IA841 – Modelagem de Sólidos

Geometria Diferencial

Farin: Capítulos 10 e 19

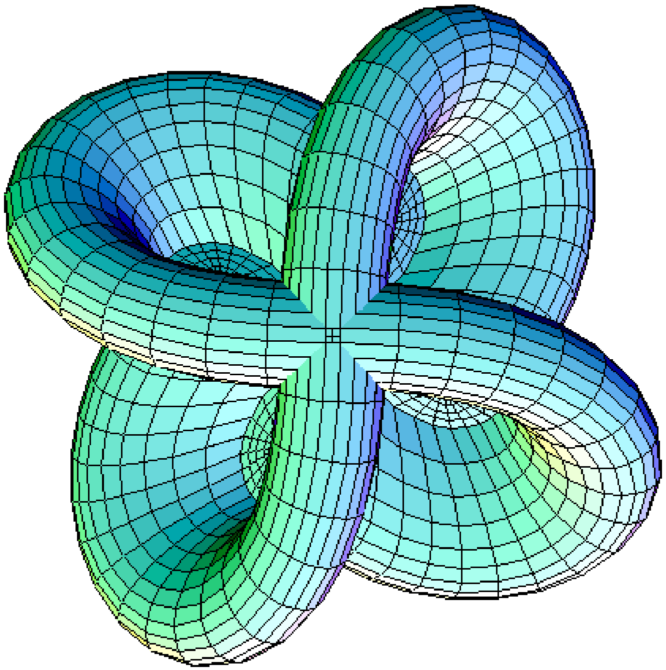
“Geometria Diferencial das curvas e superfícies tem dois aspectos.

… Gemetria Diferencial Clássica é o estudo de propriedades locais das curvas e superfícies. Por propriedades locais entendemos aquelas propriedades que dependem apenas do comportamento da curva ou superfície nas proximidades de um ponto …

O outro aspecto é chamado geometria diferencial global. Estuda-se aqui a influência das propriedades locais sobre o comportamento da curva ou superfície como um todo.”

Extraído do livro “Geometria Diferencial de Curvas e Superfícies”,
Manfredo Perdigão do Carmo

Geometria Diferencial



Funções diferenciáveis sobre as quais pode-se utilizar os métodos do Cálculo Diferencial para análise

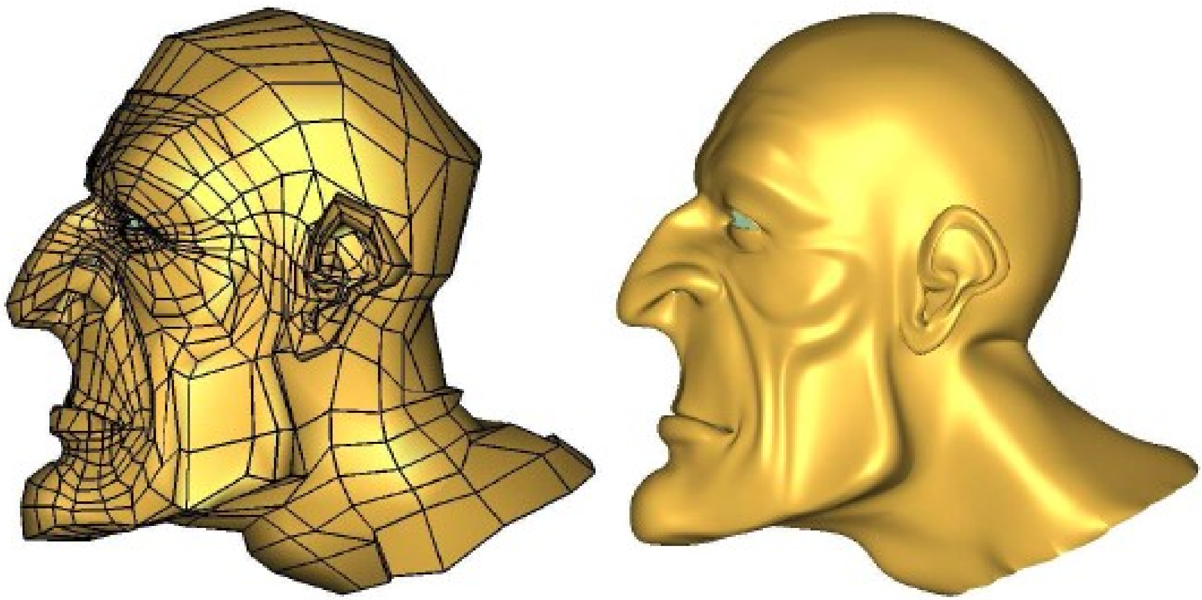
Onde é aplicada?

- Renderização não-fotorealística

http://cgtools.net/analysis_npr_pen_and_ink.php

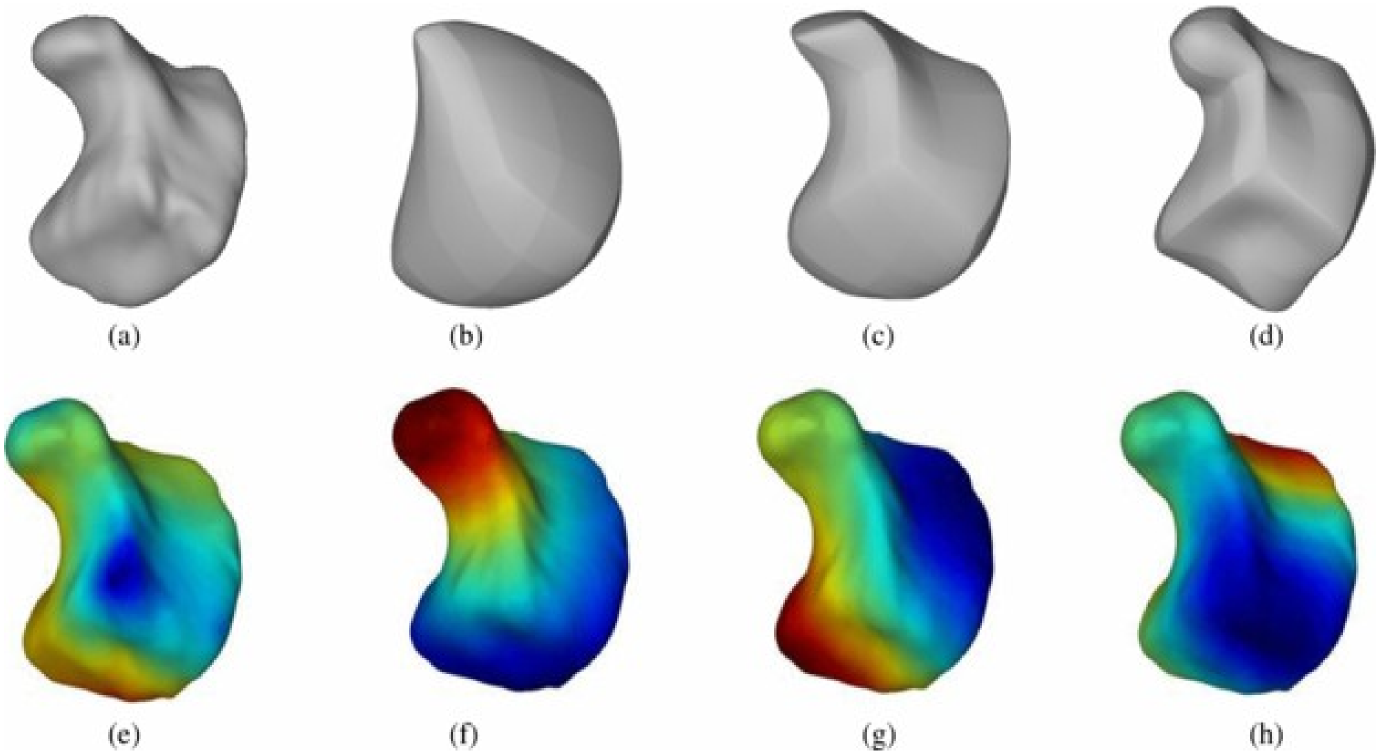
Onde é aplicada?

- Triangulação adaptativa



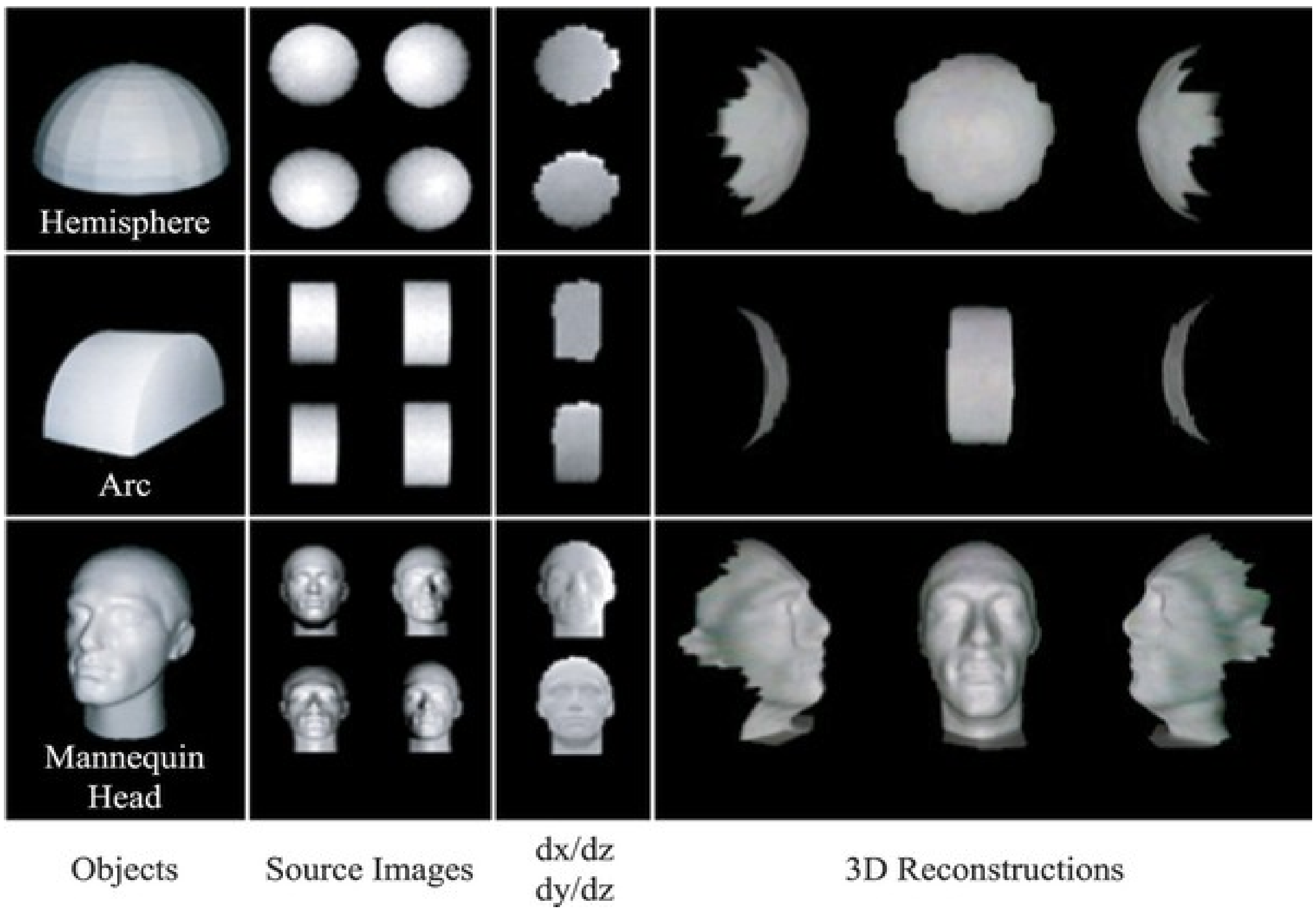
Onde é aplicada?

- Análise de formas



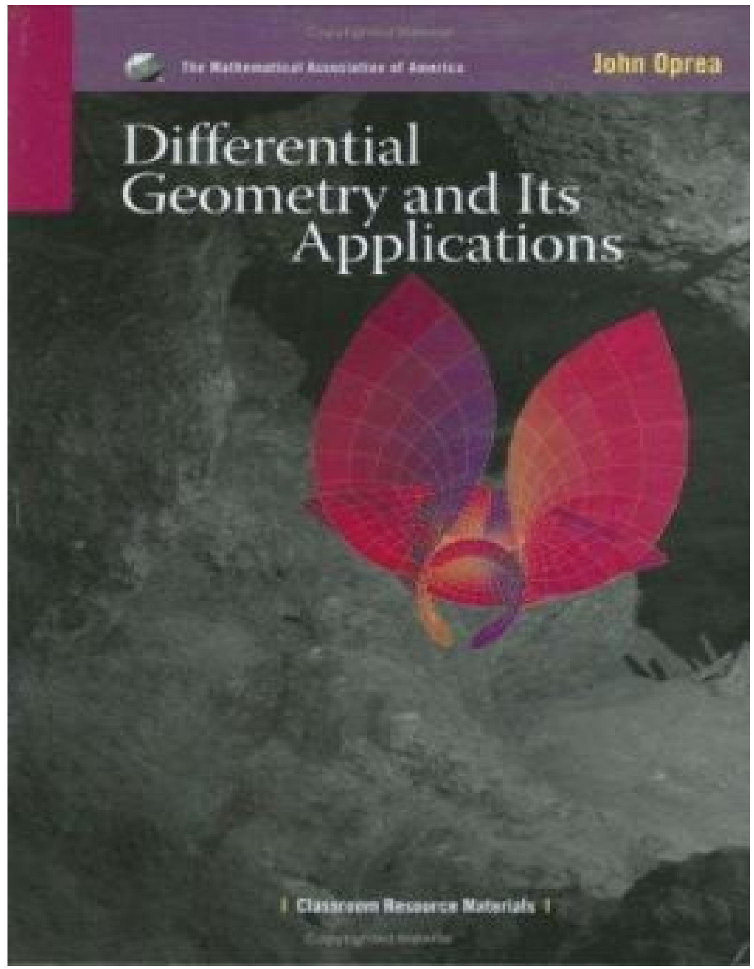
Onde é aplicada?

- Reconstrução 3D

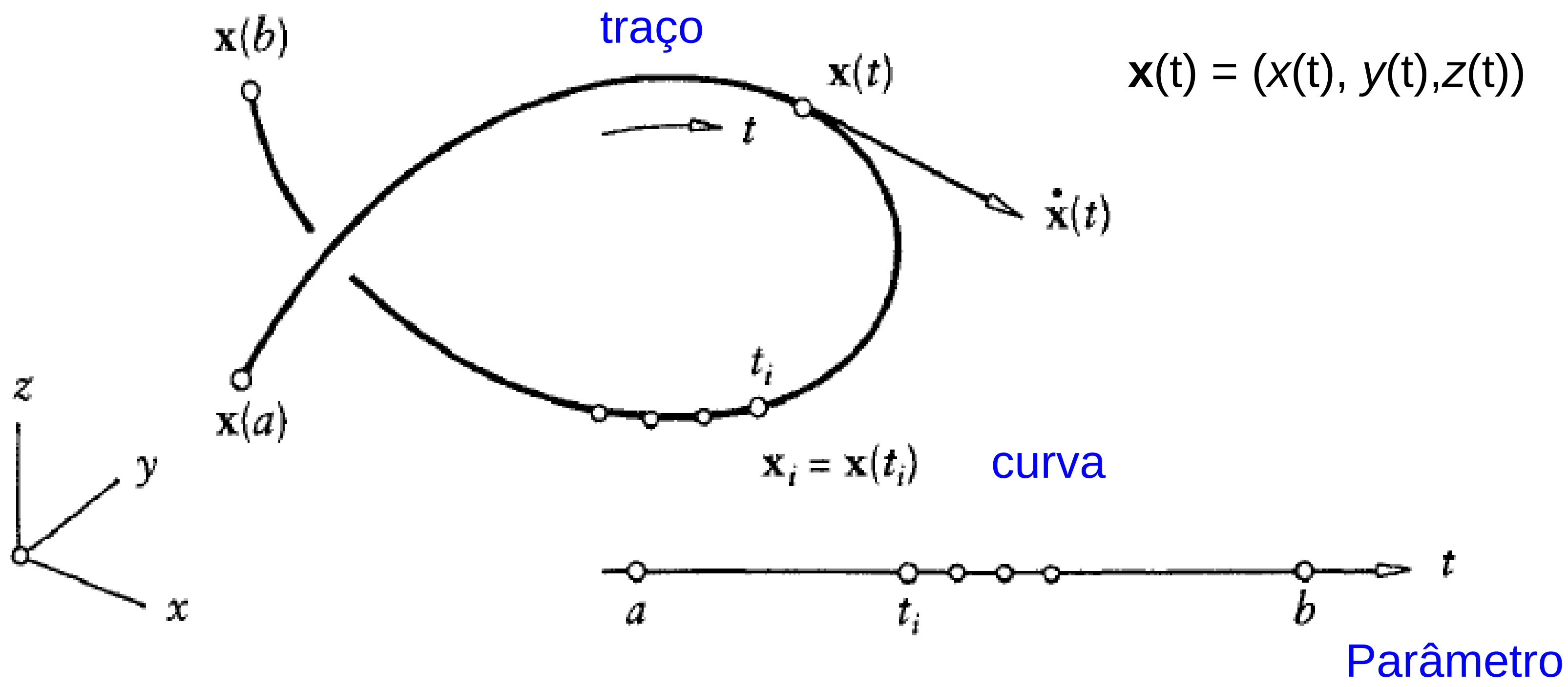


Onde é aplicada?

- etc.



Função Paramétrica Diferenciável

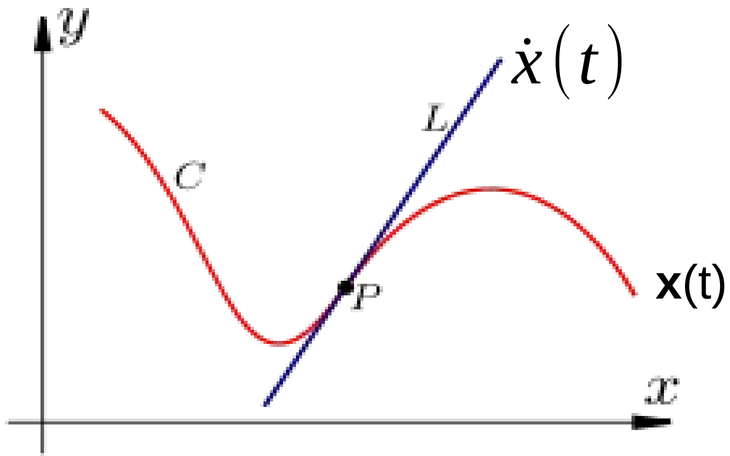


x(t) = (*x*(t), *y*(t),*z*(t))

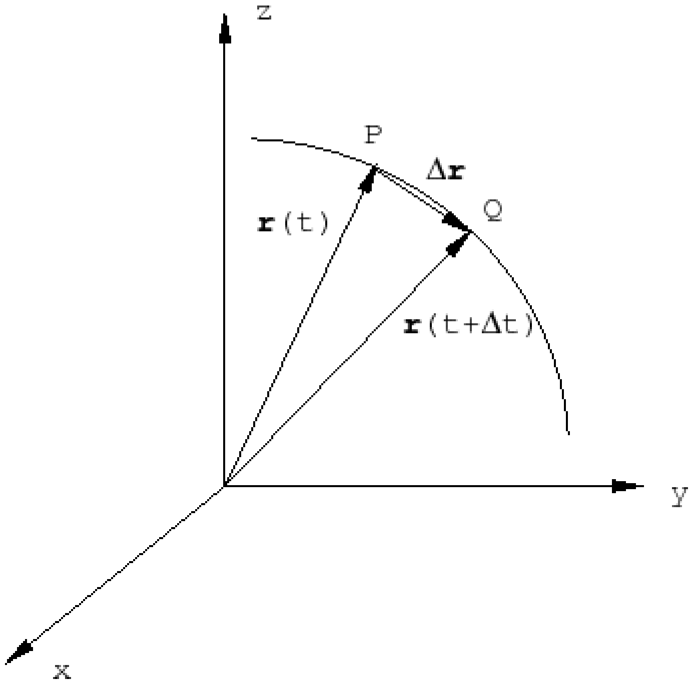
x(t), *y*(t),*z*(t) são funções diferenciáveis (ou suaves), se elas possuem em todos os pontos derivadas de todas as ordens.

Curvas Regulares

- São curvas para as quais $\dot{\mathbf{x}}(t) = d\mathbf{x}/dt \neq 0$ para todo t do intervalo em que elas estão definidas.



Comprimento de Arco



Aproximação do comprimento da curva

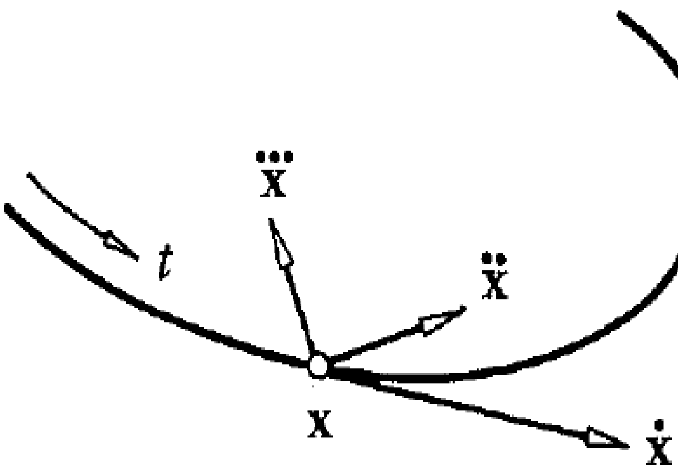
$$\sum_{t_0}^{t} \left|\frac{\Delta r}{\Delta t}\right| \Delta t$$

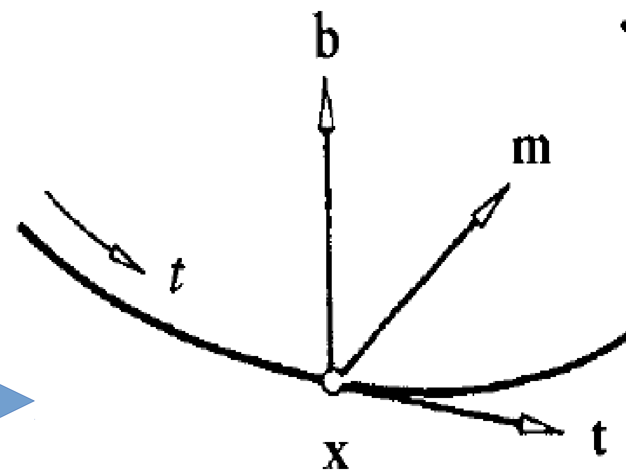
$$\Delta t \rightarrow 0 \Rightarrow s(t) = \int_{t_0}^{t} |\dot{r}(t)| dt$$

com $|\dot{r}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$

Variações Diferenciais

$$\boldsymbol{x}(t+\Delta t)=\boldsymbol{x}+\dot{\boldsymbol{x}}\,\Delta t+\ddot{\boldsymbol{x}}\,\frac{1}{2}\Delta t^2+\dddot{\boldsymbol{x}}\,\frac{1}{6}\Delta t^3+\dots$$



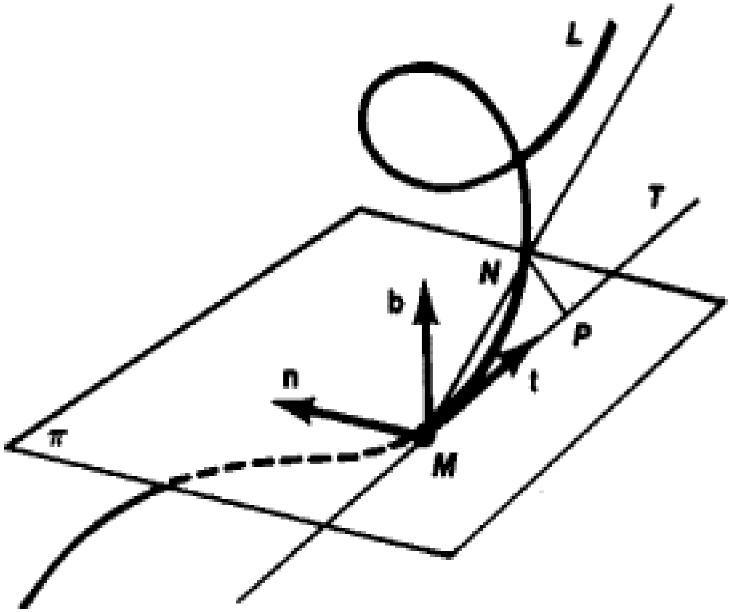


$$\begin{bmatrix}\dot{\boldsymbol{x}} & \ddot{\boldsymbol{x}} & \dddot{\boldsymbol{x}}\end{bmatrix}\begin{bmatrix}\Delta t\\ \frac{1}{2}\Delta t^2\\ \frac{1}{6}\Delta t^3\end{bmatrix}$$

$$\boldsymbol{m}=\frac{\dot{\boldsymbol{x}}\,\dot{\boldsymbol{x}}\cdot\ddot{\boldsymbol{x}}-\dot{\boldsymbol{x}}\,\ddot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}}{|\dot{\boldsymbol{x}}\,\dot{\boldsymbol{x}}\cdot\ddot{\boldsymbol{x}}-\dot{\boldsymbol{x}}\,\ddot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}|}$$

$$\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{m}$$

Triedro de Frenet



$$\boldsymbol{t} = \frac{\dot{\boldsymbol{x}}}{|\dot{\boldsymbol{x}}|}$$ tangente

$$\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t}$$ normal

$$\boldsymbol{b} = \frac{\dot{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}}{|\dot{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}|}$$ binormal

Parametrização por s

Δs corresponde ao comprimento da curva $\Delta\mathbf{x}$, ou seja,

$$\frac{d\,\mathbf{x}}{ds} = \mathbf{x}'(s) = 1 \Rightarrow |\mathbf{x}'(s)| = 1 \Rightarrow \mathbf{x}'(s) = \mathbf{t}$$

Como $\quad |\mathbf{x}'(s)| = \mathbf{x}'(s) \cdot \mathbf{x}'(s) = 1$

$$\mathbf{x}'(s) \cdot \mathbf{x}''(s) = 0$$

$$\Downarrow$$

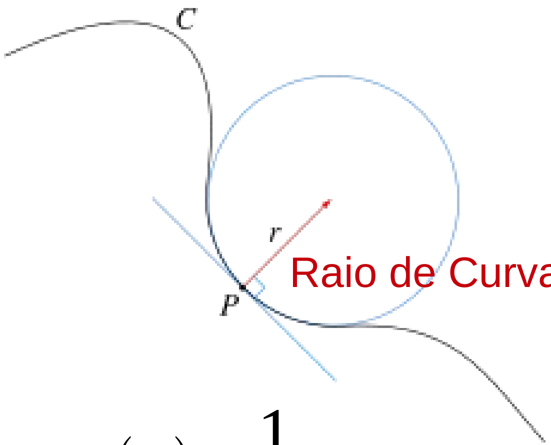
$$\mathbf{x}'(s) \perp \mathbf{x}''(s)$$

$$\Downarrow$$

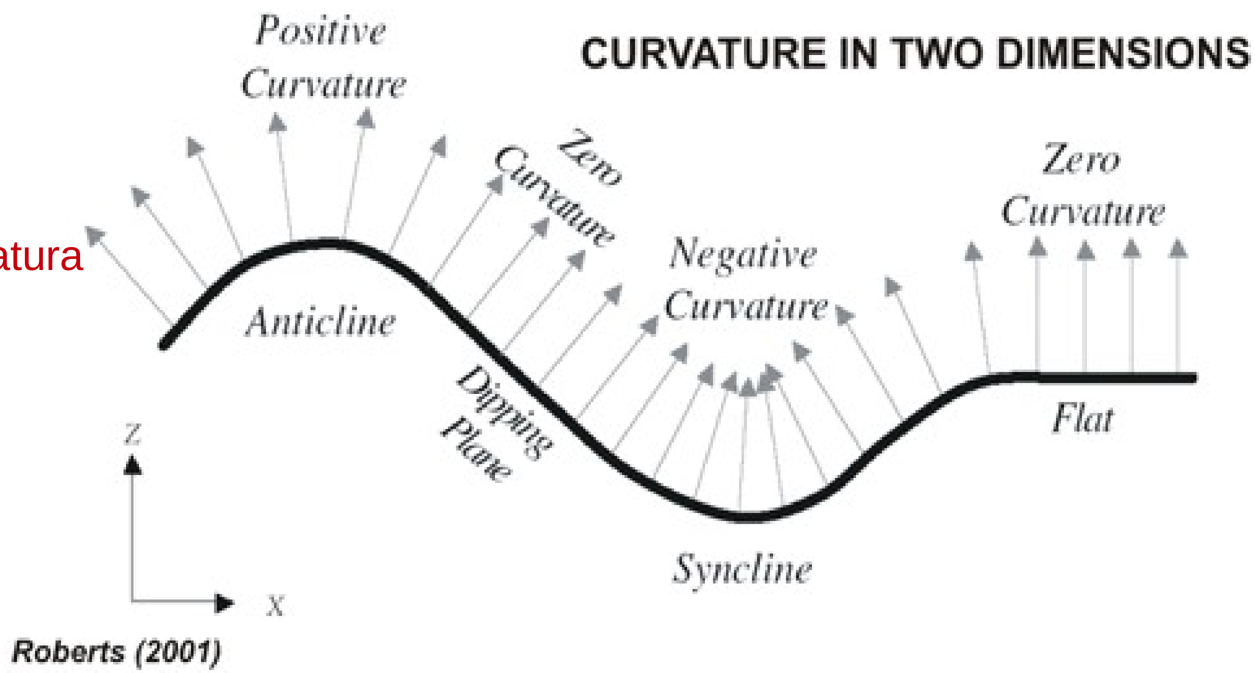
Curvatura

$$\mathbf{x}''(s) = \mathbf{t}' = \kappa(s)\,\mathbf{n}$$

Curvatura



$$\kappa(s)=\frac{1}{r}$$

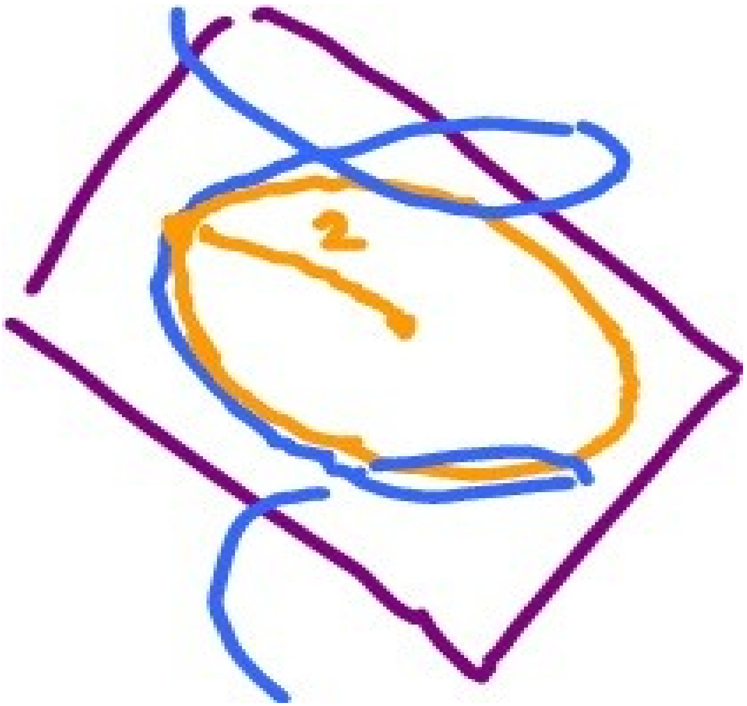


$$\beta(-s)=\alpha(s)$$

$$\frac{d\beta(-s)}{d(-s)}(-1)=\frac{d\alpha(s)}{ds}\Rightarrow\frac{d\beta(-s)}{d(-s)}=-\frac{d\alpha(s)}{ds}$$

$$\frac{d^2\beta(-s)}{d^2(-s)}(-1)=\frac{-d^2\alpha(s)}{d^2 s}\Rightarrow-\frac{d^2\beta(-s)}{d^2(-s)}=-\frac{d^2\alpha(s)}{d^2 s}$$

Plano Osculador



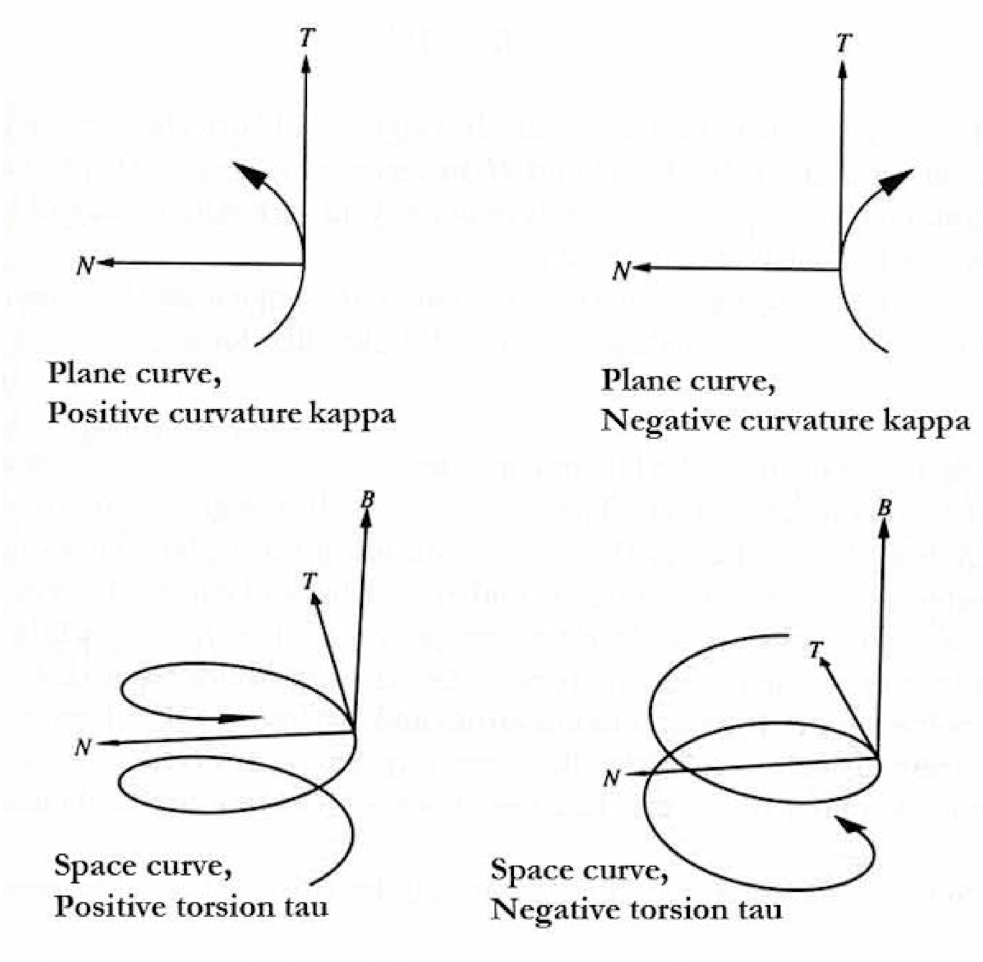
O círculo osculador está contido no plano osculador. É determinado pelos vetores $\boldsymbol{t}$ e $\boldsymbol{n}$

Círculo osculador tem contato de ordem 2 com a curva.

http://inperc.com/wiki/index.php?title=Curvature

Torção

$$\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}\Rightarrow\boldsymbol{b}'=\boldsymbol{t}'\times\boldsymbol{n}+\boldsymbol{t}\times\boldsymbol{n}'=\boldsymbol{t}\times\boldsymbol{n}'\Rightarrow\boldsymbol{b}'=-\tau\,\boldsymbol{n}$$



Indica quão rapidamente a curva se afasta, em uma vizinhança do ponto, do plano osculador.

Curvatura e Torção

$$\kappa(s) = |\mathbf{x}''|$$

$$\kappa(t) = \frac{|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}|}{|\dot{\mathbf{x}}|^3}$$

$$\tau(s) = \frac{1}{\kappa^2} det[\mathbf{x}', \mathbf{x}'', \mathbf{x}''']$$

$$\tau(t) = \frac{det[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dddot{\mathbf{x}}]}{|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}|^2}$$

Fórmulas de Frenet

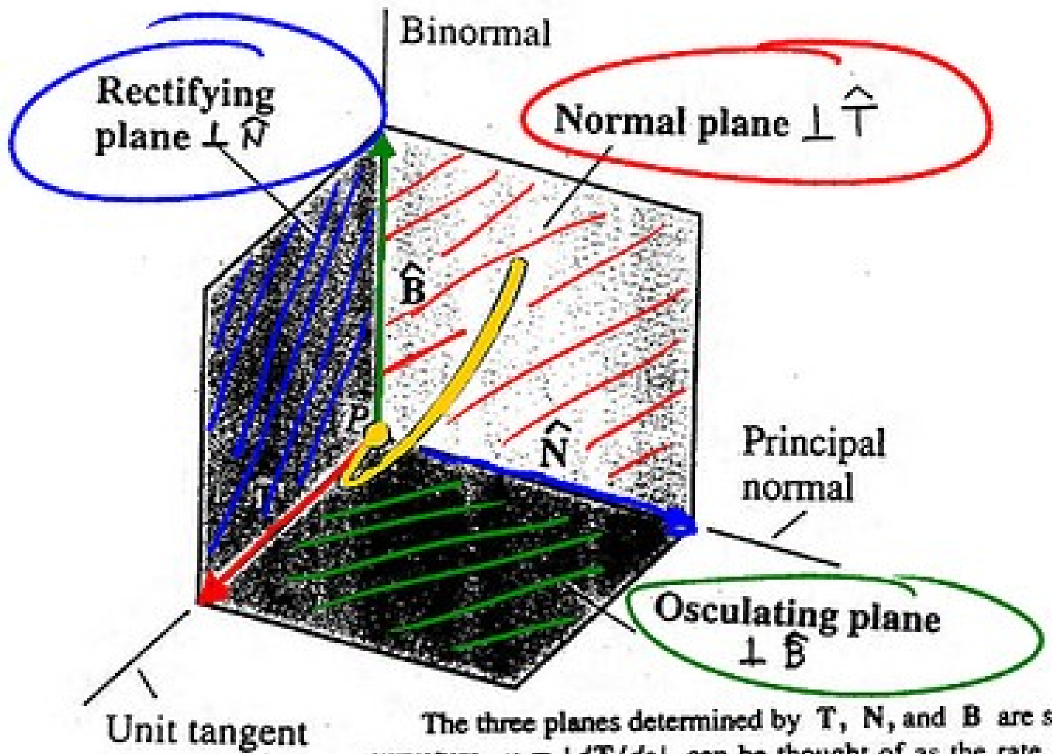
$$\boldsymbol{n}=\boldsymbol{b}\times\boldsymbol{t}\Rightarrow\boldsymbol{n}'=\boldsymbol{b}'\times\boldsymbol{t}+\boldsymbol{b}\times\boldsymbol{t}'\Rightarrow\boldsymbol{n}'=\tau\boldsymbol{n}-\kappa\boldsymbol{t}$$

$$\boldsymbol{t}'=\kappa\boldsymbol{n}$$
$$\boldsymbol{n}'=\tau\boldsymbol{n}-\kappa\boldsymbol{t}$$
$$\boldsymbol{b}'=-\tau\boldsymbol{n}$$

As derivadas podem ser expressas em termos do triedro de Frenet.

O comportamento de uma curva pode ser descrito completamente por curvatura e torção.

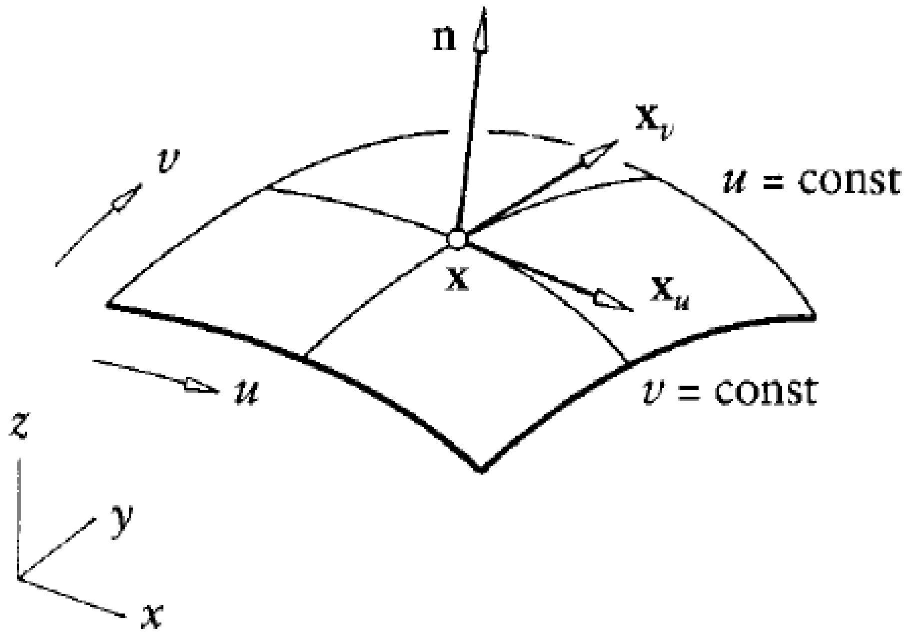
Planos

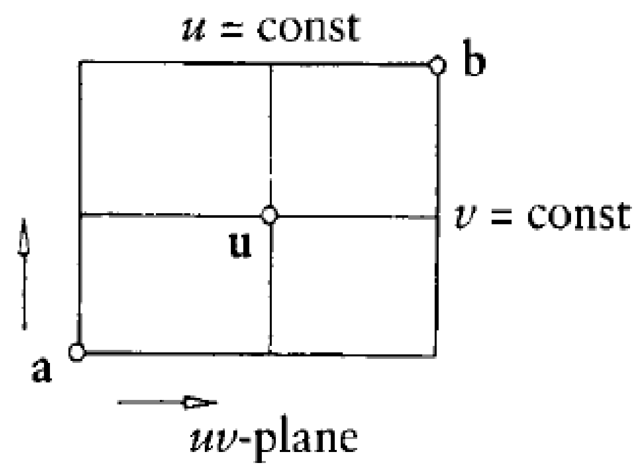


The three planes determined by **T**, **N**, and **B** are shown. The curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about **T** as P moves along the curve. Torsion measures how the curve twists.

If P is a train climbing up a curved track, the rate at which the headlight turns from side to side per unit distance is the curvature of the track. The rate at which the engine tends to twist out of the plane formed by **T** and **N** is the torsion.

Superfícies Paramétricas Regulares

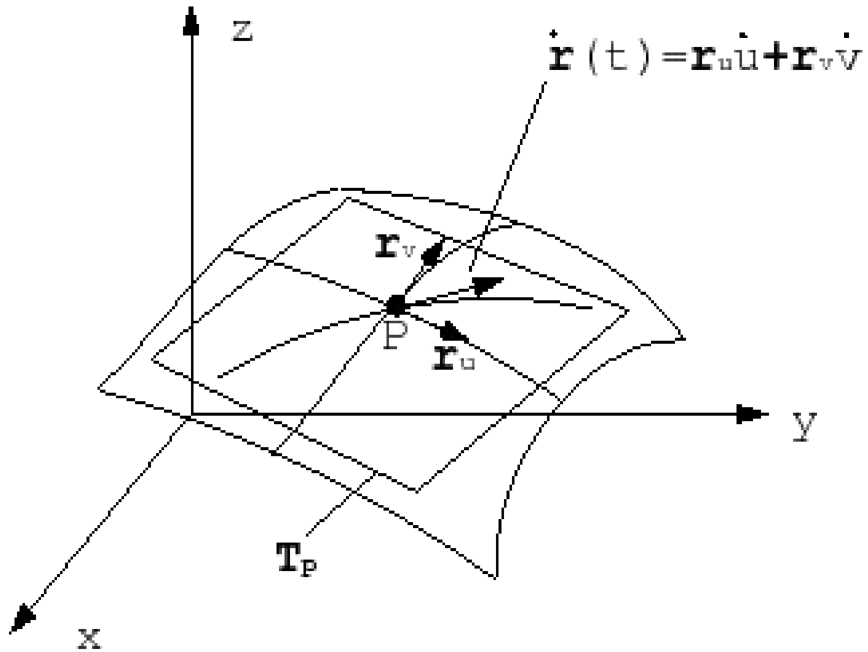




$$\boldsymbol{x}(u,v)=\begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

$$\boldsymbol{x}_u \times \boldsymbol{x}_v \neq 0$$

Comprimento de Arco de uma Curva



$$ds = \boldsymbol{r}_u \dot{u} + \boldsymbol{r}_v \dot{v}$$

$$ds^2 = (\boldsymbol{r}_u \dot{u} + \boldsymbol{r}_v \dot{v}) \cdot (\boldsymbol{r}_u \dot{u} + \boldsymbol{r}_v \dot{v})$$

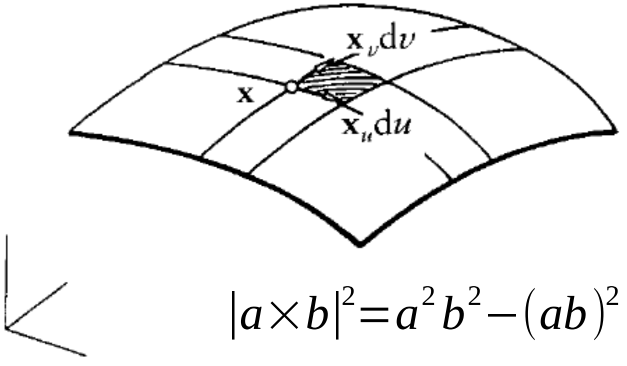
Tensor Métrico

$$ds^2 = \overbrace{\boldsymbol{r}_u \boldsymbol{r}_u}^{E} \dot{u}^2 + 2 \overbrace{\boldsymbol{r}_u \boldsymbol{r}_v}^{F} \dot{u}\,\dot{v} + \overbrace{\boldsymbol{r}_v \boldsymbol{r}_v}^{G} \dot{v}^2$$

$$ds^2 = \begin{bmatrix} \dot{u} & \dot{v} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}$$

Comprimento da curva $\int_{t_0}^{t} |\dot{\boldsymbol{r}}|\, dt = \sqrt{E\,\dot{u}^2 + 2F\,\dot{u}\,\dot{v} + G\,\dot{v}^2}\, dt$

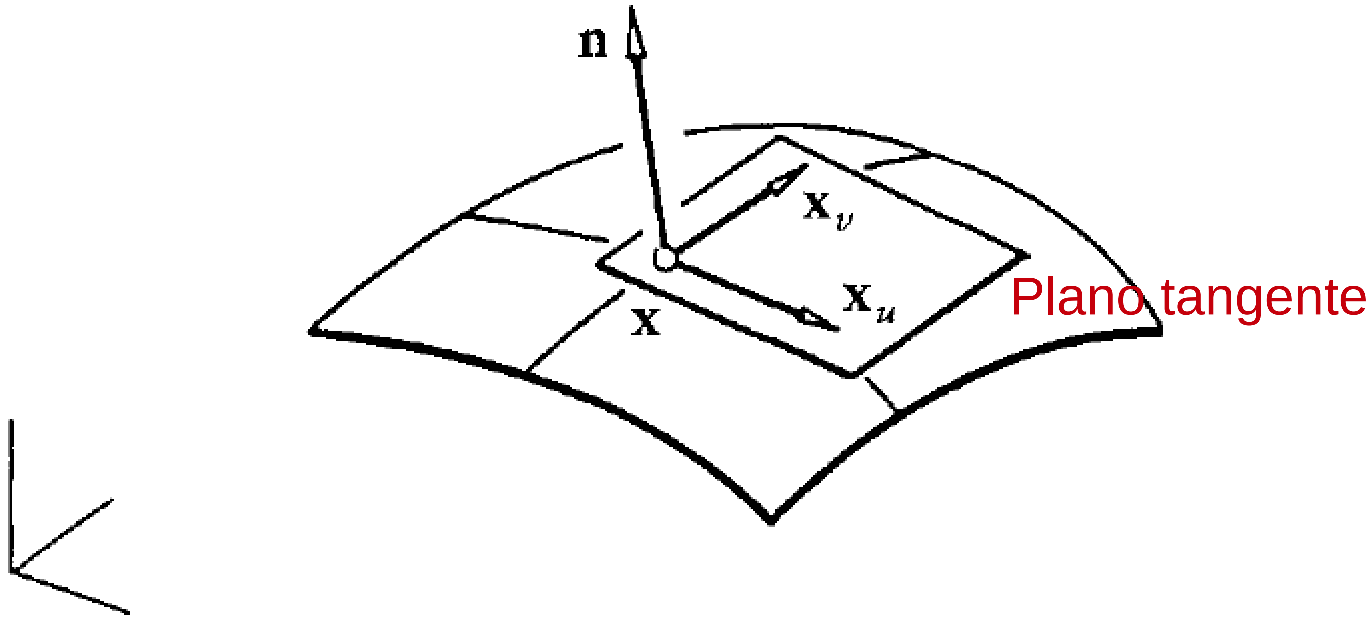
Área $|\boldsymbol{r}_u \dot{u} \times \boldsymbol{r}_v \dot{v}| = \sqrt{EG - F^2}\, du\, dv$



Ângulo $\cos\alpha = \frac{\boldsymbol{r}_u \cdot \boldsymbol{r}_v}{|\boldsymbol{r}_u||\boldsymbol{r}_v|} = \frac{F}{\sqrt{EG}}$

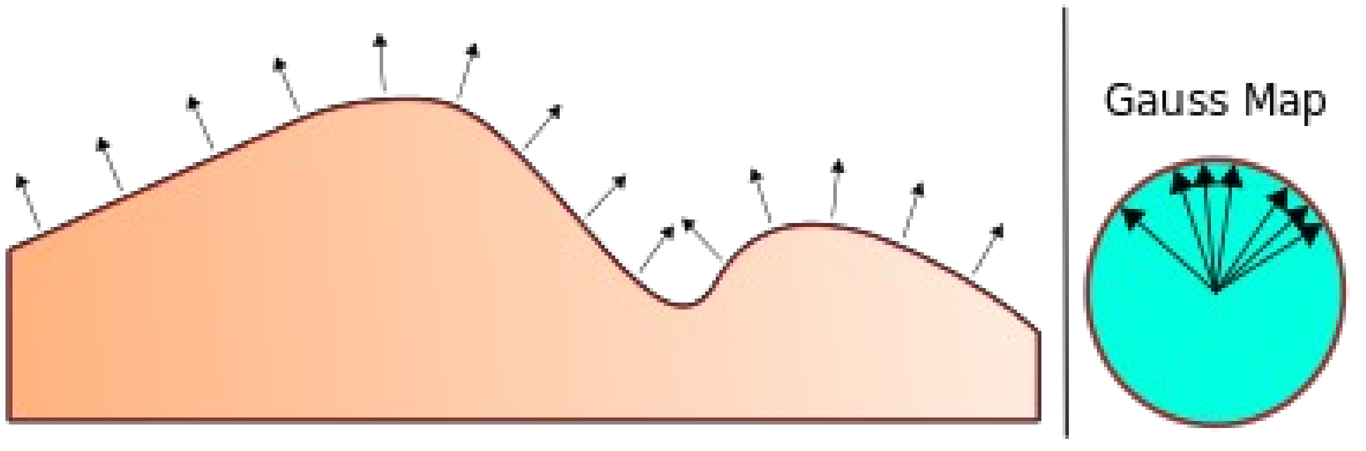
$|a \times b|^2 = a^2 b^2 - (ab)^2$

Triedro



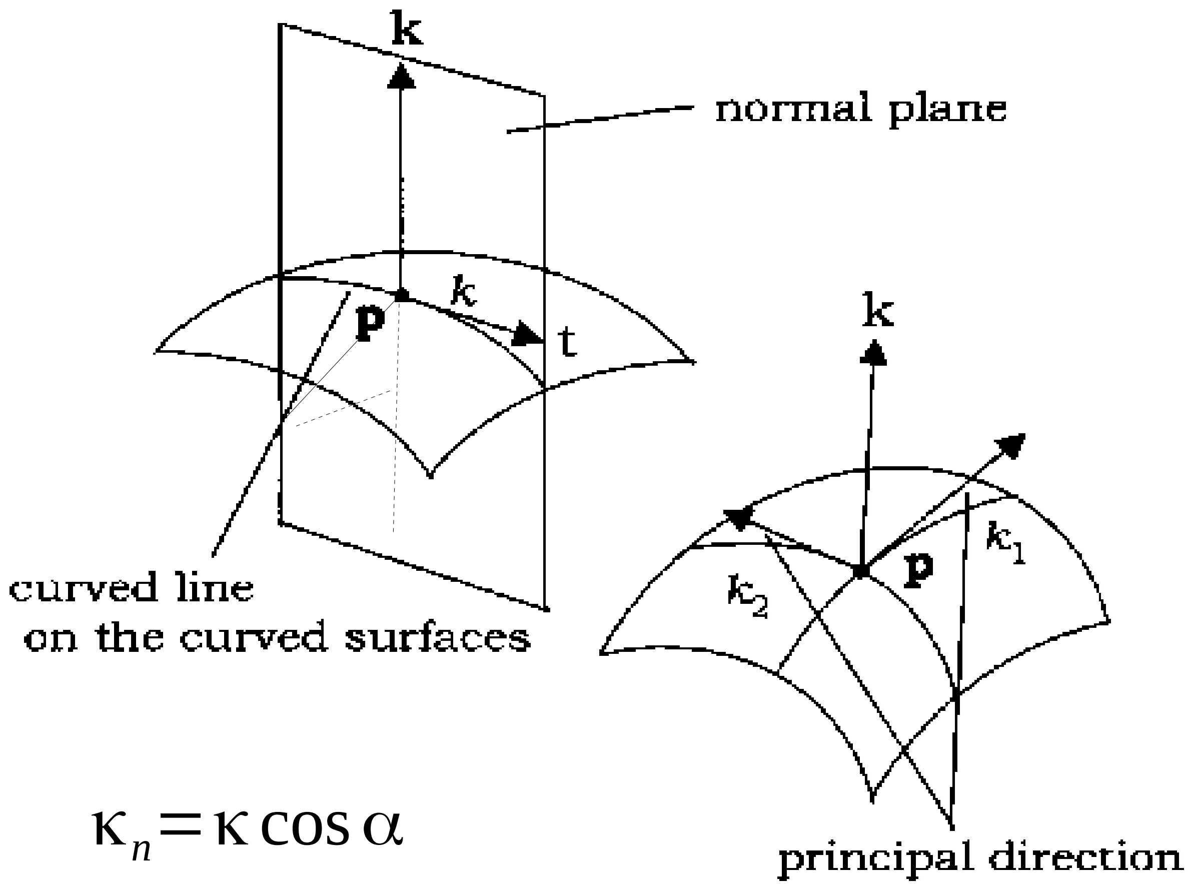
$$\boldsymbol{n} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\left|\boldsymbol{x}_u \times \boldsymbol{x}_v\right|}$$

Aplicação de Gauss



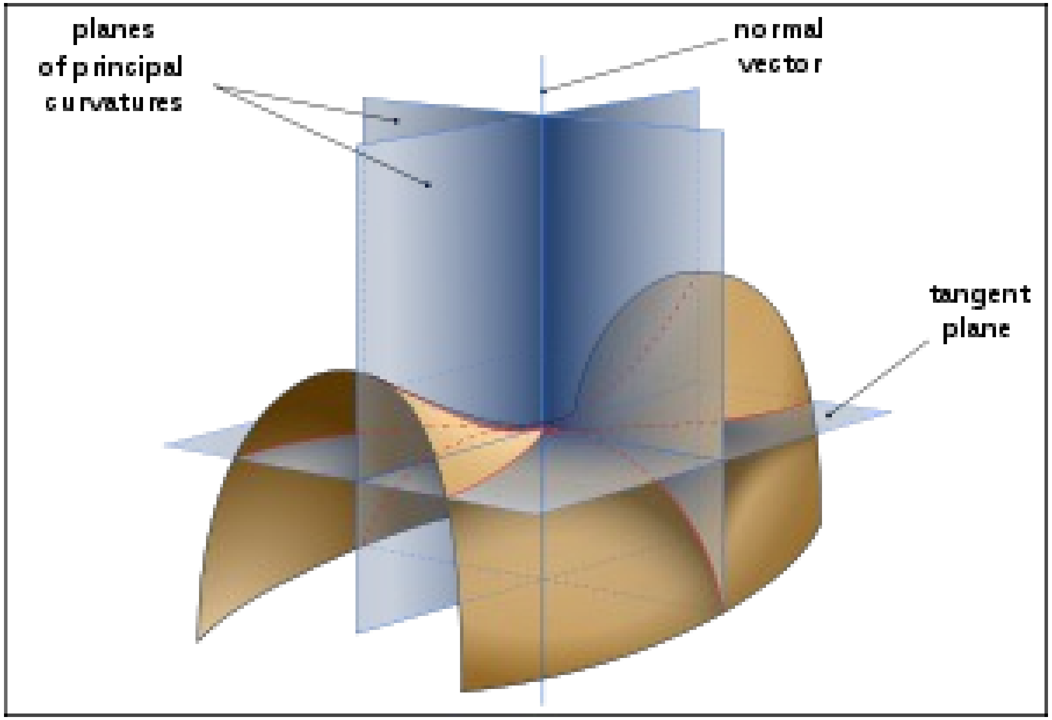
Associa cada ponto da superfície a um ponto de uma esfera unitária

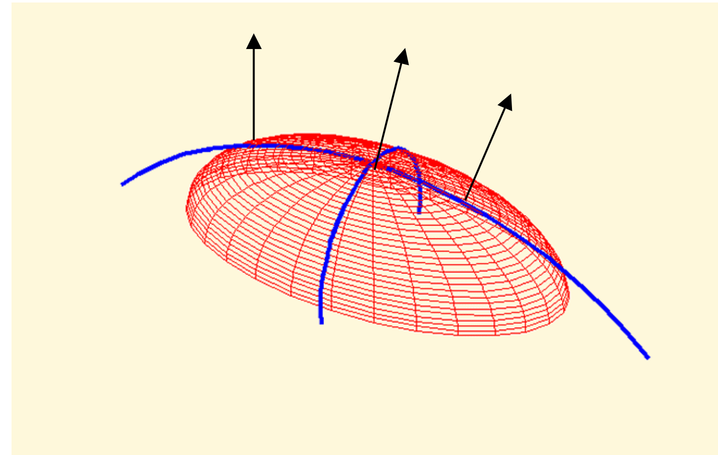
Curvatura Normal



$$\kappa_n = \kappa \cos \alpha$$

Tensor de Curvatura





$$-ds \cdot dn = -(\frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv) \cdot (\frac{\partial n}{\partial u} du + \frac{\partial n}{\partial v} dv)$$

$$-ds \cdot dn = -\frac{\partial r}{\partial u}\frac{\partial n}{\partial u} du^2 - \frac{\partial r}{\partial u}\frac{\partial n}{\partial v} dudv - \frac{\partial n}{\partial u}\frac{\partial r}{\partial v} dudv - \frac{\partial r}{\partial v}\frac{\partial n}{\partial v} dv^2$$

$$-ds \cdot dn = -\underbrace{\frac{\partial r}{\partial u}\frac{\partial n}{\partial u}}_{L} du^2 - 2\underbrace{\frac{\partial r}{\partial u}\frac{\partial n}{\partial v}}_{M} dudv - \underbrace{\frac{\partial r}{\partial v}\frac{\partial n}{\partial v}}_{N} dv^2$$

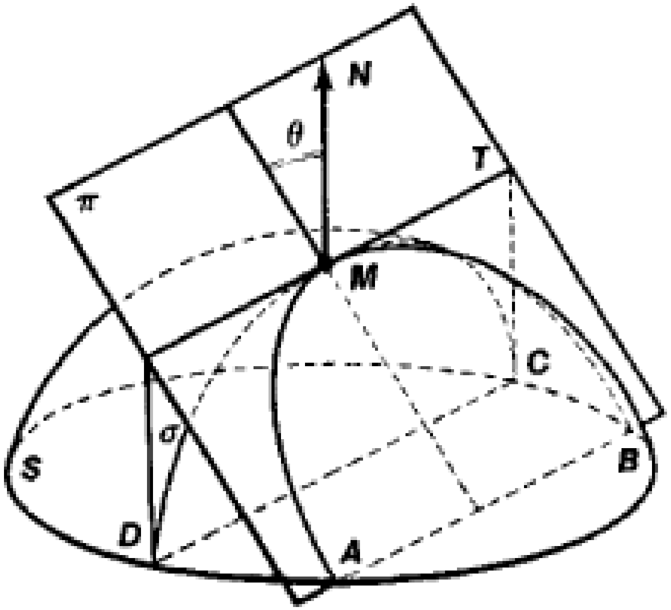
Teorema de Meusnier

- Todas as curvas de uma superfície que tem num ponto a mesma reta tangente, tem neste ponto a mesma curvatura normal

$$\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$

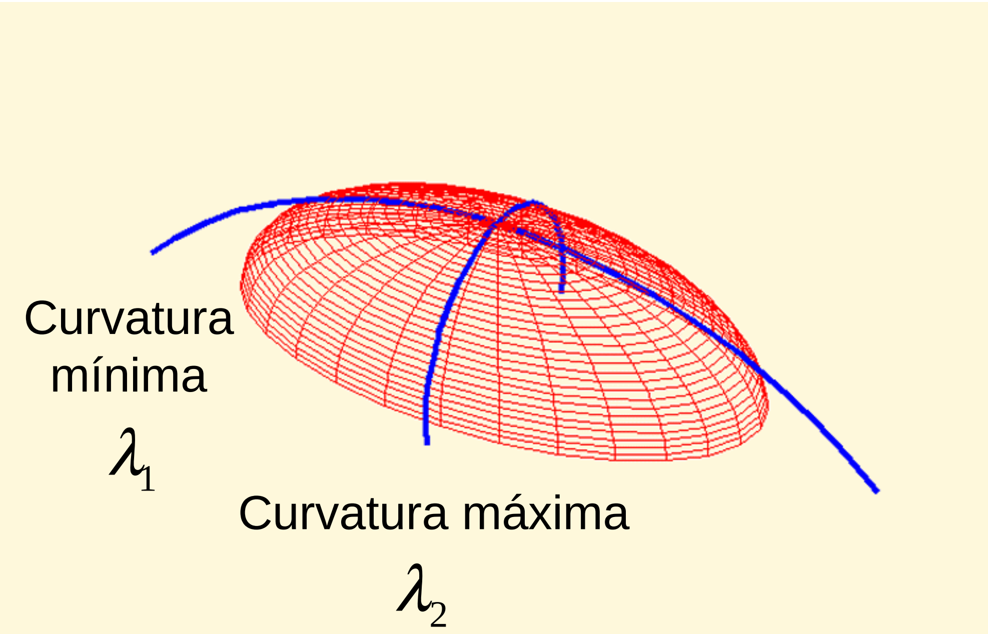
$$\lambda = \frac{dv}{du}$$

direção do vetor tangente da curva



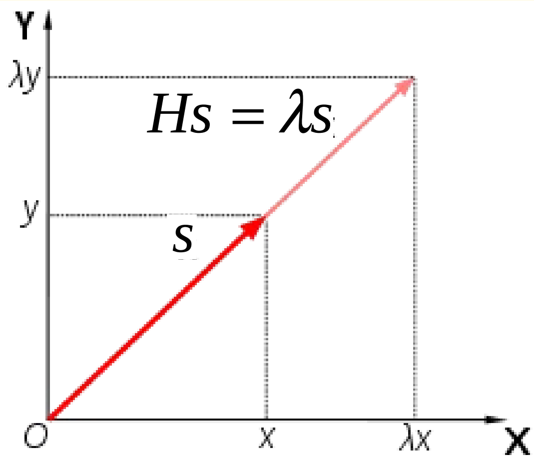
http://encyclopedia2.thefreedictionary.com/Meusnier%27s+Theorem

Linhas de Curvatura



$$\begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} = -\begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

$$Hs = \lambda s$$



$$\det\left(\begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) =$$

$$(h_{11} - \lambda)(h_{22} - \lambda) - h_{12}h_{21} = 0$$

Soluções λ_1, λ_2: autovalores

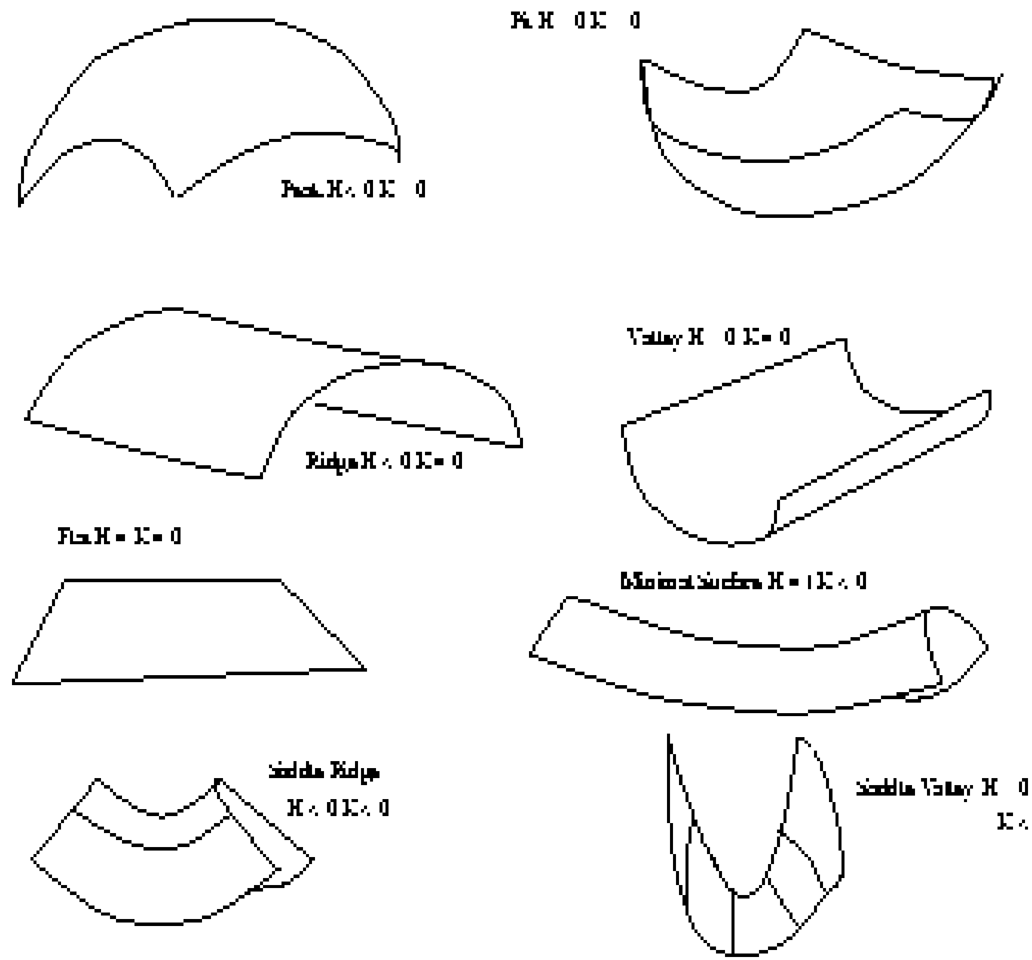
λ_i → um autovetor s_i

Curvaturas Gaussiana e Média

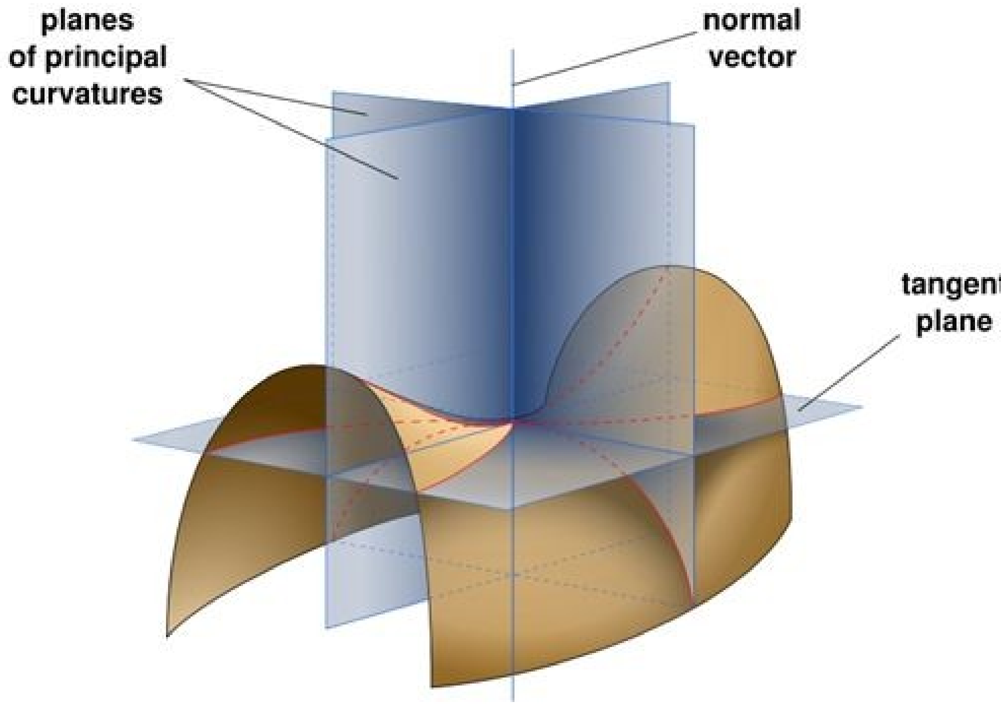
$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2}$$

K > 0: pontos elípticos
K <0: pontos hiperbólicos
K=0: pontos parabólicos
ou planares

$$2H = \kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2}$$



Principal Directions



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \quad \varphi = \text{angle with } \mathbf{t}_1$$