of Eq. 28 are the quadratic forms of the strain and bending measures, respectively, while the third term, containing products of strain and bending measures, represents a coupling of stretching and bending effects.

Furthermore, some of the material parameters in Eq. 28 satisfy certain symmetry conditions, if the elastic Cosserat surface possesses holohedral isotropy

$$
\begin{align*}
H^{\alpha \beta \lambda \rho} & =H^{\beta \alpha \lambda \rho}=H^{\alpha \beta \rho \lambda}=H^{\lambda \rho \alpha \beta} \\
& =\beta_{1} A^{\alpha \beta} A^{\lambda \rho}+\beta_{2}\left(A^{\alpha \lambda} A^{\beta \rho}+A^{\alpha \rho} A^{\beta \lambda}\right) \\
B^{\alpha \beta \lambda \rho} & =B^{\beta \alpha \lambda \rho}=B^{\alpha \beta \rho \lambda}=B^{\lambda \rho \alpha \beta} \\
& =\beta_{3} A^{\alpha \beta} A^{\lambda \rho}+\beta_{4}\left(A^{\alpha \lambda} A^{\beta \rho}+A^{\alpha \rho} A^{\beta \lambda}\right) \\
C^{\alpha \beta \lambda \rho} & =C^{\beta \alpha \lambda \rho}=C^{\alpha \beta \rho \lambda}=C^{\lambda \rho \alpha \beta} \\
& =\beta_{5} A^{\alpha \beta} A^{\lambda \rho}+\beta_{6}\left(A^{\alpha \lambda} A^{\beta \rho}+A^{\alpha \rho} A^{\beta \lambda}\right) \tag{29}
\end{align*}
$$

where $\beta_{i}$ are the elasticity constants. This is because that the holohedral isotropic materials are formed of crystalline substances having all faces symmetric.

### 3.2 Internal Force $\mathbf{N}^{\alpha}{ }_{\mid \alpha}$

The most important term in Eq. 18 is the covariant differentiation of $\mathbf{N}^{\alpha}$ with respect to $x^{\alpha}$. This term represents the fabric's internal force at a point $\mathcal{P}$. According to the shell theory, the variation of the length at a point $\mathcal{P}$ along a curve $\mathcal{C}$ is due to the internal force

$$
\begin{align*}
\mathbf{N}^{\alpha}{ }_{\mid \alpha} & \equiv\left(\mathbf{N}_{, \alpha}^{\alpha}+\Gamma_{\beta \alpha}^{\alpha} \mathbf{N}^{\beta}\right)=\left(\mathbf{N}_{, \alpha}^{\alpha}+\Gamma_{\alpha \lambda}^{\lambda} \mathbf{N}^{\alpha}\right) \\
& \equiv\left[\left(N^{i \alpha} \mathbf{a}_{i}\right)_{, \alpha}+\Gamma_{\alpha \lambda}^{\lambda} N^{i \alpha} \mathbf{a}_{i}\right], \tag{30}
\end{align*}
$$

where

$$
\mathbf{N}^{\alpha} \equiv N^{\beta \alpha} \mathbf{a}_{\beta}+N^{3 \alpha} \mathbf{a}_{3} .
$$

Considering $\overline{\mathbf{L}} \approx 0$ in Eq. 24, we have

$$
N^{3 \alpha}=M^{\alpha \beta}{ }_{\mid \beta} .
$$

By substituting it in Eq. 30, we get

$$
\begin{align*}
\mathbf{N}^{\alpha}{ }_{\mid \alpha} & \equiv\left[\left(N^{\beta \alpha} \mathbf{a}_{\beta}\right)_{, \alpha}+\Gamma_{\alpha \lambda}^{\lambda} N^{\beta \alpha} \mathbf{a}_{\beta}\right] \\
& +\left[\left(M^{\alpha \beta}{ }_{\mid \beta} \mathbf{a}_{3}\right)_{, \alpha}+\Gamma_{\alpha \lambda}^{\lambda} M^{\alpha \beta}{ }_{\mid \beta} \mathbf{a}_{3}\right] . \tag{31}
\end{align*}
$$

The covariant differentiation $M^{\alpha \lambda}{ }_{\mid \lambda}$ may be obtained in terms of $\Gamma_{\alpha \lambda}^{\lambda}$ and $M^{\beta \alpha}$

$$
\begin{equation*}
M_{\mid \lambda}^{\alpha \lambda}=\left(M^{\alpha \lambda}\right)_{, \lambda}+\left(\Gamma_{\lambda \rho}^{\alpha} M^{\rho \lambda}+\Gamma_{\lambda \rho}^{\lambda} M^{\alpha \rho}\right) . \tag{32}
\end{equation*}
$$

$N^{\beta \alpha}$ depends, in its turn, on the terms $N^{* \beta \alpha}$ and $M^{\beta \alpha}$ (Eq. 26). These two latter terms are related with the internal energy $\mathcal{A}$ by the constitutive equations given in Eq. 25. In Section 3.3 we will derive equivalent, but easily implementable, expressions to them.

### 3.3 Tensor Components $N^{* \beta \alpha}$ and $M^{\alpha \beta}$

One way for deriving component equations to $N^{* \beta \alpha}$ and $M^{\alpha \beta}$ is to use the constitutive equations given by Eq. 25 . This requires, however, the determination of the internal energy $\mathcal{A}$.

From several experimentations we observe that Eq. 29 may be further simplified without degrading the visual effects. We might assume that solely tensors with indices $\rho=\alpha$ and $\lambda=\beta$, or $\rho=\beta$ and $\lambda=\alpha$ are non-vanishing terms. Thus,

$$
\begin{align*}
\Phi^{\alpha \beta} & =\Phi^{\beta \alpha}=H^{\alpha \beta \alpha \beta}=H^{\beta \alpha \beta \alpha} \\
& =\beta_{1} A^{\alpha \alpha} A^{\beta \beta}+2 \beta_{2}\left(A^{\alpha \beta}\right)^{2} \\
\Psi^{\alpha \beta} & =\Psi^{\beta \alpha}=B^{\alpha \beta \alpha \beta}=B^{\beta \alpha \beta \alpha} \\
& =\beta_{3} A^{\alpha \alpha} A^{\beta \beta}+2 \beta_{4}\left(A^{\alpha \beta}\right)^{2} \\
\Theta^{\alpha \beta} & =\Theta^{\beta \alpha}=C^{\alpha \beta \alpha \beta}=C^{\beta \alpha \beta \alpha} \\
& =\beta_{5} A^{\alpha \alpha} A^{\beta \beta}+2 \beta_{6}\left(A^{\alpha \beta}\right)^{2} \tag{33}
\end{align*}
$$

In addition, we also suppose that cloth is an isotropic material, which implies that

$$
\begin{align*}
& \beta_{1}=\beta_{2}=\zeta_{\alpha \beta} \\
& \beta_{3}=\beta_{4}=\xi_{\alpha \beta} \\
& \beta_{5}=\beta_{6}=\phi_{\alpha \beta} . \tag{34}
\end{align*}
$$

Eq. 33 lends, hence, itself to

$$
\begin{align*}
& \Phi^{\alpha \beta}=\Phi^{\beta \alpha}=\zeta_{\alpha \beta}\left(A^{\alpha \alpha} A^{\beta \beta}+2\left(A^{\alpha \beta}\right)^{2}\right) \\
& \Psi^{\alpha \beta}=\Psi^{\beta \alpha}=\xi_{\alpha \beta}\left(A^{\alpha \alpha} A^{\beta \beta}+2\left(A^{\alpha \beta}\right)^{2}\right) \\
& \Theta^{\alpha \beta}=\Theta^{\beta \alpha}=\phi_{\alpha \beta}\left(A^{\alpha \alpha} A^{\beta \beta}+2\left(A^{\alpha \beta}\right)^{2}\right) \tag{35}
\end{align*}
$$

which depend on the reciprocal metric tensors $A^{\alpha \beta}$ of the cloth's initial configuration and its strain and bending elasticity coefficients, which are $\zeta_{\alpha \beta}$ and $\xi_{\alpha \beta}$, respectively. We call $\Phi^{\alpha \beta}$ and $\Psi^{\alpha \beta}$ the material properties. The term $\phi_{\alpha \beta}$ affects prevalently the out-of-plane behavior; thus, $\Theta^{\alpha \beta}$ is called the buckling factor.

Replacing Eq. 35 in Eq. 25, discarting negligible terms, and after simple partial derivations, we get $N^{* \alpha \beta}$ and $M^{\alpha \beta}$ in terms of the surface areas, the strain measures, the bending measures, and the material properties.

$$
\begin{align*}
N^{* \alpha \beta} & =\mu \frac{\partial \mathcal{A}}{\partial \varepsilon_{\alpha \beta}}=\frac{\mu}{\mu_{0}}\left(2 \Phi^{\alpha \beta} \varepsilon_{\alpha \beta}+\Theta^{\alpha \beta} \kappa_{\alpha \beta}\right)=\frac{\mathcal{S}_{0}}{\mathcal{S}}\left(2 \Phi^{\alpha \beta} \varepsilon_{\alpha \beta}+\Theta^{\alpha \beta} \kappa_{\alpha \beta}\right) \\
M^{\alpha \beta} & =\mu \frac{\partial \mathcal{A}}{\partial \kappa_{i \alpha}}=\frac{\mu}{\mu_{0}}\left(2 \Psi^{\alpha \beta} \varepsilon_{\alpha \beta}+\Theta^{\alpha \beta} \kappa_{\alpha \beta}\right)=\frac{\mathcal{S}_{0}}{\mathcal{S}}\left(2 \Psi^{\alpha \beta} \varepsilon_{\alpha \beta}+\Theta^{\alpha \beta} \kappa_{\alpha \beta}\right) \tag{36}
\end{align*}
$$

Further substituting the first expression of Eq. 36 in Eq. 27 we get the component equation $N^{\beta \alpha}$ of the internal force in terms of the strain and bending measures at time $t$

$$
\begin{equation*}
N^{\beta \alpha}=\frac{\mathcal{S}_{0}}{\mathcal{S}}\left[\left(2 \Phi^{\beta \alpha} \varepsilon_{\beta \alpha}+\Theta^{\beta \alpha} \kappa_{\beta \alpha}\right)-b_{\lambda}^{\alpha}\left(2 \Psi^{\beta \lambda} \kappa_{\beta \lambda}+\Theta^{\beta \lambda} \varepsilon_{\beta \lambda}\right)\right] \tag{37}
\end{equation*}
$$

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Fig. 9. Alternative approximation to a normal vector: (a) Eq. 58; (b) finite difference approach; (c) average normal; (d) Eq. 62.
and an expression component to $N^{3 \alpha}$

$$
\begin{equation*}
N^{3 \alpha}=M_{\mid \beta}^{\alpha \beta}=\frac{\mathcal{S}_{0}}{\mathcal{S}}\left[\left(2 \Psi^{\alpha \beta} \kappa_{\beta \alpha}+\Theta^{\alpha \beta} \varepsilon_{\beta \alpha}\right)\right]_{\mid \beta} \tag{38}
\end{equation*}
$$

## 4. AN APPROXIMATION TO NORMAL VECTORS

As already stated in Section 2, in most works the fundamental problem to be solved for computing more exact bending measures is to derive an expression for computing more exact normal vectors. In our proposal, besides the first and the second surface derivative, Eq. 31 also requires the surface normal $\mathbf{a}_{3}$. Eq. 11 is the classic formulation of normal vector that uses the first derivative of the surface equation (Figure 9.(a)). When the surface's analytical partial derivatives are not available, they are approximated using finite differences which only considers two directions (Figure 9.(b)). To capture the surface's behavior in all the vicinity of a point on the surface, computer graphics practicioners usually compute the average of the normal vectors of its adjacent faces, as shows Figure 9.(c). This technique may deliver exact solution if the curvature tensors vanish $\left(b_{\alpha \beta}=0\right)$.

With the purpose of maintaining "linear structure" of the constitutive equations given by Eq. 31, [Wu and de Melo 2003] derive from Eq. 62 an alternative exact expression to the normal vectors of the surface for every point where the curvature tensors do not vanish $\left(b_{\alpha \beta} \neq 0\right)$

$$
\begin{align*}
\mathbf{a}_{3}(t) & =\frac{1}{b_{\alpha \beta}(t)}\left[\frac{\partial^{2} \mathbf{r}}{\partial x^{\alpha} \partial x^{\beta}}(t)-\Gamma_{\alpha \beta}^{1}(t) \frac{\partial \mathbf{r}}{\partial x^{1}}(t)-\Gamma_{\alpha \beta}^{2}(t) \frac{\partial \mathbf{r}}{\partial x^{2}}(t)\right] \\
& =\frac{1}{b_{\alpha \beta}(t)}\left[\frac{\partial^{2} \mathbf{r}}{\partial x^{\alpha} \partial x^{\beta}}(t)-\Gamma_{\alpha \beta}^{1}(t) \mathbf{a}_{1}(t)-\Gamma_{\alpha \beta}^{2}(t) \mathbf{a}_{2}(t)\right] \tag{39}
\end{align*}
$$

Since Eq. 39 considers all the vicinity of a point $\mathcal{P}$, the average of the four normal vectors determined from it is closer to the exact normal vector than the value we obtain by simply applying the discrete version of Eq. 58. This is because that the latter takes into account
only the variations of the position vectors $\mathbf{r}$ along two directions. Comparing the vectors we obtain from Eq. 39 with the average of the normal vectors of the adjacent plane faces of $\mathcal{P}$, the latter presents poorer performance for the vertices of higher curvature, because it does not consider the curvature built by the adjacent faces as the former approach. Moreover, we also observe that Eq. 39 preserves the compatibility condition expressed by Eq. 67.

Then, our proposal consists in switching between the two methods according to the local geometry in the vicinity of each point. We define a curvature tensor threshold $\delta$, above which one formulation is used and below which another one is applied.

$$
\mathbf{a}_{3}=\left\{\begin{array}{l}
\frac{1}{4} \sum \frac{1}{b_{\alpha \beta}(t)}\left[\frac{\partial^{2} \mathbf{r}}{\partial x^{\alpha} \partial x^{\beta}}(t)-\Gamma_{\alpha \beta}^{1}(t) \frac{\partial \mathbf{r}}{\partial x^{1}}(t)-\Gamma_{\alpha \beta}^{2}(t) \frac{\partial \mathbf{r}}{\partial x^{2}}(t)\right], \text { if } b_{\alpha \beta}>\delta \\
\text { the average of adjacent faces' normals, otherwise. }
\end{array}\right.
$$

## 5. IMPLEMENTATION

In this section, as a proof of concept, we describe a finite difference implementation of our proposed model based on the explicit time integration scheme. Our goal is to evaluate the impact of this simple integration scheme on the visual results of a geometrically exact cloth's model. It is also worth remarking that we use the sign $\sum$ in all finite difference expressions to explicitly indicate the number of times that a summation must be carried out, in order to facilitate their implementations.

To improve the realism in the motion of a deforming surface $\mathcal{S}=\mathbf{r}\left(x^{1}, x^{2}\right)$ under the force $\mathbf{F}$ per unit mass, we may consider the damping phenomenon $\varrho$ that acts against the motion of $\mathcal{S}$, that is,

$$
\begin{equation*}
\mu \dot{\mathbf{v}}(t)+\varrho \mathbf{v}(t)=\mu \mathbf{F}(t)=\mathbf{f}(t) \tag{40}
\end{equation*}
$$

with $\mathbf{v}=\frac{\partial \mathbf{r}\left(x^{1}, x^{2}\right)}{\partial t}$. For sufficiently small variations, in which the linearity is observed, we may apply the superposition principle and sum the effects due to Eq. 18 and Eq. 40 to find the total resultant on the position vetor $\mathbf{r}$ at time $t$

$$
\begin{equation*}
\mu \dot{\mathbf{v}}(t)+\varrho \mathbf{v}(t)-\mathbf{N}_{\mid \alpha}^{\alpha}(t)=\mathbf{f}(t) \tag{41}
\end{equation*}
$$

A corresponding finite differencing equation may be obtained by choosing spaced points along $t$-, $x^{1}$ - and $x^{2}$-axes.

$$
\begin{aligned}
x_{k}^{1} & =x_{0}+k \Delta^{1}, \quad k=0,1, \cdots, m \\
x_{l}^{2} & =x_{0}+l \Delta^{2}, \quad l=0,1, \cdots, n \\
t_{i} & =t_{0}+i \Delta t, \quad i=0,1, \cdots, T
\end{aligned}
$$

where $\Delta^{1}$ and $\Delta^{2}$ are the grid spacings and $\Delta t$, the time step. We then consider the deformation $\mathbf{r}\left(x^{1}, x^{2}, t\right)$ by its values at the discrete points of a grid with mesh size $m \times n$ and at the discrete time point.

From now on,
—we write $(k, l, i)$ for denoting each node $\mathbf{r}\left(x_{k}^{1}, x_{l}^{2}, t\right)$;
-for the spatial backward first-order finite difference of $f$ with respect to $x^{\alpha}$ at time point
$i$, we define

$$
\begin{align*}
& D_{1}^{-}(f)(k, l, i)=\frac{(f)(k, l, i)-(f)(k-1, l, i)}{\Delta^{1}} \\
& D_{2}^{-}(f)(k, l, i)=\frac{(f)(k, l, i)-(f)(k, l-1, i)}{\Delta^{2}} \tag{42}
\end{align*}
$$

-for the spatial forward first-order difference of $f$ with respect to $x^{\alpha}$ at time point $i$,

$$
\begin{align*}
D_{1}^{+}(f)(k, l, i) & =\frac{(f)(k+1, l, i)-(f)(k, l, i)}{\Delta^{1}} \\
D_{2}^{+}(f)(k, l, i) & =\frac{(f)(k, l+1, i)-(f)(k, l, i)}{\Delta^{2}} \tag{43}
\end{align*}
$$

-for the spatial forward second-order differences of $f$ at time point $i$,

$$
\begin{align*}
D_{\alpha \alpha}^{+}(f)(k, l, i) & =D_{\alpha}^{+} D_{\alpha}^{+}(f)(k, l, i) \\
D_{12}^{+}(f)(k, l, i) & =D_{21}^{+}(f)(k, l, i)=D_{1}^{+} D_{2}^{+}(f)(k, l, i) \tag{44}
\end{align*}
$$

-for the spatial backward second-order differences of $f$ at time point $i$,

$$
\begin{align*}
D_{\alpha \alpha}^{-}(f)(k, l, i) & =D_{\alpha}^{-} D_{\alpha}^{-}(f)(k, l, i) \\
D_{12}^{-}(f)(k, l, i) & =D_{21}^{-}(f)(k, l, i)=D_{1}^{-} D_{2}^{-}(f)(k, l, i) \tag{45}
\end{align*}
$$

-for the spatial central second-order differences of $f$ at time point $i$,

$$
\begin{align*}
D_{\alpha \alpha}^{*}(f)(k, l, i) & =D_{\alpha}^{-} D_{\alpha}^{+}(f)(k, l, i) \\
D_{12}^{-+}(f)(k, l, i) & =D_{21}^{+-}(f)(k, l, i)=D_{1}^{-} D_{2}^{+}(f)(k, l, i) \\
D_{21}^{-+}(f)(k, l, i) & =D_{12}^{+-}(f)(k, l, i)=D_{2}^{-} D_{1}^{+}(f)(k, l, i) \tag{46}
\end{align*}
$$

Using these notations and replacing Eqs. 37 and 38 in Eq. 31, the finite difference of the covariant derivative $\mathbf{N}_{\mid \alpha}^{\alpha}(k, l, i)$ at each node $(k, l, i)$ takes the form

$$
\begin{align*}
-\sum_{\alpha} \mathbf{N}_{\mid \alpha}^{\alpha} & =-\frac{S_{0}}{S} \sum_{\alpha, \beta=1}^{2} D_{\alpha}^{-}\left(\left(2 \Phi^{\alpha \beta} \varepsilon_{\alpha \beta}+\Theta^{\alpha \beta} \kappa_{\alpha \beta}\right)(k, l, i) \mathbf{a}_{\beta}(k, l, i)\right) \\
& +\frac{S_{0}}{S} \sum_{\alpha, \beta, \lambda=1}^{2} D_{\alpha}^{-}\left(\left(b_{\lambda}^{\beta}\left(2 \Psi^{\alpha \lambda} \kappa_{\alpha \lambda}+\Theta^{\alpha \lambda} \varepsilon_{\alpha \lambda}\right)\right)(k, l, i) \mathbf{a}_{\beta}(k, l, i)\right) \\
& -\frac{S_{0}}{S} \sum_{\alpha, \beta, \lambda=1}^{2}\left(\Gamma_{\alpha \lambda}^{\lambda}\left(2 \Phi^{\alpha \beta} \varepsilon_{\alpha \beta}+\Theta^{\alpha \beta} \kappa_{\alpha \beta}\right)\right)(k, l, i) \mathbf{a}_{\beta}(k, l, i) \\
& +\frac{S_{0}}{S} \sum_{\alpha, \beta, \lambda, \rho=1}^{2}\left(\Gamma_{\alpha \lambda}^{\lambda} b_{\rho}^{\beta}\left(2 \Psi^{\alpha \rho} \kappa_{\alpha \rho}+\Theta^{\alpha \rho} \varepsilon_{\alpha \rho}\right)\right)(k, l, i) \mathbf{a}_{\beta}(k, l, i) \\
& -\frac{S_{0}}{S} \sum_{\alpha, \lambda=1}^{2} D_{\alpha}^{-}\left(\left(2 \Psi^{\alpha \lambda} \kappa_{\alpha \lambda}+\Theta^{\alpha \lambda} \varepsilon_{\alpha \lambda}\right)_{\mid \lambda}(k, l, i) \mathbf{a}_{3}(k, l, i)\right) \\
& -\frac{S_{0}}{S} \sum_{\alpha, \beta, \lambda=1}^{2}\left(\Gamma_{\alpha \beta}^{\beta}\left(2 \Psi^{\alpha \lambda} \kappa_{\alpha \lambda}+\Theta^{\alpha \lambda} \varepsilon_{\alpha \lambda}\right)_{\mid \lambda}\right)(k, l, i) \mathbf{a}_{3}(k, l, i) \tag{47}
\end{align*}
$$



Fig. 10. Finite-difference of the derivatives of a point of $\mathcal{S}$ : (a) interior points; (b) points on the top and left-hand side of the boundary; and (c) points on the bottom and right-hand side of the boundary.

The vectors $\mathbf{a}_{\beta}(k, l, i)$ and $\mathbf{a}_{3}(k, l, i)$ are, in their turn, given by Eq. 10 and Eq. 40, respectively. To compute them, the first and the second derivatives of $\mathbf{r}(k, l, i)$ are required. These derivatives may be computed by approximate finite difference equations. Whenever the adjacent vertices of a node $\mathbf{r}(k, l, i)$ are well defined, namely $\mathbf{r}(k+1, l, i), \mathbf{r}(k, l+1, i)$, $\mathbf{r}(k-1, l, i), \mathbf{r}(k, l-1, i)$, and $\mathbf{r}(k+1, l+1, i)$ (Figure 10.(a), we approximate the first derivatives by Eq. 43 and the second ones by Eq. 46.

However, for the nodes on the boundary of the grid, some of required adjacent points do not exist. On the top side we miss $\mathbf{r}(k, l-1, i)$; on the left-hand side $\mathbf{r}(k-1, l, i)$ are not defined. As an alternative solution, we use Eq. 44 to evaluate the second derivatives (Figure 10.(b)). Finally, for the nodes on the right-hand and bottom sides we apply Eqs. 42 and 45 to calculate the necessary derivatives (Figure 10.(c)). In this way, we assure that the derivatives are computable at any grid node.

Another question that we should solve is how to determine the coefficients $b_{\alpha \beta}(k, l, i)$ that appear in Eq. 40. We consider that the time steps are sufficiently small such that we may apply the normal vectors of the previous iteration $\mathbf{a}_{3}(k, l, i-1)=\frac{D_{1}^{+} \mathbf{r}(k, l, i-1) \times D_{1}^{+} \mathbf{r}(k, l, i-1)}{\left|D_{1}^{+} \mathbf{r}(k, l, i-1) \times D_{1}^{+} \mathbf{r}(k, l, i-1)\right|}$ in Eq. 57 for computing them.

Algorithm 1 outlines a procedure that generates, on the basis of this paradigm, a set of configurations from the deformable surface's physical properties, its initial values (position and velocity), and the simulation parameters. At the end of each iteration, the normal vectors are updated with use of Eq. 40. Without loss of generality, we assume that $\Delta^{1}=$ $\Delta^{2}=\Delta$. Moreover, for conciseness, we omit, from now, the indices $k$ and $l$ when we refer to the quantities of all the $m n$ grid nodes.

Observe that instead of explicitly providing the spring's stiffiness and the oscillation damping forces, as in the classic mass-spring approach [Provot 1995; Baraff and Witkin 1998; Desbrun et al. 1999; Choi and Ko 2002; Grispun et al. 2003; Bridson et al. 2003; Ngoc 2004; Ji et al. 2006], our model implicitly represent such quantities as a function of the material properties, buckling factor, and variations on the coefficients of the first and the second fundamental forms.

The kernel of Algorithm 1 is the function Next_configuration, which derives the displacements of all points of the $i$-th configuration on the basis of the geometry of the $i-1$-th configuration. Theoretically, our model is more accurate in comparison with most of cloth's models that have been published. We conjecture, therefore, that we may apply the simple explicit integration method to track how the cloth propagates itself forward in time and our

```
Function Deformation \(\left(\mathcal{S}_{0}\right.\) : Geometry, mass: double;
                \(\zeta_{\alpha \beta}, \xi_{\alpha \beta}, \phi_{\alpha \beta}\) :double; damping coefficient: double; \(\mathcal{V}_{0}\) : Velocity; f: Force; \(\Delta\) : double;
                \(\Delta t\), T: double; deformedStates [] : Geometry): void
    Begin
        Compute the area \(S_{0}\) of the initial configuration;
        Discretize \(\mathcal{S}_{0}\) into a grid with \(\Delta\) spacing to get \(\mathbf{r}(0)\);
        For each grid node \((k, l)\) do
        Compute its initial velocity \(\mathbf{v}(k, l)\) from \(\mathcal{V}\);
        Compute its normal vector \(\mathbf{n}(k, l)\) of \(\mathcal{S}_{0}\);
        Compute metric and bending tensors of \(\mathcal{S}_{0}: A_{\alpha \beta}, A^{\alpha \beta}, B_{\alpha \beta}, B^{\alpha \beta}\);
        Using Eq. 35, compute material properties: \(\Phi^{\alpha \beta}, \Psi^{\alpha \beta}, \Theta^{\alpha \beta}\);
        End for
        \(i \leftarrow 1\);
        While the time point \(i\) is less than \(T\) do
        Determine the area \(S\) of the configuration \(i-1\);
        \(\mu \leftarrow \frac{\text { mass }}{S}\);
        \(\varrho \leftarrow \frac{\text { dampingcoefficient }}{S}\);
        ratio \(\leftarrow \frac{S_{0}}{S}\);
        For each grid node \((k, l)\) do
            Compute \(a_{\alpha \beta}(k, l, i-1)=D_{\alpha}^{+} \mathbf{r}(k, l, i-1) \cdot D_{\beta}^{+} \mathbf{r}(k, l, i-1)\);
            Using Eq. 55, compute \(a^{\alpha \beta}(k, l, i-1)\) from \(a_{\alpha \beta}(k, l, i-1)\);
            Compute \(\Gamma_{\alpha \beta}^{\gamma}=\sum_{\rho=1}^{2} a^{\gamma \rho}(k, l, i-1)\left(D_{\rho}^{+} \mathbf{r}(k, l, i-1) D_{\alpha \beta}^{+} \mathbf{r}(k, l, i-1)\right)\);
            Using Eq. 8 , compute \(\varepsilon_{\alpha \beta}(k, l, i-1)\) from \(a_{\alpha \beta}(k, l, i-1)\) and \(A_{\alpha \beta}\);
            Compute \(b_{\alpha \beta}(k, l, i-1)=D_{\alpha}^{+} D_{\beta}^{+} \mathbf{r}(k, l, i-1) \cdot \frac{D_{1}^{+} \mathbf{r}(k, l, i-1) \times D_{1}^{+} \mathbf{r}(k, l, i-1)}{\left|D_{1}^{+} \mathbf{r}(k, l, i-1) \times D_{1}^{+} \mathbf{r}(k, l, i-1)\right|}\);
            Using Eq. 65 , compute \(b_{\alpha}^{\beta}(k, l, i-1)\) from \(a^{\alpha \beta}(k, l, i-1)\) and \(b_{\alpha \beta}(k, l, i-1)\);
            Using Eq. 9, compute \(\kappa_{\alpha \beta}(k, l, i-1)\) from \(b_{\alpha \beta}(k, l, i-1)\) and \(B_{\alpha \beta}\);
        End for
        Next_configuration \(\left(\mathbf{r}(i-1), \varepsilon_{\alpha \beta}(i-1), \kappa_{\alpha \beta}(i-1), \Phi^{\alpha \beta}, \Psi^{\alpha \beta}, \Theta^{\alpha \beta}\right.\),
            \(\mathbf{n}\), ratio, \(\left.\mu, \Gamma_{\alpha \beta}^{\gamma}(i-1), b^{\alpha \beta}(i-1), \Delta, \Delta t, \varrho, \mathbf{v}, \mathbf{f}(i), \mathbf{r}(i)\right)\);
        deformedStates[i] \(\leftarrow \mathbf{r}(i)\);
        For each grid node \((k, l)\) do
            \(\mathbf{v}(k, l) \leftarrow \frac{\mathbf{r}(k, l, i)-\mathbf{r}(k, l, i-1)}{\Delta t}\);
            Update \(\mathbf{n}(k, l)\) with the average normals computed from Eq. 40;
        End for
        Increment \(i\);
        End while
    End
```

Algorithm 1: Deformation.
results should be comparable to the ones delivered by the semi-implicit integration scheme of the known cloth models. If it is the fact, we may be benefited from the performance of this method. To underpin our conjecture, we developed an explicit integration scheme for our model.

The simple Euler integration method presented in [Provot 1995] is applied to each sample point $\mathbf{r}(k, l)$ for predicting its position at time $t+\Delta t$. To cast Eq. 41 in the form appropriate for the Euler integration method, we should put the term $\mathbf{v}(t)$ alone on the left-hand side as follows

$$
\mu \dot{\mathbf{v}}(t)=\mathbf{f}(t)-\varrho \mathbf{v}(t)+\mathbf{N}_{\mid \alpha}^{\alpha}(t)
$$

and assume that $\mathbf{v}(t)$ and $\mathbf{N}_{\mid \alpha}^{\alpha}(t)$ are already known, that is, the expression is differenced as follows:

$$
\mu \frac{\mathbf{v}(k, l, i)-\mathbf{v}(k, l, i-1)}{\Delta t}=\mathbf{f}(k, l, i)-\varrho \mathbf{v}(k, l, i-1)+\mathbf{N}_{\mid \alpha}^{\alpha}(k, l, i-1),
$$

which leads to

$$
\mathbf{v}(k, l, i)=\mathbf{v}(k, l, i-1)+\frac{\Delta t\left(\mathbf{f}(k, l, i)-\varrho \mathbf{v}(k, l, i-1)+\mathbf{N}_{\mid \alpha}^{\alpha}(k, l, i-1)\right)}{\mu}
$$

Considering the differential approximation

$$
\mathbf{v}(k, l, i)=\frac{\mathbf{r}(k, l, i)-\mathbf{r}(k, l, i-1)}{\Delta t}
$$

we determine the position vectors $\mathbf{r}(k, l, i)$ from the quantities at time step $i-1$ from the expression

$$
\begin{align*}
\mathbf{r}(k, l, i) & =\mathbf{r}(k, l, i-1)+\Delta t(\mathbf{v}(k, l, i-1)+ \\
& \left.+\frac{\Delta t\left(\mathbf{f}(k, l, i)-\varrho \mathbf{v}(k, l, i-1)+\mathbf{N}_{\mid \alpha}^{\alpha}(k, l, i-1)\right)}{\mu}\right) . \tag{48}
\end{align*}
$$

Algorithm 2 codifies the ideas presented.

```
Function Next_configuration(r(i-1): Geometry; }\mp@subsup{\varepsilon}{\alpha\beta}{},\mp@subsup{\kappa}{\alpha\beta}{}\mathrm{ :double;
                                    \Phi}\mp@subsup{}{}{\alpha\beta},\mp@subsup{\Psi}{}{\alpha\beta},\mp@subsup{\Theta}{}{\alpha\beta}:\mathrm{ double; n(i-1):Normal Vector; rat io, }\mu\mathrm{ : double; }\mp@subsup{\Gamma}{\alpha\beta}{\gamma},\mp@subsup{b}{}{\alpha\beta}:\mathrm{ double;
                                    \Delta,\Deltat, \varrho: double; }\mathbf{v}(i-1):Velocity; f(i): Force, r(i):Geometry): voi
    Begin:
    For each grid node ( }k,l\mathrm{ ) do:
        Using Eq. 47, determine N}\mp@subsup{\mathbf{N}}{}{\alpha}(k,l,i-1) |\alpha
        Using Eq. 48, compute r(k,l,i);
    End for;
    End;
```

Algorithm 2: Explicit Integration.

## 6. RESULTS

In this section we present some experiments that we carried out to measure experimentally the time complexity of our implementation. Moreover, we also present some results for visually demonstrating that our proposed cloth model satisfies both the intuitiveness and the controllability of the bending behaviors. It is worth emphasizing here that, despite its theoretical foundation, the main concern of our proposal is to provide realistic visual effects. Hence, we did not perform any comparison between the simulation numerical data and the physical data obtained experimentally, such as from the Kawabata Evaluation System [Kawabata 1980]. Morover, we remark that the provided running times are of the explicit integration scheme and that the simulated objects do not exhibit complex interactions with its surrounding or with itself, because our implementation cannot handle collisions yet.

### 6.1 Performance

In order to evaluate the performance of the explicit integration scheme for solving our model, we measure the overall processing times $(T)$ of the simulations shown in Figures 1, 4, and 5. All the simulations were run on an AMD Athlon $643200+$ with 512MB of memory, equipped with a NVIDIA GeForce FX 5900 with 128 MB. In the simulations presented in Figure 1, the following parameters are used: damping coefficient $=0.2 \frac{\mathrm{~kg}}{\mathrm{~m}^{2} s}, \Delta t=0.001 \mathrm{~s}$, number of renderable frames $=400$, total simulation time $=20 \mathrm{~s}$. The size of the grid is $60 \times 60(3600$ vertices) with $\Delta=0.01$ unit. The axial forces $|\mathcal{F}|=28.28 \mathrm{~N}$ are applied at the nodes as depicts Figure 11.(a). Their duration was calibrated in such a way that the self-collisions are avoided and they are sufficient for building the bucklings.


Fig. 11. Axial forces on a $60 \times 60$ panel.

In the simulations shown in Figure 5, we fixed the four corners of the panels and let them stretch under the gravity force. Other parameters are: damping coefficient $=0.2 \frac{\mathrm{~kg}}{\mathrm{~m}^{2} \mathrm{~s}}$ and $\Delta t=0.001 \mathrm{~s}$.

| Images | Fabric | Weight $\left(\mathrm{g} / \mathrm{m}^{2}\right)$ | $\zeta_{\alpha \beta}$ | $\xi_{\alpha \beta}$ | $\phi_{\alpha \beta}$ | duration (s) | time (s)/frame |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Figure 1(a) | silk satin | 78 | 100 | 0.01 | 0.1 | 6 | 0.265 |
| Figure 1(b) | cotton | 88 | 100 | 0.02 | 0.1 | 10 | 0.265 |
| Figure 1(c) | jeans | 111 | 100 | 0.03 | 0.1 | 10 | 0.265 |
| Figure 1(d) | sailcloth | 133 | 100 | 0.05 | 0.1 | 10 | 0.265 |
| Figure 4 | silk satin | 78 | 100 | 0.01 | 0.1 | 10 | 0.265 |
| Figure 5(a) | cheese | - | 10 | 0.1 | 0.01 | 10 | 0.265 |
| Figure 5(b) | metal | - | 250 | 0.3 | 0.0 | 10 | 0.265 |

Table I. Running timings

Differently from the previsouly reported results [Baraff and Witkin 1998], the timings in Table I reveal that the time performance of our implementation is quite insensitive to the range of material properties and buckling factor. Our explanation for this behavior is that our time step must be very small in order to ensure accurate estimation to $b_{\alpha \beta}$.

We further estimate the time complexity of the explicit integration scheme in function of the number of grid nodes, by measuring the simulation running times for the four clothes


Fig. 12. Time $\times$ Number of nodes for the cloth in Figure 1.(a) (red), Figure 1.(b) (blue), Figure 1.(c) (green), and Figure 1.(d) (black).
presented in Figure 1. Preserving the simulation parameters, the material properties, and the buckling factor, we vary the number of grid nodes $(N=m n)$ from $100(10 \times 10)$ to $10000(100 \times 100)$ and collect some time samples. From the interpolated curves Time $\times$ Number of nodes shown in Figure 12 we may estimate that the complexity might almost be linear with respect to the number of nodes $(O(N))$ for all four fabrics.

### 6.2 Comparisons with Experimental Results

In this section we provide the photos of the experimental results and the images of the finite difference solutions of our proposed model, both in the equilibrium state, for illustrating the adequacy of our proposal to cloth modeling - at least visually. We maintain the following simulation parameters: damping coefficient $=0.2 \frac{\mathrm{~kg}}{\mathrm{~m}^{2} \mathrm{~s}}, \Delta t=0.001 \mathrm{~s}$, number of renderable frames $=400$, and the total simulation time $=20 \mathrm{~s}$. Table II provides the time necessary to achieve the state that is presented in each figure. It is worth remarking that we maintain the values of $\zeta_{\alpha \beta}$ and $\xi_{\alpha \beta}$, as far as the fabric is of the same material, and only increase the parameters $\phi_{\alpha \beta}$ to favor the buckling formation.

Figure 13.(a) presents the classical draping experiment of a (silk satin) tablecloth $\mathcal{S}_{0}=$ $60 \mathrm{~cm} \times 60 \mathrm{~cm}$ over a circular table with radius equal to 18 cm . Figure 13.(b) is the image of the associated simulation. To reproduce the equivalent visual effect, we fixed all the nodes in contact with the table and applied the gravitational force on the remaining nodes that are initially laid horizontally.

Figure 14.(a) presents the same draping experiment with another cloth sample: linen. Figure 14.(b) is the image of its simulation. As expected, for simulating the behavior of a more rigid fabric, both the stretching and the bending elasticity constants must be larger than those used in the former simulation.

Figure 15.(a) presents the draping experiment of a polyester cloth ( $60 \mathrm{~cm} \times 60 \mathrm{~cm}$ ) hanging at two corners. Figure 15.(b) is the image of the associated simulation. Besides the gravitational force, we applied forces on the nodes $(0,0)$ and $(0,29)$ forces that displace them horizontally towards the node $(0,14)$ to produce the beautiful round drape.

Table II collects all the simulation parameters we used to generate the results presented in this section.

(b) Actual

(c) Simulation

Fig. 13. The draping of a satin tablecloth.


Fig. 14. The draping of a linen tablecloth.

(b) Actual

(c) Simulation

Fig. 15. The draping of a polyester cloth hanging at two corners.

| Images | Fabric | Weight $\left(\mathrm{g} / \mathrm{m}^{2}\right)$ | grid nodes | $\zeta_{\alpha \beta}$ | $\xi_{\alpha \beta}$ | $\phi_{\alpha \beta}$ | time (s)/frame |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Fig. 13 | satin | 98 | $40 \times 40$ | 100 | 0.01 | 0.1 | 0.517 |
| Fig. 14 | linen | 88 | $40 \times 40$ | 100 | 0.02 | 0.1 | 0.517 |
| Fig. 15 | polyester | 160 | $30 \times 30$ | 0.375 | 0.00037 | 0.00025 | 0.265 |

Table II. Simulations of realistic behaviors.

### 6.3 Realistic Dynamics

To demonstrate the realistic dynamic behavior of our proposed model, we provide in this section some intermediary images of the simulation output of the set of panels depicted in Figure 1 and how a panel of distinct material evolutes itself forward in time under the gravity force.

First of all, we observe distinguishing inertial behaviors of the four fabrics. For the stiffer materials, such as a sailcloth, we need much more force to overcome its internal frictions in comparison with the the silk satin fabric (Figure 16.(a) and (e)).

Another dynamic behavior of the most fabrics is their ease in restoring their initial configurations when the external forces are removed. As expected, the time interval that each fabric requires to restore its initial state is dependent on its mass and internal structure, as shown in Figure 16.(d) and (h).

We also observed that when the initial configuration is almost restored, the fabrics also reaches the equilibrium state, that is the reached configuration is preserved in the subsequent frames as far as no more external forces are applied, as illustrate Figure 16.(d) and (h). No unrealistic oscillations are noticeable.


Fig. 16. Realistic dynamics.

We evaluate whether a surface can achieve an equilibrium configuration when it is always under external forces. Figure 16(1) presents the final deforming state of a tablecloth and

Figure $16(\mathrm{p})$ shows the equilibrium state of a hanging panel also subjected some perpendicular isolated forces. We did not perceive any residual forces that cause the trembling effects, as in the mass-spring models.

Finally, it is worth observing that in all simulations the variation of the area of the deformed surface in its final configuration is less than $5 \%$ with respect to the original configuration.

### 6.4 Intuitiveness and Controllability

Tables I and II provide a summary of the relation between the simulated objects and their material parameters $\zeta_{\alpha \beta}, \xi_{\alpha \beta}, \phi_{\alpha \beta}$. The data in these tables substantiate the intuitiveness of our model. Both the environment damping factor (in the case, air) and the mass per unit area are physically exact values. The weight of the materials has been obtained from the fabrics suppliers.

For simulating materials with high resistance to the metric and bending variations, such as a metallic plate (Figure 5.(b)), the $\zeta_{\alpha \beta}$ and $\xi_{\alpha \beta}$ should assume higher values. On the contrary, for simulating materials with low resistance to stretching and high resistance to curving, such as a melting cheese (Figure 5.(a)), we should set lower values to $\zeta_{\alpha \beta}$ and higher values to $\xi_{\alpha \beta}$.

To simulate the behavior of distinct fabrics, which consists in high resistance to stretching, we should set larger values to $\zeta_{\alpha \beta}$. From the values that we should set to get valid simulations, it seems that $\zeta_{\alpha \beta}$ are dependent on the weight of the cloth. The heavier it is, the higher should be the value.

As the fabrics have a peculiar bending property (buckling) under axial loadings, we should ensure that $\phi_{\alpha \beta}$ are non-zero. From the values we used in all simulations, we are prone to conclude that, for getting beautiful buckling effects, we should set higher values to $\phi_{\alpha \beta}$.

The distinct bending behavior of the fabrics is, neverthless, determined by the parameters $\xi_{\alpha \beta}$. Observe in Tables I and II that we have attributed the same value to the coefficients $\zeta_{\alpha \beta}$ of a silk satin, polyester, and a cotton, which are lighter materials, and varied their parameters $\xi_{\alpha \beta}$, in order to get different curving effects. We adopted the similar strategy for controlling the curving behavior of heavier textile materials, such as a jeans and a sailcloth.

### 6.5 Complex Simulations

We conclude this section by providing more complex simulations: a hanging curtain (Figure 17) and a cloth drapping over a seat chair (Figure 18).

In the first case, axial forces (in red arrows) are applied on the points that are supposed on the curtain rails. These forces are responsible for the regular ondulations along the horizontal extension. To preserve each wrinkle along the vertical extension, we also applied the perpendicular forces on the bottom (in red dashed arrow), as shows Figure 17.(a). For legibility, we remark that the number of drawn nodes in the figure does not represent the actual number.

In the second case, the blanket is thrown over an imaginary seat. It slides gently into the center of the seat until the equilibrium is achieved. To overcome the collision problem that our algorithm cannot handle yet, we just fix the points on the back of the imaginary seat and apply very weak repulsion forces along the imaginary arms, as illustrates Figure 18.(a). These points are colored in red in Figure 18.(a). Naturally, the blanket assumes an expected


Fig. 17. Curtain.
shape.


Fig. 18. A covered seat chair.

Table III summarizes the simulation parameters.

|  | $\mu \mathcal{S}$ | $\zeta_{\alpha \beta}$ | $\xi_{\alpha \beta}$ | $\phi_{\alpha \beta}$ | \# vertices | time (s)/frame |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Curtain | 320 | 100 | 0.35 | 0.1 | 10,000 |  |
| Seat | 25 | 100 | 0.04 | 0.1 | 2500 | 0.975 |

Table III. Simulation parameters for complex simulations.

## 7. CONCLUSIONS

The investigation of Eischen et al. [Eischen et al. 1996] led to the conclusion that the fabric modeling based on the continuum mechanics delivers more faithful numerical results. Unfortunately, one of the barrier to its application is its complexity, involving terms that are not convenient for numerical computations. With their work [Simo and Fox 1989a; 1989b; 1989c], Simo and Fox succeed to overcome this barrier, but their solution, consisting in getting rid of the explicit differential geometrical quantities, does not seem suitable to the graphics community that has strong appeal for intuitiveness. This paper contributes with a formulation, also based on the theory of a Cosserat surface, that preserves all the geometrical quantities. We show that it is possible to approximate the awkward quantities with a numerically tractable component equations.

The main feature of our proposed model is its underlying theoretical foundation, allowing us to interpret correctly each term that appears in it. From the implementational standpoint, our proposal has equivalent complexity as the existing ones with the advantage that no term is empirical or fictitious ones. This makes the control on the bending behaviors of a variety of fabrics much more intuitive. The simulation results confirm this hypothesis.

Computational efficiency is also an important issue. In this work we implement our proposal with the well-known explicit integration scheme. As already expected, since our proposed model is much more accurate than the models that we found in the literature, we obtain a lifelike photorealistic results. From the timings provided in Section 6.1, we may claim that our proposed model is not only accurate but has good performance as well.

In Section 4 we propose one paradigm to the normal vector estimation, which is based on the geometrically exact local differential geometry properties. In Section 5, a finite difference formulation we used in our implementation is given. It seems to do a good job in practice, but it imposes the restriction on the time step. We conjecture that the alternative normal and curvature estimation methods, such as the ones proposed in [Meyer et al. 2003; Agam and Tang 2005], may deliver better performance. This is due to the fact that they use the current information in their computations. We have not yet pursued this idea in detail. It is in our plan to compare the knwon methods with our proposal, in terms of the efficiency, the accuracy, and the visual effects.

One apparent limitation of a continuum mechanics based technique, such as our proposed model, is that the mesh must be rectangular (vertex valence equal to 4). All of its formulations are based on the derivatives with respect to the two linearly independet curvilinear coordinates. We, however, believe that with the explicit integration scheme this restriction is easily got round. This is because that our proposed model depends essentially on accurate and robust local geometry estimates (normal vectors, first and second derivatives). Sharing the paradigm adopted by Agam and Tang [Agam and Tang 2005], we may consider that the vicinity of a point is covered by a set of curves passing through it and only the curves of the two linearly independent (not necessarily orthogonal) directions are of relevance. This means that, when a point possesses valence more than four, only four of them should be used. We will include this pre-processing in our system in the next future to validate this idea.

The finite difference formulation of our proposed model still suffers from stability problems. We believe that it is due to the initial and boundary conditions that have not been appropriately established. As a further work, we plan to improve such conditions or experiment finite element formulation that are used in most works on cloth modeling.

We present in this work only dynamics simulation of simple-shaped surfaces. This is because that it is easier to highlight the cloth's response under a specific external force. In actual fashion and entertainment applications, clothes or garments consist of several sewn fabric panels [Fontana et al. 2006]. They are used to cloth the virtual actors or avatars. It is of our interest to integrate in our cloth model the sewing process, and the method that attaches the deformable fabric pieces to rigid objects and handles their contact with rigid surfaces.

Collision detection and response are essential for realistic animation of any kind of cloth. To integrate the well-known collision methods or to devise a new yet suitable one to our algorithm is one of our priorities.

Finally, to make our model a useful tool for the accurate simulations in the textile and
apparel applications, it is necessary to establish an appropriate mapping between the material parameters ( $\Phi^{\alpha \beta}$ and $\Psi^{\alpha \beta}$ ) and the fabric properties usually given in terms of mechanical parameters, such as tensile modulus, shear modulus, bending modulus, Poisson's ratio, and the elasticity constant. In addition, the validation of our model with respect to the experimental data should also be performed.

## A. TENSOR CALCULUS

Tensor Calculus is concerned with the behavior of entities under the transition from a given coordinate system to another system. It is, in fact, a generalization of classical vector analysis and enables one to express geometrical or physical relationships in a concise manner, independently of a coordinate system. Differently from the other sections, the summation must be performed from 1 to $n$ in this appendix.

Any vector $\mathbf{T}$ in $\mathbf{R}^{n}$ can be expressed in the components form:

$$
\mathbf{T}=T_{1} \mathbf{e}_{1}+T_{2} \mathbf{e}_{2}+T_{3} \mathbf{e}_{3}+\cdots+T_{n} \mathbf{e}_{n}
$$

where $\mathbf{e}_{1}=(1,0,0, \cdots, 0), \mathbf{e}_{2}=(0,1,0, \cdots, 0)$, and $\mathbf{e}_{n}=(0, \cdots, 0,0,1)$ are linearly independent unit basis vectors. The $n$-tuple notation, $\mathbf{T}=\left(T_{1}, T_{2}, T_{3}, \cdots, T_{n}\right)$, is also widely used.

In the tensorial calculus, a still shorter notation is employed - an index notation: $T_{i}, i=$ $1,2,3, \cdots, n$. This notation emphasizes the components of the vectors and uses a dummy index $i$, whose range covers all the components of the vectors. When these components obey certain transformation laws, they are referred to as tensor systems. The subscripts and the superscripts are used to distinguish the types of systems. The number of subscripts and superscripts determines the order of the systems. A system with one index is a first order, two indices is a second order, etc. A system with no indices is called a scalar.

Consider a set of any $n$ independent vectors $\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \cdots, \mathbf{a}_{n}\right\}$, which are not necessarily orthogonal nor the unit length. We may represent the vector $\mathbf{T}$ in terms of these vectors. For example:

$$
\mathbf{T}=T^{1} \mathbf{a}_{1}+T^{2} \mathbf{a}_{2}+T^{3} \mathbf{a}_{3}+\cdots+T^{n} \mathbf{a}_{n}
$$

The components $\left(T^{1}, T^{2}, T^{3}, \cdots, T^{n}\right)$ can be determined from the components by taking appropriate projections and getting $n$ equations with $n$ unknowns. A way to compute these components is to construct a reciprocal basis $\left\{\mathbf{a}^{1}, \mathbf{a}^{2}, \mathbf{a}^{3}, \cdots, \mathbf{a}^{n}\right\}$ of $\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \cdots, \mathbf{a}_{n}\right\}$. Two bases are reciprocal if

$$
\mathbf{a}_{i} \mathbf{a}^{j}=\delta_{i}^{j}=\left\{\begin{array}{l}
1 \text { if } i=j  \tag{49}\\
0 \text { if } i \neq j
\end{array} .\right.
$$

This means that the dot product of $\mathbf{a}^{i}$ with $\mathbf{a}_{i}$ is unitary and $\mathbf{a}^{i}(t)$ is orthogonal to all the other base vectors $\mathbf{a}_{j}(t), j \neq i$.

We may represent $\mathbf{T}$ with respect to either of these bases. The components $\left(T^{1}, T^{2}, T^{3}, \cdots, T^{n}\right)$ of $\mathbf{T}$ relative to the basis $\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \cdots, \mathbf{a}_{n}\right\}$ are denoted the contravariant components and the components $\left(T_{1}, T_{2}, T_{3}, \cdots, T_{n}\right)$ of $\mathbf{T}$ relative to the basis $\left\{\mathbf{a}^{1}, \mathbf{a}^{2}, \mathbf{a}^{3}, \cdots, \mathbf{a}^{n}\right\}$ are denoted the covariant components. The covariant and contravariant components are different ways to represent the same vector with respect to a set of basis or reciprocal basis vectors. Figure 19 ilustrates the decomposition of a vector $\mathbf{v}$ in its covariant and contravariant components.


Fig. 19. A set of reciprocal basis vectors.
Whenever $n$ components $T^{i}$ in a coordinate system $\left(x^{1}, x^{2}, \cdots, x^{n}\right)$ transform to the $n$ components $\bar{T}^{i}$ in a coordinate system $\left(\bar{x}^{1}, \bar{x}^{2}, \cdots, \bar{x}^{n}\right)$ according to:

$$
\bar{T}^{i}=\frac{\partial \bar{x}^{i}}{\partial x^{j}} T^{j}
$$

we way that it is a contravariant tensor of first order. If the $n$ components $T_{i}$ in a coordinate system $\left(x^{1}, x^{2}, \cdots, x^{n}\right)$ transform to the $n$ components $\bar{T}_{i}$ in a coordinate system $\left(\bar{x}^{1}, \bar{x}^{2}, \cdots, \bar{x}^{n}\right)$ according to:

$$
\bar{T}_{i}=\frac{\partial x^{j}}{\partial \bar{x}^{i}} T_{j}
$$

we way that it is a covariant tensor of first order.
The transformation law of the differentials of a contravariant tensor $T^{i}$ with respect to the coordinate $x^{j}$ does not satisfy the tensorial transformation. Hence, the covariant derivative is defined. This is a special differential which has some of the properties associated with the usual concept of a differential, but which is also tensorial. In terms of the basis vectors and the reciprocal basis vectors, it is given, respectively, by

$$
\begin{align*}
T_{i \mid j} & =\frac{\partial T_{i}}{\partial x^{j}}+\Gamma_{i j}^{k} T^{k} \\
T^{i}{ }_{\mid j} & =\frac{\partial T^{i}}{\partial x^{j}}+\Gamma_{j k}^{i} T^{k} \tag{50}
\end{align*}
$$

where $\Gamma_{j k}^{i}$ is a connection coefficient or a Christoffel symbol. The additional term involving $\Gamma_{j k}^{i}$ ensures that the expression is a tensorial one.
Let a system of $n^{2}$ real numbers $T_{i j}$ be associated with the coordinates ( $x^{1}, x^{2}, \cdots, x^{n}$ ) and a system of $n^{2}$ real numbers $\bar{T}_{i j}$ be associated with the coordinates $\left(\bar{x}^{1}, \bar{x}^{2}, \cdots, \bar{x}^{n}\right)$. If these numbers are related by

$$
\bar{T}_{i j}=\frac{\partial x^{k}}{\partial \bar{x}^{i}} \frac{\partial x^{l}}{\partial \bar{x}^{j}} T_{k l}
$$

we say that a covariant tensor of second order.
Let a system of $n^{2}$ real numbers $T^{i j}$ be associated with the coordinates $\left(x^{1}, x^{2}, \cdots, x^{n}\right)$ and a system of $n^{2}$ real numbers $\bar{T}^{i j}$ be associated with the coordinates $\left(\bar{x}^{1}, \bar{x}^{2}, \cdots, \bar{x}^{n}\right)$. If these numbers are related by

$$
\bar{T}^{i j}=\frac{\partial \bar{x}^{i}}{\partial x^{k}} \frac{\partial \bar{x}^{j}}{\partial x^{l}} T^{k l}
$$

we say that a contravariant tensor of second order.
Let a system of $n^{2}$ real numbers $T_{j}^{i}$ be associated with the coordinates $\left(x^{1}, x^{2}, \cdots, x^{n}\right)$ and a system of $n^{2}$ real numbers $\bar{T}_{j}^{i}$ be associated with the coordinates $\left(\bar{x}^{1}, \bar{x}^{2}, \cdots, \bar{x}^{n}\right)$.

If these numbers are related by

$$
\bar{T}_{j}^{i}=\frac{\partial x^{l}}{\partial \bar{x}^{j}} \frac{\partial \bar{x}^{i}}{\partial x^{k}} T_{l}^{k}
$$

we say that a mixed tensor of second order.
The covariant derivative of the tensors $T_{i j}, T^{i j}$ and $T_{j}^{i}$ with respect to the coordinate $x^{k}$ are given by

$$
\begin{align*}
T_{i j \mid k} & =\frac{\partial T_{i j}}{\partial x^{k}}-\left(\Gamma_{i k}^{l} T_{l j}+\Gamma_{j k}^{l} T_{i l}\right) \\
T_{{ }_{\mid k}}^{i j} & =\frac{\partial T^{i j}}{\partial x^{k}}+\left(\Gamma_{k l}^{i} T^{l j}+\Gamma_{k l}^{j} T^{i l}\right) \\
T_{j \mid k}^{i} & =\frac{\partial T_{j}^{i}}{\partial x^{k}}+\left(\Gamma_{k l}^{i} T_{j}^{l}-\Gamma_{j k}^{l} T_{\lambda}^{i}\right) . \tag{51}
\end{align*}
$$

The vectors $\mathbf{T}^{i}, i=1,2, \cdots, n$, are said contravariant when they obey the law of contravariance of the contravariant tensors of first order.

## B. DIFFERENTIAL GEOMETRY

Let $\mathbf{r}: \Omega \rightarrow \mathbf{R}^{3}$ be a regular surface $\mathcal{S}$ [Carmo 1976] given by $\mathbf{r}\left(a_{1}, a_{2}\right)=\left(x\left(x^{1}, x^{2}\right)\right.$, $\left.y\left(x^{1}, x^{2}\right), z\left(x^{1}, x^{2}\right)\right), x^{1}, x^{2} \in \Omega$ (Figure 20). As we have

$$
\begin{equation*}
d \mathbf{r}=\frac{\partial \mathbf{r}}{\partial x^{1}} d x^{1}+\frac{\partial \mathbf{r}}{\partial x^{2}} d x^{2} \tag{52}
\end{equation*}
$$



Fig. 20. Parametrizations of a surface.
the squared length $I(\mathbf{w})$ of an arc of a parameterized curve $\alpha(t)=\mathbf{r}\left(x^{1}(t), x^{2}(t)\right)$, $t \in(-\delta, \delta)$, with $P=\alpha(0)=\mathbf{r}\left(x^{1}(0), x^{2}(0)\right)$ and $\mathbf{w}=\alpha^{\prime}(0)$, can be expressed by

$$
\begin{align*}
I(\mathbf{w}) & =d \mathbf{r} \cdot d \mathbf{r} \equiv a_{\alpha \beta} d x^{\alpha} d x^{\beta} \\
& \equiv\left[\begin{array}{ll}
d x^{\alpha} & d x^{\beta}
\end{array}\right]\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]\left[\begin{array}{c}
d x^{\alpha} \\
d x^{\beta}
\end{array}\right] \tag{53}
\end{align*}
$$

where

$$
\begin{equation*}
a_{\alpha \beta}\left(\mathbf{r}\left(x^{1}, x^{2}\right)\right)=\mathbf{a}_{\alpha} \cdot \mathbf{a}_{\beta}, \tag{54}
\end{equation*}
$$

where $\mathbf{a}_{\lambda}=\frac{\partial \mathbf{r}}{\partial x^{\lambda}}$.
The quadratic form, defined by Eq. 53, is the first fundamental form or metric tensor and their components $a_{\alpha \beta}$, given by Eq.54, are called the metric coefficients. Since the inner product is symmetric, we have $a_{12}=a_{21}$.

With the first fundamental form we can treat metric questions on a regular surface without further reference to the ambient space $\mathbf{R}^{3}$ where the surface lies. Therefore, all geometric properties expressed in terms of the metric coefficients, such as length, area, and angle, are invariant under isometries and are called intrinsic geometric properties. The geometric properties that are solely observable by a viewer located in the surrounding space are extrinsic ones.

The coefficients $a^{\alpha \beta}$ of the inverse matrix of the matrix given in Eq. 53 are

$$
\begin{equation*}
a^{11}=\frac{a_{22}}{a}, a^{12}=a^{21}=-\frac{a_{12}}{a}, a^{22}=\frac{a_{11}}{a} \tag{55}
\end{equation*}
$$

where

$$
a=a_{11} a_{22}-a_{12} a_{21}
$$

The superscripts and subscripts denote, respectively, contravariant and covariant tensors.
It can be shown that the normal curvature $k_{n}$ of an arbitrary curve $\alpha(t)=\mathbf{r}\left(x^{1}(t), x^{2}(t)\right)$ on $\mathcal{S}$, passing the point $P=\alpha(0)$ with $\mathbf{w}=\alpha^{\prime}(0)$, can be expressed as

$$
\begin{equation*}
I I(\mathbf{w})=k_{n}(\mathbf{w})=\frac{b_{\alpha \beta} d x^{\alpha} d x^{\beta}}{a_{\alpha \beta} d x^{\alpha} d x^{\beta}} \tag{56}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{\alpha \beta}=\mathbf{n} \cdot \frac{\partial^{2} \mathbf{r}}{\partial x^{\alpha} \partial x^{\beta}} \tag{57}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{n}=\mathbf{a}_{3}=\frac{\mathbf{a}_{1} \times \mathbf{a}_{2}}{\left\|\mathbf{a}_{1} \times \mathbf{a}_{2}\right\|} \tag{58}
\end{equation*}
$$

corresponding to the normal vector of $\mathcal{S}$ at $P$. The term

$$
b_{\alpha \beta} d x^{\alpha} d x^{\beta}=\left[\begin{array}{ll}
d x^{\alpha} & d x^{\beta}
\end{array}\right]\left[\begin{array}{ll}
b_{11} & b_{12}  \tag{59}\\
b_{21} & b_{22}
\end{array}\right]\left[\begin{array}{l}
d x^{\alpha} \\
d x^{\beta}
\end{array}\right]
$$

is called the second fundamental form or curvature tensor and the elements $b_{\alpha \beta}$ are the curvature coefficients. The curvature coefficients are symmetric, that is, $b_{12}=b_{21}$.

The directions for which the normal curvatures are the minimum or maximum are called principal directions. We call the normal curvatures in the principal directions the principal curvatures, and denote them by $k_{1}$ and $k_{2}$. A line of curvature is a curve on the surface with the property that, at any point of the curve, the tangent is a principal direction of the surface at that point. Two lines of curvature intersecting at right-angles.

In terms of the principal curvatures, which are not intrinsic, we may define the mean curvature

$$
\begin{equation*}
H=\frac{k_{1}+k_{2}}{2}=\frac{a_{11} b_{22}-2 a_{12} b_{12}+a_{22} b_{11}}{2\left(a_{11} a_{22}-\left(a_{12}\right)^{2}\right)} \tag{60}
\end{equation*}
$$

and the Gaussian curvature

$$
\begin{equation*}
K=k_{1} k_{2}=\frac{b_{11} b_{22}-b_{12}^{2}}{\left(a_{11} a_{22}-\left(a_{12}\right)^{2}\right)} \tag{61}
\end{equation*}
$$

In some sense, Gaussian curvature measures how far a surface is from being Euclidean plane, since it relates the small radius $\epsilon$ around the point $\mathcal{P}$ and their circunference $L(\mathcal{C})$ (Figure 21.(a)). While Gaussian curvature is related with the ratio between the coverage of the normal vectors and the area defined by the corresponding points on a unitary sphere, the mean curvature may be associated to the variation of the area $\mathcal{A}$ bounded by a closed curve $\mathcal{C}(u(t), v(t))$ on $\mathcal{S}$, with respect to the length on the direction $\mathbf{n}(u(t), v(t))$ (Figure 21.(b)).


Fig. 21. Geometrical interpretation of (a) Gaussian curvature and (b) mean curvature.

A point of a surface is called elliptic, if $K>0$, hyperbolic, if $K<0$, parabolic, if $K=0$, and planar, if $K=0$ and $H=0$.

If $\mathcal{S}$ is orientable, it is possible to assign to each point a basis given by the vectors $\mathbf{a}_{1}, \mathbf{a}_{2}$ and $\mathbf{a}_{3}$. By expressing the derivatives of $\mathbf{a}_{1}$ and $\mathbf{a}_{2}$ in the basis $\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}\right\}$, we obtain the Gauss formula

$$
\begin{equation*}
\mathbf{a}_{\alpha, \beta}=\Gamma_{\alpha \beta}^{\lambda} \mathbf{a}_{\lambda}+b_{\alpha \beta} \mathbf{a}_{3}, \tag{62}
\end{equation*}
$$

where $\Gamma_{\alpha \beta}^{\lambda}$ are the Christoffel symbols that depend exclusively on the coefficients of the first fundamental form and their derivatives as follows

$$
\begin{align*}
\Gamma_{\alpha \beta}^{\gamma} & =\frac{1}{2} a^{\gamma \lambda}\left(\frac{\partial a_{\beta \lambda}}{\partial x^{\alpha}}+\frac{\partial a_{\lambda \alpha}}{\partial x^{\beta}}-\frac{\partial a_{\alpha \beta}}{\partial x^{\lambda}}\right) \\
& =a^{\gamma \lambda}\left(\mathbf{a}_{\alpha, \beta} \cdot \mathbf{a}_{\lambda}\right)=\mathbf{a}_{\alpha, \beta} \cdot \mathbf{a}^{\gamma} \tag{63}
\end{align*}
$$

with $\mathbf{a}^{\gamma}=a^{\gamma \lambda} \mathbf{a}_{\lambda}$ are the reciprocal basis vectors of the basis $\left\{\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}\right\}$. Hence, the Christoffel symbols are also intrinsic properties of $\mathcal{S}$ and all geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.

Since $\mathbf{a}_{\alpha, \beta}=\mathbf{a}_{\beta, \alpha}$, the Christoffel symbols are symmetric relative to the lower indices; that is, $\Gamma_{\alpha \beta}^{\gamma}=\Gamma_{\beta \alpha}^{\gamma}$.

Once $\mathbf{a}_{3, \alpha}=\partial \mathbf{n} / \partial x^{\alpha}$ at $P$ lies in the tangent plane $T_{P}(S)$, we may also express them in terms of the natural basis

$$
\begin{equation*}
\mathbf{a}_{3, \alpha}=-b_{\alpha}^{\beta} \mathbf{a}_{\beta}, \tag{64}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{\alpha}^{\beta}=a^{\beta \lambda} b_{\alpha \lambda} . \tag{65}
\end{equation*}
$$

These equations are called Weingarten equations. The mean and the Gaussian curvature correspond, respectively, to the half of the trace and the determinant of the matrix

$$
\left[\begin{array}{ll}
b_{1}^{1} & b_{2}^{1} \\
b_{1}^{2} & b_{2}^{2}
\end{array}\right]
$$

The most surprising result is the (Gauss) equation

$$
\begin{align*}
\frac{\partial \Gamma_{12}^{2}}{\partial x^{1}}-\frac{\partial \Gamma_{11}^{2}}{\partial x^{2}}+\Gamma_{12}^{1} \Gamma_{11}^{2}+\Gamma_{12}^{2} \Gamma_{12}^{2}-\Gamma_{11}^{2} \Gamma_{22}^{2} & -\Gamma_{11}^{1} \Gamma_{12}^{2} \\
& =-a_{11} K \tag{66}
\end{align*}
$$

which proves the Theorema Egregium
Theorem B. 1 Egregium. The Gaussian curvature $K$ of a surface is invariant by local isometries.

This theorem says that all geometric properties given in terms of the Gaussian curvature $K$ are bending invariant, that is, the properties that are unchanged by deformations which do not involve stretching, shrinking, or tearing. For example, a cylinder and a plane have the same Gaussian curvature. For distinguishing these two classes of shapes, we should use other measurements, such as the mean curvature $H$ : the mean curvature of a plane is zero while that of a cylinder is nonzero. Unlike the Gaussian curvature, the mean curvature depends on the embedding and is closely related to the first variation of surface area.

When the intrinsic metric properties (distances of points along curvilinear coordinates or angles of their tangent directions at a point) and the extrinsic geometric properties (the way that the normal vectors vary along curvilinear coordinates) change, the coefficients of the first and the second fundamental forms cannot vary independently. They should obey the Mainardi-Codazzi equations

$$
\begin{aligned}
& \frac{\partial b_{11}}{\partial a_{2}}-\frac{\partial b_{12}}{\partial a_{1}}=b_{11} \Gamma_{12}^{1}+b_{12}\left(\Gamma_{12}^{2}-\Gamma_{11}^{1}\right)-b_{22} \Gamma_{11}^{2} \\
& \frac{\partial b_{12}}{\partial a_{2}}-\frac{\partial b_{22}}{\partial a_{1}}=b_{11} \Gamma_{22}^{1}+b_{12}\left(\Gamma_{22}^{2}-\Gamma_{12}^{1}\right)-b_{22} \Gamma_{12}^{2}
\end{aligned}
$$

which may be derived from Eq. 62 and the compatibility conditions

$$
\begin{align*}
& \frac{\partial^{2} \mathbf{r}}{\partial^{2} x^{1}} \\
& \partial x^{2} \tag{67}
\end{align*}=\frac{\frac{\partial^{2} \mathbf{r}}{\partial x^{1} \partial x^{2}}}{\partial x^{1}}, ~\left(\frac{\partial^{2} \mathbf{r}}{\partial^{2} x^{2}}=\frac{\frac{\partial^{2} \mathbf{r}}{\partial x^{1} \partial x^{2}}}{\partial x^{1}}=\frac{1}{2}\right.
$$

A natural question is whether the converse also holds, that is whether the knowledge of the first and second fundamental form determines a surface locally. The answer of this question is due to O . Bonnet.

Theorem B. 2 Fundamental Theorem. If $a_{11}, a_{12}, a_{22}$ and $b_{11}, b_{12}, b_{22}$ are given as functions of $u$ and $v$, sufficiently differentiable, which satisfy the Mainardi-Codazzi equations, while $a_{11} a_{22}-a_{12}^{2} \neq 0$, then there exists a surface which admits as its first and second fundamental forms $I=a_{11} d u^{2}+2 a_{12} d u d v+a_{22} d v^{2}$ and $I I=b_{11} d u^{2}+2 b_{12} d u d v+b_{22} d v^{2}$, respectively. This surface is uniquely determied except for its position is space.

## ACKNOWLEDGMENTS

The second and the third authors would like to acknowledge the Brazilian Coordination for the Improvement of Higher Education Personnel Foundation (CAPES/PICD) for financial support, respectively, during the period 2000 - 2004 in the context of the doctorate program at Unicamp, and duraing the period 2004 - 2006 in the context of the master program at the same institution.

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Received March 2005; revised May 2006; accepted Month Year


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[^1]:    ACM Transactions on Graphics, Vol. V, No. N, Month 20YY.

[^2]:    ${ }^{1}$ Relations that describe the connections between two physical quantities. Examples of constitutive equations are the Hooke's law, the Ohm's law, the thermal condutiviy, and the Navier's equations.

[^3]:    ${ }^{2}$ The coordinate curves of $\mathcal{S}$ will not generally be lines of curvature for $\mathcal{S}$.

